Noncommutative resolutions and intersection cohomology for quotient singularities

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Varieties

In algebraic geometry one studies **varieties** which are zero loci of some given polynomials in several variables. We will consider varieties over the complex numbers.

Smooth varieties are complex manifolds. Many interesting varieties are **singular**.

Example of a singular variety: zero locus X of $xy - zt$ in $\mathbb{C}^4$.

Intersection cohomology

Vector spaces associated to varieties: **singular cohomology**, **K-theory**.

If the variety is smooth and proper, singular cohomology satisfies Poincaré duality.

For singular varieties, **intersection cohomology** (defined by Goresky–MacPherson) has better properties than singular cohomology, for example it satisfies Poincaré duality.

For a resolution of singularities $Y \to X$, a consequence of the Beilinson–Bernstein–Deligne–Gabber decomposition theorem is that $IH^{\cdot}(X)$ is a direct summand of $H^{\cdot}(Y)$. When the resolution is *small*, $IH^{\cdot}(X) = H^{\cdot}(Y)$.

Intersection K-theory?

Question 1. Is there a K-theoretic version of intersection cohomology?

Application in representation theory: for 3d Cohomological Hall algebras, the number of generators is given by the dimension of intersection cohomology of some singular moduli spaces.

Noncommutative resolutions

variety $X \rightsquigarrow$ (dg) category $D^b(X)$

One can recover $K_\cdot(X)$ or a periodic version of $H^\cdot(X)$ from $D^b(X)$.

A **noncommutative resolution** (NCR) of a variety X is a smooth dg category $\mathbb{D}$ with a pair of adjoint functors

$$F : \mathbb{D} \to D^b(X),\ G : \mathrm{Perf}(X) \to \mathbb{D}$$

such that $FG = \mathrm{id}$.

Example: category $D^b(Y)$ for $f : Y \to X$ a resolution of singularities of X with rational singularities.

There are more NCRs than standard resolutions.

Strategy for finding NCRs: look at semi-orthogonal decompositions of $D^b(Y)$ for a resolution of singularities $f : Y \to X$

Minimal NCR resolutions

Bondal–Orlov conjecture. For X a variety, there exists a minimal NCR $\mathbb{M}(X)$, i.e. for any NCR $\mathbb{M}'$ of X, there is a semi-orthogonal decomposition

$$\mathbb{M}' = \langle \mathbb{M}(X), - \rangle.$$

In particular, if X is singular and has resolutions of singularities $Y_1, Y_2 \to X$ which are Calabi-Yau, then $D^b(Y_1) \cong D^b(Y_2)$.

Categorification of intersection cohomology?

Question 2. For X a variety, is there a natural dg category $\mathbb{I}(X)$ such that

$$HP_{\cdot}(\mathbb{I}(X)) = \bigoplus_{i \in \mathbb{Z}} IH^{\cdot+2i}(X)?$$

For varieties X for which Question 2 has a positive answer, its K-theory will be a version of intersection K-theory.

Categories $\mathbb{I}(X)$ answering Question 2 are natural candidates to be minimal NCRs in the sense of Bondal–Orlov.

Quotient singularities

Let G be a reductive group and V a linear representation of G. Consider the stack $\mathcal{X} = V/G$ with coarse space $X = V /\!\!/ G$.

Example: Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^2_1 \oplus \mathbb{C}^2_{-1}$ such that G acts with weight 1 on $\mathbb{C}^2_1$ and weight -1 on $\mathbb{C}^2_{-1}$. Then

$$V /\!\!/ G = (xy - zt = 0) \subset \mathbb{C}^4.$$

Remark: A large class of Artin stacks $\mathcal{X}$ admits good moduli spaces X (Alper et. al.) such that X is étale locally a quotient as above.

Strategy for finding NCRs: consider the "resolution" $\pi : \mathcal{X} \to X$, search for NCRs inside $D^b(\mathcal{X})$.

NCRs of quotient singularities

Theorem (P). There exist NCRs $\mathbb{D}(X)$ of X such that

$$D^b(\mathcal{X}) = \langle \mathbb{D}(X), - \rangle,$$

the complement is generated by complexes supported on attracting loci $\mathcal{S} \to \mathcal{X}$, and $\mathbb{D}(X)$ is minimal with these properties.

Question. When are these categories $\mathbb{D}(X)$ minimal in the sense of Bondal–Orlov?

Example. For $X = \mathbb{C}^4 /\!\!/ \mathbb{C}^* = (xy - zt = 0) \subset \mathbb{C}^4$, the above categories are

$$\langle \mathcal{O}_{\mathbb{C}^4}(w), \mathcal{O}_{\mathbb{C}^4}(w+1) \rangle \subset D^b(\mathbb{C}^4/\mathbb{C}^*)$$

for $w \in \mathbb{Z}$. They are equivalent to $D^b(X^+)$ and $D^b(X^-)$, where $X^+, X^- \to X$ are the small resolutions of X obtained by variation of GIT (van den Bergh).

Categorification of intersection cohomology for quotient singularities

Theorem (P). There exist natural subcategories $\mathbb{I}(X) \subset \mathbb{D}(X)$ such that

$$HP_{\cdot}(\mathbb{I}(X)) = \bigoplus_{i \in \mathbb{Z}} IH^{\cdot+2i}(X).$$

We can thus define a version of intersection K-theory of X by $IK(X) := K(\mathbb{I}(X))$.

When $\mathcal{X} = V/G$, there is a version of the above result for noncommutative motives. This implies that $IK(X)$ is a direct summand of $K(\mathcal{X})$.

Application in representation theory (P): for 3d K-theoretic Hall algebras, the number of generators is given by intersection K-theory of some singular moduli spaces.

Thank you for your attention!