Hodge-theory: Abel to Deligne

Phillip Griffiths

Talk given at the IAS, October 14, 2013.

- this will be mostly an historical talk — it is not intended as an overview nor a current state of affairs of Hodge theory — it will also be informal — intent is to give a sense of things and how they fit together, not to present precise statements
- the central point of the talk will be the analogy
  - Abel took a subject (elliptic integrals) in which there were very interesting special cases, and set it in a new and general context that opened up the subject
  - similarly, Deligne's theory of mixed Hodge structures took the subject of classical Hodge theory for smooth, projective varieties and recast it for arbitrary complex varieties and maps between them, thus similarly opening up the subject of Hodge theory

- will begin by recounting Abel's theorem in its original form and then trace some history, emphasizing especially the work of Picard which is perhaps less familiar, up through Lefschetz, Hodge and Deligne's theory, concluding with some comments on the recent work of M. Saito who took mixed Hodge theory and variations of Hodge structure and fused them into the currently penultimate state of the subject
- my presentation of the historical development is subjective and personal and does not reflect a researched account of the development of the subject — I will also draw on some recollections of what was told to me as a student by Lefschetz, Kodaira and Spencer, and later by Hodge when I visited him in Cambridge (UK)

- finally, as we all know there are issues about what was by current standards actually proved by the classical mathematicians — here I will draw three levels of distinction
  - an argument that can readily be made into a modern proof (e.g., Abel's theorem or Picard's version of the algebraic de Rham theorem)
  - an argument that is at least to me convincing but I have not written out all the details (the existence of the Picard variety of ∞^q rationally inequivalent curves on an algebraic surface S, where $q = h^0(\Omega^1_S)$)
  - an argument that is incomplete but which was later done (e.g., Lefschetz's for the hard Lefschetz theorem and Hodge's for the Hodge theorem)

$\langle \circ \rangle$

Abel

- 18^{th} century mathematicians were interested in integrals of algebraic functions

$$\int r(x, y(x))dx, \qquad f(x, y(x)) = 0;$$

they arise in geometry

$$\int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1 - x^2}} \text{ for } x^2 + y^2 = 1$$

and in mechanics

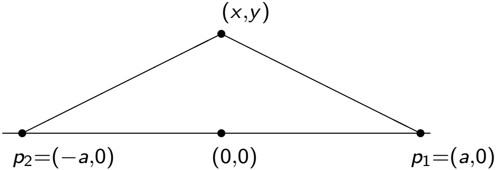
$$\int \sqrt{p(x)}dx \text{ for } \dot{y}(x)^2 = p(x)$$

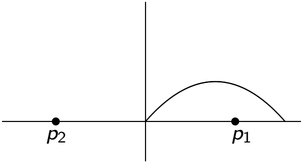
- of particular interest were elliptic integrals

$$x^2/a^2 + y^2/b^2 = 1, \qquad a > b$$

$$\int \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta = a\int \frac{(1-k^2\sin^2\theta)d\theta}{\sqrt{1-k^2\sin^2\theta}} = a\int \frac{(1-k^2x^2)dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

  where $k^2 = (a^2-b^2)/a^2$ and $x = \sin\theta$; by analogy with the formulas for $\sin 2\theta, \cos 2\theta$ for doubling the arc length of a circle, the mathematicians of that period (Count Fagnano, Euler, Lagrange, Gauss, ...) found formulas for doubling the arc length of the ellipse;
- also much studied was the arclength of lemniscate = locus of a point $p = (x, y)$ such that product of distances from two fixed points p_1, p_2 is constant



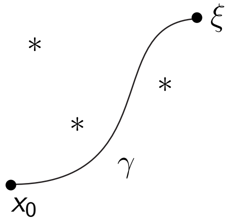


$$\int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1 - x^4}}$$

Here we have taken $2a^2 = 1$

- The integral above was understood as the "function" of the upper limit ξ of integration and by choosing a path γ of integration from x_0 to ξ together with a branch of the multi-valued function $y(x)$ along γ

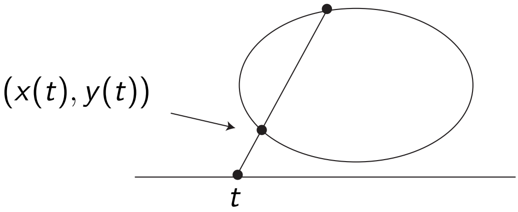
$$F(\xi) = \int_{x_0}^{\xi} r(x, y(x))dx$$



$*$ = branch points

A different choice of path changes $F(\xi)$ by a *period* (which is where we now think of as Hodge theory started). When $f(x, y) \in \mathbb{Q}[x, y]$ and $\xi \in \overline{\mathbb{Q}}$ the transcendence of $F(\xi)$, as well as of the periods, is a much studied and very beautiful subject (Siegel, Bombieri, etc.)

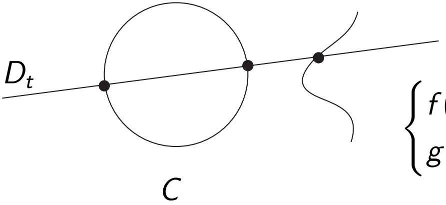
- when $\deg f = 1$ such integrals of algebraic functions may be expressed in terms of elementary functions (partial fractions), and when $\deg f = 2$ they may also be explicitly evaluated, essentially because conics are rational curves



but when $\deg f \geqq 3$, e.g., $f(x,y) = y^2 - p(x)$ where $\deg p(x) \geqq 3$, these integrals were of a different character;

- as noted above, there were a number of specific results about elliptic integrals when Abel recast the whole subject (1820) — he introduced two general concepts
  - abelian sums
  - inversion

  for the first he considered the curve $C = \{f(x,y) = 0\}$ together with a family of curves $D_t = \{g(x,y,t) = 0\}$ depending rationally on a parameter t —



$$\begin{cases} f(x,y) = y^2 - p_3(x) \\ g(x,y,t) = ax + by - t \end{cases}$$

and setting

$$C \cdot D_t = \sum_i (x_i(t), y_i(t))$$

he considered the abelian sum

$$u(t) = \sum_i \int_{(x_0,y_0)}^{x_i(t),y_i(t)} r(x, y(x))dx$$

Theorem

$u(t)$ is an elementary function; i.e., $u'(t) \in \mathbb{C}\{t\}$

This is a very general result; the only implicit assumption is that $f(x, y)$ has no multiple factors

Translated into a current framework, Abel's proof was the following:

$$
\begin{array}{ccc}
 & I & \\
 {}^{p}\swarrow & & \searrow^{q} \\
C & & \mathbb{P}^1 = t\text{-line}
\end{array}
$$

$$
\begin{aligned}
I &= \{(x,y,t) : f(x,y) = 0, \\
&\qquad g(x,y,t) = 0\} = \text{incidence variety};
\end{aligned}
$$

for $\omega = r(x,y)dx\,|_C$, the *trace*

$$\mathrm{Tr}\,\omega = q_* p^* \omega$$

is a rational 1-form on $\mathbb{P}^1$ — by local considerations it is everywhere meromorphic (no essential singularity), and hence is rational —

$$u(t) = \int_{t_0}^{t} \mathrm{Tr}\,\omega;$$

- when $\deg f \geqq 3$, a new phenomenon arises — setting

$$\omega = \left. \frac{p(x,y)dx}{f_y(x,y)} \right|_C$$

$$\deg p \leqq n-3 \Rightarrow u(t) = \text{ constant;}$$

  the point is that $\deg p \leqq n-3 \Rightarrow \operatorname{Tr}\omega$ is holomorphic on $\mathbb{P}^1$;

Note

The particular form arises from

$$\operatorname{Res}_C\left(\frac{p(x,y)dx \wedge dy}{f(x,y)}\right) = \left. \frac{p(x,y)dx}{f_y(x,y)} \right|_C$$

- for inversion, if we define $x(u), y(u)$ by

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \omega$$

then Abel's theorem translates into an *addition theorem* for the $x(u)$ and $y(u)$ — e.g. for the Weierstrass cubic, and choosing (x_0, y_0) properly, Abel's theorem is

$$u_1 + u_2 + u_3 = 0;$$

which gives

$$\begin{cases} x(-(u_1 + u_2)) = R(x(u_1), x(u_2), y(u_1), y(u_2)) \\ y(-(u_1 + u_2)) = S(x(u_1), x(u_2), y(u_1), y(u_2)); \end{cases}$$

using $f_y = 2y$ and the fundamental theorem of calculus

$$\left.\frac{dx}{2y}\right|_C = du$$

which for the Weierstrass p-function $x(u)$ and its derivative $y(u)$ gives the usual addition theorems; Abel viewed his result in part as a very general addition theorem;

- an important point is that C may be singular — essentially and in hindsight Abel was using the *mixed Hodge structure* on C to describe what we now call the *generalized Jacobian* — for C smooth of degree n

$$H^0(\Omega^1_C) \cong \{\text{polynomials of degree } n-3\} \cong \mathbb{C}^{\frac{(n-1)(n-2)}{2}},$$

  and for C possibly singular the RHS gives the F^1 for the *limiting mixed Hodge structure* as a smooth C_s specializes as $s \to 0$ to $C = C_0$;
- Abel's theorem opened the gateway to the modern theory of Hodge structures of weight one associated to compact Riemann surfaces — Jacobi, who formulated the Jacobian variety

$$J(C) = H^0(\Omega^1_C)^*/\text{periods}$$

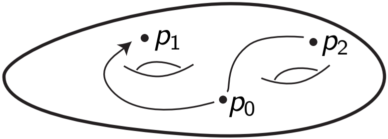
  and proved the inversion theorem and a converse to Abel's theorem

- In more detail, note that when C is smooth it is complex submanifold of $\mathbb{P}^2$ and hence is a compact Riemann surface with space $H^0(\Omega^1_C)$ of holomorphic differentials — the periods are

$$\int_\gamma \omega, \qquad \omega \in H^0(\Omega^1_C) \text{ and } \gamma \in H_1(C, \mathbb{Z});$$

choosing a base point $p_0 \in C$, abelian sums correspond to

$$\sum_{i=1}^{d} \int_{p_0}^{p_i} \omega \qquad \text{modulo periods,}$$



it is the periods that correspond to what is usually thought of as Hodge theory.
The object $J(C)$ is a complex torus and the geometry of the above map

$$C^{(d)} \to J(C)$$

was the main interest of classical theory.

— Riemann, who conceived Riemann surfaces and showed that

- $h^0(\Omega^1_C) = \left(\frac{1}{2}\right) b_1(C)$
- $H^1(C, \mathbb{C}) \cong H^0(\Omega^1_C) \oplus \overline{H^0(\Omega^1_C)}$
  $$= H^{(1,0)}(C) \oplus H^{0,1}(C),$$

although these mathematicans of the 19^{th} century were able to work with some singular curves as necessary, there was no general formulation of the intrinsic Hodge theory and of the relation of the Hodge theory of a general smooth C_s to the Hodge theory of a specialization — for this we had to wait for Deligne, whose worked opened the way for Schmid, Clemens-Schmid, Steenbrink, and Cattani-Kaplan-Schmid, ...

Picard

- before discussing him I want to say something about Lefschetz — he was born in Moscow (1884) and family moved to France — went to school there and loved mathematics but studied engineering because he thought that, not being French, it would be hard for him to get an academic position there — came to the US to get practical experience and lost both of his hands in a boiler accident at a foundry in St. Louis — so enrolled in graduate school in mathematics at Worcester Polytechnic where there were three faculty, one of whom worked on the classical theory of plane curves (how many double points and cusps can an irreducible curve have?)

— took a job at the University of Kansas where he was for 10 years and did his work in the topology of algebraic varieties, and also his fixed point theorem in topology — he took with him the works of Picard and wrote "these provided the wellspring for me" (he also told me that he was not allowed to teach introductory complex analysis there because they didn't think he knew the subject well enough)

- Picard studied algebraic surfaces, which for him were given by

$$S = \{f(x, y, z) = 0\} \subset \mathbb{P}^3,$$

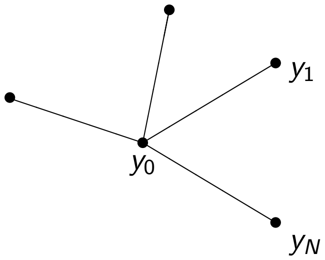
a surface with ordinary singularities in $\mathbb{P}^3$ — he wanted to extend the study of integrals of the $1^{\text{st}}, 2^{\text{nd}}$ and 3^{rd} kind on a curve to this case — for him the method was to fibre the surface by curves

$$C_y = \{f(x, y, z) = 0,\ y = \text{constant}\};$$

in a modern framework

$$\begin{array}{c} \widetilde{S} \\ {\scriptstyle\pi}\big\downarrow \\ P^1 \end{array} = \left\{ \begin{array}{c} \text{desingularization of } S \\ \text{blown up along the} \\ \text{base locus of the pencil} \end{array} \right\}$$

- he introduced the picture

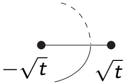


where C_{y_0} is a reference curve and the C_{y_i} have a single node —

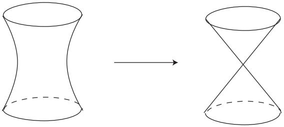
- the complement of the above configuration is contractible in $\mathbb{P}^1 =$ Riemann sphere, and so the whole surface retracts onto the part lying over the slits in the complex plane.

- for the node

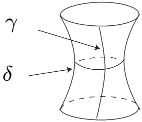
  analytically: $u^2 = v^2 - t$ $\qquad$ $u = \sqrt{v^2 - t}$



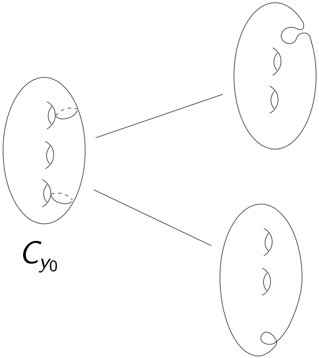
  topologically



  locally the topology is

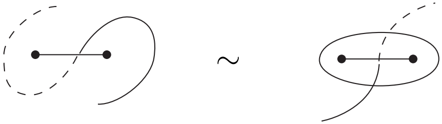


- thus we may topologically picture the surface as something like



- the key to understanding this is the monodromy action of $\pi_1(\mathbb{P}^1 \backslash \{y_1, \ldots, y_N\})$ on $H_1(C_{y_0}, \mathbb{Z})$;
- for this the first step is the local monodromy around each of the singular curves;

Picard drew the picture below and wrote
"as t turns around the origin we get for γ"



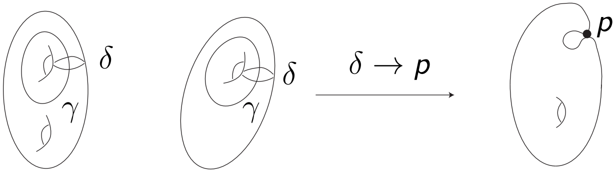
Lefschetz said that he spent hours staring at that picture
— Picard first seems to have arrived at it analytically

$$\frac{1}{2\pi i}\int_\gamma \frac{du}{v} = \log t + \text{holomorphic function of } t;$$

i.e.

$$\frac{d}{dt}\left(\int \frac{du}{v}\right) = \frac{1}{t} + h(t)$$

- this gives the local monodromy



$$\begin{cases} \delta \to \delta \\ \gamma \to \gamma + \delta \end{cases}$$

(Picard-Lefschetz transormation);

- the cycles in $H_1(C_{y_0})$ were divided into

$$\begin{cases} H_1(C_{y_0})_{\mathrm{ev}} = \text{vanishing cycles} \\ H_1(C_{y_0})_{\mathrm{inv}} = \text{invariant cycles} \end{cases}$$

- the global monodromy is another story that we will come to;

- Turning to differentials on the surface, Picard studied

$$\text{1-forms} \qquad \varphi = \frac{p(x,y,z)dx}{f_z(x,y,z)} + \frac{q(x,y,z)dy}{f_z(x,y,z)}$$

$$\text{2-forms} \qquad \omega = \frac{r(x,y,z)dx \wedge dy}{f_z(x,y,z)},$$

for the holomorphic 1-forms he divided them into two groups

$$\underbrace{\varphi_1, \ldots, \varphi_q} \ \underbrace{\varphi_{q+1}, \ldots, \varphi_g}$$

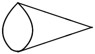
that varied holomorphically in $y \in \mathbb{P}^1$ — in modern language

$$R^1_\pi \omega_{\widetilde{S}/\mathbb{P}^1} \cong \bigoplus^q \mathcal{O}_{R^1} \oplus \left(\bigoplus_{i=q+1}^{q} \mathcal{O}_{\mathbb{P}^1}(k_i) \right)$$

where all $k_i \geqq 1$ — Lefschetz identified those where $k_i \geqq 2$ as being $H^0(\Omega_S^2)$ under $\varphi \to \varphi \wedge dy$ – will return to this later, all of the above is correct

- on the topological side, Picard also gave a correct proof that

$$H_1(C_{y_0}, \mathbb{Z}) \to H_1(\widetilde{S}, \mathbb{Z}) \to 0$$

$H_2(\widetilde{S}, C_{y_0}, \mathbb{Z})$ is generated by the $\Delta_i = \left\{ \begin{array}{c} \text{locus of } \delta_i \\ \text{along the} \\ \text{segment } y_0 y_i \end{array} \right\} =$  ;

here things get a little murkey as he essentially asserts that

$$\begin{cases} H_1(C_{y_0}, \mathbb{Q})_{\mathrm{ev}} \oplus H_1(C_{y_0}, \mathbb{Q})_{\mathrm{inv}} = H_1(C_{y_0}, \mathbb{Q}) \\ \dim H_1(C_{y_0}, \mathbb{Q})_{\mathrm{inv}} = 2q; \end{cases}$$

where "ev" = vanishing and "inv" = invariant in any case we know this as a consequence of regular Hodge theory, and the vast generalization of the above direct sum is a consequence of the semi-simplicity of global monodromy (theorem of the fixed part) which results from Deligne's mixed Hodge theory;

- he did, together with Poincaré, give a correct proof that

  "there exist ∞^q rationally inequivalent curves on S";

  this is an *existence* theorem, whose proof was a step in Lefschetz' proof of his famous $(1, 1)$ theorem — if time permits will sketch this argument, now much extended and recast by Saito and others;

- one of the interests was to classify rational differentials on surfaces extending the situation for curves as
    - first kind = holomorphic differentials
    - second kind = differentials without residues (and thus may give cohomology classes)
    - third kind = everything else
- the classification of 1-forms was a fairly direct extension of the case of curves

- turning to the 2-forms, Picard gives a correct proof of the affine algebraic de Rham theorem

$$H^2(\widetilde{S}^0, \mathbb{C}) \cong H^2_{\mathrm{DR}}\big(\Gamma(\Omega^\bullet_{\widetilde{S}^0,\mathrm{alg}}), d\big)$$

(he did this also for 1-forms) — for 2-forms he found the new phenomenon that there were ω which are holomorphic on $\widetilde{S}^0$ and where

$$\omega = d\psi$$

with ψ rational on $\widetilde{S}$ but $(\psi)_\infty \supsetneqq (\omega)_\infty$; this cannot happen for 1-forms in fact, the number of such relations was equal to $\rho - 1$ where, in modern language

$$\rho = \dim \mathrm{Hg}^1(\widetilde{S}) = \mathrm{rank}\, NS(\widetilde{S}) \otimes \mathbb{Q}$$

— this focused attention on the homology classes carried by algebraic cycles — Picard's statement was

$$\dim \{\text{space of differentials of the 2nd kind/exact}\} = b_2(\widetilde{S}) - \rho$$

— all the classical results of this kind are now "explained" using mixed Hodge structures and the various exact sequences that result from that theory;

- As for Lefschetz, who wrote that he "put the harpoon of topology into the whale of algebraic geometry," I will not say much as his basic results on the topology of algebraic varieties are well known — will comment on the so-called "hard Lefschetz theorem" — in the above discussion of algebraic surfaces, for an invariant 1-cycle λ we let

$$\Lambda = \{\text{3-cycle on } \widetilde{S} \text{ traced out by } \lambda\}$$
$$\Lambda \cdot C_{y_0} = \lambda;$$

this is defined as class in $H_3(\widetilde{S}, \mathbb{Q})$, and every class there arises in this way; hard-Lefschetz is the statement that

$$\cap C_{y_0} : H_3(\widetilde{S}, \mathbb{Q}) \to H_1(\widetilde{S}, \mathbb{Q})$$

is *injective* (and must then be an isomorphism by Poincaré duality) — this is the same as

$$H_1(C_{y_0}, \mathbb{Q})_{\mathrm{ev}} \cap H_1(C_{y_0}, \mathbb{Q})_{\mathrm{inv}} = (0)$$

just as with the issue discussed above, it seems that Lefschetz' argument for hard Lefschetz is incomplete (and there are philosophical reasons why this should be the case) — Lefschetz had of course incredible geometric intuition, and it was said of him that he never stated a false theorem nor gave a correct proof — when I was a graduate student he once gave a talk in the old Fine Hall on the topology of algebraic varieties and after giving an intuitive "cycle traced out by the locus of..." argument one of the young turks in the audience objected that this was not a proof, to which Lefschetz replied "come now, we're not babies after all"

— the point though is that hard Lefschetz remained unproved until Hodge — moreover, in general the cycle traced out by the locus of the invariant cycle λ closes up only if $d_2\lambda = 0$ in a spectral sequence, which may not be the case unless we are in Kähler case;

- Hodge did his work in the 1930's and the stated main purpose of his book "Harmonic integrals" was to prove, using differential forms and the recent de Rham theorem, Lefschetz's results on the topology of algebraic varieties — he had as a model Hermann Weyl's book "Die Idee der Riemannaschen Flächen," in which he formulated in to this day very modern terms Riemann's results on the existence of a polarized Hodge structure

$$H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C), \quad H^{0,1}(C) = \overline{H^{1,0}(C)}$$

$$\begin{cases} Q(H^{1,0}(C), H^{1,0}(C)) = 0 \\ iQ(H^{1,0}(C), \overline{H^{1,0}(C)}) > 0 \end{cases}$$

where $C =$ compact Riemann surface, $H^{1,0}(C) = H^0(\Omega^1_C)$ and $Q =$ cup product (intersection form via Poincaré duality) using harmonic theory — minimize

$$\|\varphi\|^2 = \int_C \varphi \wedge \overline{*\varphi}, \qquad \varphi \in A^1(C)$$

in its cohomology class — Hodge sought to do the same in general, the difference being that in higher dimensions a Riemannian metric was needed to define $\|\varphi\|^2$;

- Hodge's work fell into two parts:
  - A. $H^q_{\mathrm{DR}}(X) \cong \mathcal{H}^q(X) =$ harmonic q-forms where $(X, g) =$ compact Riemannian manifold
  - B. the special structure of $\mathcal{H}^q(X)$ that arises when (X, g) is a Kähler manifold

  I will mainly discuss B, but will note that A has an interesting history involving the IAS. Namely, the importance of Hodge's work was of course recognized and was the subject of a seminar conducted here by Weyl — the sense at the time seems to have been that Hodge's theorem A was almost certainly true, but his proof was incomplete

— Hodge's book was before the modern theory of linear elliptic PDE's, and it was not clear that (i) a minimum exists in L^2, and (ii) even if it does it may not be C^∞ — in putting together a complete argument Weyl devised the famous Weyl lemma

$$\varphi \in L^2 \text{ and } (\Delta\psi, \varphi) = 0 \text{ for all } C^\infty\psi \Rightarrow \varphi \text{ is } C^\infty$$

(weak solutions of linear elliptic equations are smooth)

- part B was done by calculations, using classical tensor calculus notations (indices galore), and was a major tour de force — the two main points are

$$[\Delta, \pi^{p,q}] = 0$$

where Δ = Laplacian or differential forms and $\pi^{p,q}$ = projection of a differential form

$$\varphi = \sum_{I,J} \varphi_{I,J} dz^I \wedge d\bar{z}^J$$

onto the terms where $|I| = p$, $|J| = q$; and

$$[\omega, \Delta] = 0$$

where $\omega = \left(\frac{\sqrt{-1}}{2}\right) \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is the Kähler form — both of these were lengthy, very difficult computations

— the first of these has at least been put into a conceptual framework by Chern — as for the second, it was and still is something of a miracle, as in general

$$\Delta(\varphi \wedge \psi) = \Delta\varphi \wedge \psi + \varphi \wedge \Delta\psi + \text{ big mess;}$$

Hodge told me that he proved it because it was the only way to get Lefschetz's results to come out; (I have not checked this, but my guess is that for a positive $(1,1)$ form $\varphi = \left(\frac{i}{2}\right) \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, the "big mess" $= 0 \iff d\varphi = 0$)

- in any case one has the *Kähler package*
  (i) $H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X), \quad H^{q,p}(X) = \overline{H^{p,q}(X)}$
  (*Hodge decomposition* giving a *Hodge structure of weight r*)
  (ii) $L^k, H^{n+k}(X) \xrightarrow{\sim} H^{n-k}(X)$, $\dim X = n$ (hard Lefschetz)
  (iii) $H^{r+k}(X)_{\mathrm{prim}} =: \ker\{L^{k+1} : H^{n+k}(X) \to H^{n-k-2}(X)\}$
  and
  $$H^r(X) = \bigoplus_{s \geqq 0} L^s H^{r+2s}(X)_{\mathrm{prim}}$$
  (iv) $Q : H^{n+k}(X)_{\mathrm{prim}} \otimes H^{n+k}(X)_{\mathrm{prim}} \to \mathbb{C}$
  $Q(\varphi,\psi) = \int_X L^k \varphi \wedge \psi$ gives a PHS,
  where $L =$ cup product by the Kähler form ω

- note that the polarizations depend on the choice of $[\omega] \in H^2(X, \mathbb{R})$ — when $[\omega]$ is rational (think of $X \subset \mathbb{P}^{N*}$ and $[\omega]$ = hyperplane section), (ii)–(iv) are defined over $\mathbb{Q}$ — for later use, we note two points
  - the first is that the Hodge decomposition (i) is equivalent to giving *Hodge filtration* $F^r \supset F^{r-1} \supset \cdots \supset F^0 = H^r(X, \mathbb{C})$ where for each p

$$F^p \oplus \overline{F}^{r-p+1} \xrightarrow{\sim} H^r(X, \mathbb{C}),$$

the relation to (i) is

$$\begin{cases} H^{p,q}(X) = F^p \cap \overline{F}^q \\ F^p = \bigoplus\limits_{\substack{p' \geqq p \\ q}} H^{p',q}(X) \end{cases}$$

(think of the "p" in F^p as "at least p dz^α's in the differential form representing the cohomology class)

- the second is that it is the existence of a polarization that to a large extent distinguishes the Hodge structures of geometric interest — e.g., it leads to the very strong curvature properties of the space of all Hodge structures on a $\mathbb{Q}$-vector space with a given polarizing form Q and Hodge numbers $h^{p,q}$;

- in the 1950's, Kodaira and Spencer (here at the IAS and later at the university) built on Hodge's work, among other things that resulted were the
  - Kodaira vanishing theorem

    $$H^q(X, L^*) = 0 \text{ for } q < n$$

    where $L \to X$ is a positive (= ample) line bundle

    and its equivalence in the Akizuki-Nakano formulation and in case $L = [Y]$ where Y = smooth hypersurface, to the Lefschetz theorems over $\mathbb{Q}$ about $H^*(X) \to H^*(Y)$, thereby established a connection between vanishing theorems and Hodge theory

- in more detail,

$$H^q\big(X, \Omega_X^p \otimes L^*\big) = 0 \ \text{ for } \ p + q < n$$
$$\iff \left\{ \begin{array}{ll} H^k(X) \xrightarrow{\sim} H^k(Y), & k \leqq n-2 \\ H^{n-1}(X) \hookrightarrow H^{n-1}(X) & \end{array} \right\}$$

(usual KVT is the $p = 0$ case)

- proof that

$$\begin{cases} \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_Y^*) \quad (= \text{ Lefschetz (1,1) theorem}) \\ \mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \end{cases}$$

where $\mathrm{Pic}(X) = \mathrm{CH}^1(X) = \left(\frac{\text{divisors modulo}}{\text{rational equivalence } D = (f)} \right)$;
this gives ∞^q rationally inequivalent curves on a surface

- in the mid 1960's a few fragments of Hodge theory for singular varieties were around on an ad hoc basis
  - generalized Jacobians of algebraic curves (return to Abel)
  - Hodge theory for the singular fibres in a Lefschetz pencil, and generalized intermediate Jacobians in this case; at least to my knowledge there was no concept of a functorial Hodge theory for *all* varieties — as with the earlier attempts by Picard, Poincaré, Lefschetz, the presence of singularities created major difficulties in attempts to develop a satisfactory theory of integrals of the $1^{\rm st}, 2^{\rm nd}$ and $3^{\rm rd}$ kind — then came

- Hironaka's resolution of singularities
- Deligne's mixed Hodge theory

it is interesting to note that in their paper on integrals of the 3^{rd} kind, Atiyah and Hodge used resolution of singularities to define the log complex

$$\Omega^{\bullet}(\log D), \text{ quasi-isomorphic to } \mathcal{O} \otimes \Lambda^{\bullet}\left(\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}\right)$$

where $D = \{z_1 \cdots z_k = 0\}$ — as pointed out by Grothendieck, this contained the kernel of the idea for his algebraic de Rham theorem, which together with Deligne's mixed Hodge theory effectively contains the general story of integrals of the $1^{\mathrm{st}}, 2^{\mathrm{nd}}$ and 3^{rd} kinds

Deligne

- A *mixed Hodge structure* (MHS) is given by the data $(H, W_\bullet, F^\bullet)$ where
  - H is a $\mathbb{Q}$-vector space
  - $W_\bullet$ is an increasing filtration of H
  - $F^\bullet$ is a decreasing filtration of $H_\mathbb{C}$

  which satisfies

  $F^p \cap W_{k,\mathbb{C}} + W_{k-1,\mathbb{C}}/W_{k-1,\mathbb{C}}$ induces a pure Hodge structure of weight k on $\mathrm{Gr}_k = W_k/W_{k-1}$;

  a mixed Hodge structure is thus successive extensions, of a special type, of pure Hodge structures — MHS's admit the usual operations of linear algebra, and the notion of a morphism between them is the natural one — they also have subtle linear algebraic properties, one of which is *strictness*: Given $\varphi : H \to H'$ with $\varphi(W_m) \subset W'_m, \varphi(F^p) \subset F'^p$, then

$$\varphi(H) \cap F'^p = \varphi(F^p)$$

this implies that MHS's form an abelian category

Deligne's Theorem

For X a complex algebraic variety, $H^r(X, \mathbb{Q})$ has a functorial MHS

- *for X complete, the weight filtration is $W_0 \subset \cdots \subset W_r$*
- *for X smooth and open the weights are $W_r \subset \cdots \subset W_{2r}$*

Given a particular presentation of an algebraic variety X in terms of smooth ones with normal crossing singularities, it is plausible that there is lurking in $H^*(X)$ a description as some sort of iterated extensions of pure Hodge structures. Remarkable to me is the formulation of the extension data and, especially, the independence of the particular resolution of singularities.

It is safe to say that this result, together with what I will next mention, opened up the study of the cohomology of complex algebraic varieties in a way similar to what followed from Abel's theorem for integrals of algebraic functions

- Regarding the weight in the weight filtration, *very* approximately the weight of a class φ reflects the degree of the cohomology class on a smooth variety in the stratification that traces through the various exact sequences to give φ

- a special type of MHS arises when $W_\bullet = W_\bullet(N)$ is associated to a nilpotent endomorphism

$$N : H \to H, \quad N^{r+1} = 0$$

then there is a unique $W_\bullet(N) = W_{-r}(N) \subset \cdots \subset W_r(N)$, centered at 0, where

$$\begin{cases} N : W_l(N) \to W_{l-2}(N) \\ N^k, \mathrm{Gr}_k \xrightarrow{\sim} \mathrm{Gr}_{-k}; \end{cases}$$

think of the L in the Lefschetz decomposition, only going the other way and with a shift in indices— *a limiting mixed Hodge structure* $(H, W_\bullet(N), F^\bullet)$ is one where

$$N(F^p) \subset F^{p-1},$$

here omitting the polarization conditions they arise when you have a family

$$\mathcal{X} \xrightarrow{\pi} \Delta$$

with $X_t = \pi^{-1}(t)$ smooth for $t \neq 0$ — Deligne conjectured and Schmid proved that the Hodge filtrations on $F^p H^r(X_t, \mathbb{C})$ have a limit as $t \to 0$ (this is somewhat subtle, as we need to take the limit of filtrations on a variable vector space) and the result is a LMHS;

more precisely there is a monodromy transformation

$$T : H^r(X_{t_0}, \mathbb{Q}) \to H^r(X_{t_0}, \mathbb{Q})$$

where

$$(T^l - I)^{r+1} = 0$$

(quasi-unipotency)

replacing t by t^l we may assume that T is unipotent with $N = \log T$ nilpotent — then by Schmid, for $l(t) = \left(\frac{1}{2\pi i}\right) \log t$, $e^{-l(t)N} \cdot F_t^\bullet$ is single-valued and has a limit $F_{\lim}^\bullet$ as $t \to 0$; moreover, $e^{l(t)\cdot N} \cdot F_{\lim}^\bullet$ exponentially approximates $F_t^\bullet$.

- Schmid proved his result for any variation of Hodge structure over Δ^* — the geometric result was later done by Steenbrink — in practice, this result tells us how the topology and Hodge structure behave in a 1-parameter family of varieties acquiring arbitrary singularities — the several variable version of how the Hodge structure degenerates was done by Cattani-Kaplan-Schmid with significant input by Deligne and continues to play a major role in current Hodge theory

- One of the first applications of Deligne's theorem dealt with a global family
$$\mathcal{X} \xrightarrow{\pi} S$$
of smooth projective varieties X_s — here both $\mathcal{X}$ and S are quasi-projective — the basic invariant of this situation is the monodromy representation
$$\rho : \pi_1(S, s_0) \to \mathrm{Aut}\, H^r(X_{s_0}, \mathbb{Z})$$
with image
$$\Gamma \subset \mathrm{Aut}\,(H^r(X_{s_0}, \mathbb{Z}), Q)$$
where Q is the polarizing form on $H^r(X_{s_0}, \mathbb{Q})$

— whereas the local monodromies around the branches of $\overline{S} \backslash S$ are quasi-unipotent, Deligne's result is that opposite is true globally — loosely stated it is that

$$\Gamma \text{ is semi-simple } /\mathbb{Q}$$

this is a vast generalization of the issue that Picard and Lefschetz encountered for Lefschetz pencils and was in that particular case finally settled by Hodge

— will conclude with two further topics

- brief mention of Morihiko Saito's theory of mixed Hodge modules, which is the current penultimate version of Hodge theory
- some discussions of existence issues in Hodge theory/complex algebraic geometry

- in the early 80's Beilinson-Bernstein-Deligne-Gabber initiated the development of what is now called the Hodge theory of maps

$$(*) \qquad f : \mathcal{X} \to S$$

where (to keep things simple) f is *proper* — if there are no singular fibres then Blanchard-Deligne proved that the Leray spectral sequence degenerates at E_2 — when there are singular fibres and $\dim S \geqq 2$ the issue is much more subtle

— here again the IAS played a role in that a critical ingredient is the *intersection cohomology* of a complete singular variety, which was created by Mark Goresky and Bob MacPherson and which carries a pure Hodge structure — the theory for $(*)$ as developed by BBDG used characteristic-p methods (Frobenius acting on l-adic cohomology) — Saito's stated objective was to put $(*)$ in a purely Hodge-theoretic setting — the basic building blocks of his theory are

- VHS's and admissible VMHS's (everything polarizable)
- $\mathcal{D}$-modules and perverse sheaves (the latter being sheaf-theoretic version of intersection cohomology)

very informally, on has the description

- Hodge modules are the smallest category of objects over smooth varieties that contains generically (i.e., on the complement of a divisor) polarizable VHS's *and is closed under direct images*
- mixed Hodge modules (the penultimate objects) are the same with admissible VMHS's

- application of his theory include
  - a general vanishing theorem for the graded pieces of the de Rham complex of a mixed Hodge module. If $(\mathcal{M}, F)$ underlies a mixed Hodge module on a projective variety X, and if L is an ample line bundle on X, then the hypercohomology of the complex

$$L \otimes \left[\mathrm{gr}_p^F \mathcal{M} \to \Omega_X^1 \otimes \mathrm{gr}_{p+1}^F \mathcal{M} \to \cdots \to \Omega_X^{\dim X} \otimes \mathrm{gr}_{p+\dim X}^F \mathcal{M}\right]$$

vanish in degrees greater than $\dim X$. This result includes most of the standard vanishing theorems in algebraic geometry (such as Kodaira's theorem) as special cases

- Kollár showed that for a projective morphism $f : X \to Y$, with X smooth, the higher direct image sheaves $R^k f_* \omega_{X/Y}$ of the relative canonical bundle are torsion-free, and vanish above $\dim X - \dim Y$. He conjectured that the same should be true for sheaves of the form $R^k f_*(\omega_{X/Y} \otimes F^p \mathcal{H})$, where $\mathcal{H}$ is a polarizable variation of Hodge structure. Saito proved this using this theory.

- On a smooth projective variety X, any set of the form

$$\{L \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes L) \geqq m\}$$

  is a translate of an abelian subvariety of $\mathrm{Pic}^0(X)$, and the generic vanishing theorem of Green-Lazarsfeld gives a bound for its codimension. This, and many related results, are in fact special cases of general theorems about mixed Hodge modules on abelian varieties.
- As a concluding application to illustrate how Hodge theory and mixed Hodge theory have developed beyond their original algebro-geometric roots, I will mention a conjecture/work in progress of Schmid-Villonen:

— Suppose G is a real semi-simple Lie group with Lie algebra g and a maximal compact subgroup K. The admissible (g, K) modules are certain nicely behaved bimodules for g and K, which were classified a long time ago by Langlands and later reclassified in terms of twisted D-modules on generalized flag varieties by Beilinson and Bernstein. The important problem of deciding which (g, K) modules are unitary is, however, in some sense open. There is an algorithm for deciding this question due to Vogan, however, it is complicated. (Think of defining a module A as an irreducible sub-quotient of a module B, which you know concretely, that has a filtration that you know exists but don't know concretely)

An important ingredient in making this algorithm feasible is a certain hermitian form on a (g, K) module V. In the essential cases, it always exists and the problem of deciding if V is unitary can be boiled down to a problem of the signature of the form on the (usually infinite dimensional) V.

Schmid and Villonen do the following:

(1) interpret the form as a polarization on a mixed Hodge module M on a homogeneous space for G associated to V

(2) conjecture that the polarization satisfies sign identities, with respect to the Hodge filtration that M induces on V, completely analogous to the Riemann bilinear relations on a polarized Hodge structure

This setup has nothing to do with the Hodge theory arising from a family of algebraic varieties and illustrates the extent to which formal Hodge theory has developed as a subject in its own right

- finally want to discuss existence issues — e.g., conjectures of Hodge and Bloch-Beilinson — methods for proving existence include
  - *deformation theory* — works in some cases — existence of Picard variety either by analytic methods (Kodaira-Spencer) or algebraic ones (Grothendieck) – so far not in others — e.g., for S a surface and $Z(S) =$ 0-cycles on S, the formal tangent spaces (not defined here) T^f has the property

$$p_g = 0 \Rightarrow T^f Z(S) = T^f Z(S)_{\text{rat}},$$

    but attempts to "integrate" this fall short;

- *PDE* — linear theory works well (Hodge theorem) — also non-linear theory (Yau, Donaldson-Yau-Uhlenbeck, Simpson, Tian, ...) — but geometric theory using currents again falls short — for $\xi \in \mathrm{Hg}^q(X, \mathbb{Z})$ and Y = degree $d \gg 0$ linear section let

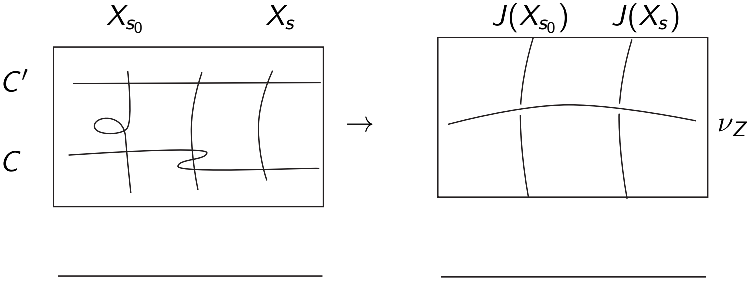
$$Z = \text{minimal positive integral current with } [Z] = \xi + [Y];$$

then Z exists non-uniquely and according to Blaine Lawson should be a complex analytic subvariety — one essential problem is "what does $d \gg 0$ mean?" Examples due to János Kollár and others, and recently studied by Claire Voisin, show that there are integral classes

$$\begin{cases} \xi \in H^{2p}(X, \mathbb{Z}) \\ \xi = [Z] \text{ for } Z = \sum n_i Z_i, \quad n_i \in \mathbb{Q} \end{cases}$$

but we cannot choose Z with $n_i \in \mathbb{Z}$

- *induction on dimension* — this is the original method of Picard-Poincaré-Lefschetz to construct curves on a surface — the picture is



where $Z = C - C'$, $\deg Z_s =: Z \cdot X_s = 0$ and $\nu_Z(s) = \mathrm{AJ}_{X_s}(Z_s)$.

Picard and Poincaré showed, successfully in my view, that

- any $\nu = \nu_Z$ for some Z (Jacobi inversion with dependence on parameters)
- the family $J(X_s)$ has a fixed part of dimension $q = h^0(\Omega^1_X)$
- varying Z by varying ν in the fixed part traces out ∞^q rationally inequivalent curves on X

Lefschetz showed that the variable part of ν_Z depends only on $[Z]$, and

$$(*) \qquad \int_{[Z]} \omega = 0 \ \text{ for } \ \omega \in H^0(\Omega^2_X);$$

obvious now but not then — his argument used that

$$H^0(\Omega^2_X) = dy \wedge \left(\bigoplus_{k \geqq 2} H^0\left(\mathcal{O}_{\mathbb{P}^1}(k)\right) \text{ part of } R^0\omega_{X/\mathbb{P}^1} \right)$$

Lefschetz then reversed the argument to show that any ν gives a class in $[\nu] \in H_2(X, \mathbb{Z})$, and that $(*)$ is satisfied with $[\nu]$ replacing $[Z]$. But then $\nu = \nu_Z$ for some Z.

- using mixed Hodge theory this argument generalizes and has been used to prove a number of *non-existence* results but (except in a few special cases) no existence ones — it does show that *associated to a Hodge class* ξ *there is an admissible normal function* ν_ξ

— recently a number of people have revisited the old idea of using, instead of just pencils, the whole family S of hypersurfaces of high degree — the geometric idea is that when $\dim X = 2n$ any primitive algebraic n-cycle Z will have support contained in *necessarily singular* hypersurfaces X_s — building on earlier work on Néron models in higher codimension, using M. Saito's theory several people have

- shown how to define $\operatorname{sing} \nu_\xi$ for an admissible normal function ν_ξ
- shown that over $\mathbb{Q}$, $\xi = [Z] \Leftrightarrow \operatorname{sing} \nu_\xi \neq \emptyset$ for $d \gg 0$

from examples one may suspect something like

$$d \gg 0 \text{ means } d \geqq C|\zeta^2|$$

where the constant is uniform in the moduli of X — but even formulating this precisely seems to run into the previous issue that even if $\zeta = [Z]$ we don't see how to bound the denominators in the n_i

What is at issue is what one might think of as the effective Hodge conjecture. If this were done and the above remark about $\mathrm{sing}\nu_\xi$ reformulated as

$$\begin{cases} S \xrightarrow{\nu_\xi} \mathcal{J} = \text{universal family of Néron models} \\ \nu_\xi^{-1}(\mathcal{J}_{\mathrm{sing}}) = \mathrm{sing}\,\nu_\xi \end{cases}$$

then at least the topological question

$$\nu_\xi^*[\mathcal{J}_{\mathrm{sing}}] \neq 0 \ \text{ for } \ d \gg 0$$

would make sense.