Uniqueness aspects of symplectic fillings

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Contact manifolds

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Example

1. $\mathbb{R}^{2n+1}$ with $\xi := \ker(\alpha = \mathrm{d}z - \sum_{i=1}^{n} p_i \mathrm{d}q_i)$.
2. S^{2n-1} with $\xi_{std} := JTS^{2n-1} \cap TS^{2n-1}$.
3. ST^*M, links of singularities, etc.

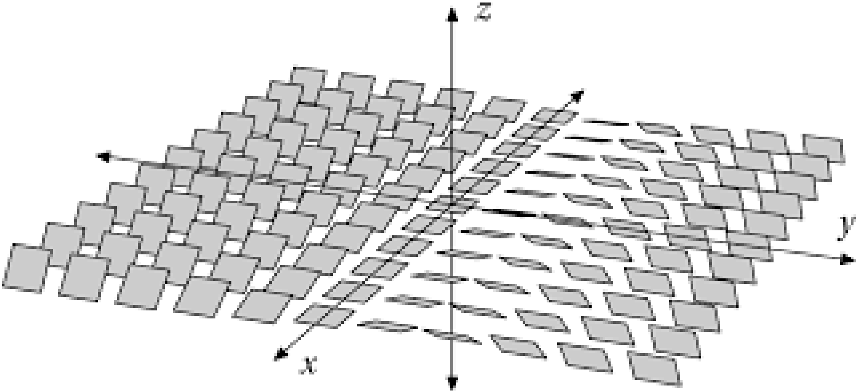

Figure: Standard contact $\mathbb{R}^3$

Symplectic fillings

Definition

(W, λ) is called a Liouville filling of contact manifold (Y, ξ) iff $\partial W = Y$ and the following holds

1. $\omega := \mathrm{d}\lambda$ is a symplectic form on W and there is a vector field X such that $L_X\omega = \omega$ and X is pointing out along boundary.
2. $\xi = \ker \lambda$.

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Example

1. The standard symplectic ball is a filling of the standard contact sphere.
2. The cotangent disk bundle DT^*M is a filling of ST^*M.
3. Smoothings of a singularity are fillings of the link of the singularity.

Symplectic cobordisms

In general, we have Liouville cobordisms between contact manifolds.

Question

Understand this cobordism category.

Symplectic Field Theory

Symplectic field theory (SFT) is a field theory on the cobordism category of contact manifolds, a special case of SFT assigns an algebra $CC_*(Y)$ called contact homology to a contact manifold Y. For the empty contact manifold, the assigned algebra is the ground field $\mathbf{k}$. Therefore given a filling W of Y, we have a map $\epsilon_W : CC_*(Y) \to \mathbf{k}$, such map is called an augmentation.

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It is much easier to classify augmentations than to classify fillings.

Slogan

If a contact manifold admits only the trivial augmentation, then many Floer theoretic properties and topological properties of the filling is independent of the filling.

Asymptotically dynamically convex manifolds

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Let Y *be a simply connected subcritically fillable contact manifold of dimension* ≥ 5*, then exact fillings of* Y *have the unique diffeomorphism type.*

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Let Y be a simply connected subcritically fillable contact manifold of dimension ≥ 5, then exact fillings of Y have the unique diffeomorphism type.

The strategy of proof is finding a "homological foliation" by pseudo-holomorphic curves, the existence of foliation is hinted by the fact that $W = V \times \mathbb{C}$, when W is subcritical.

Uniqueness of symplectic fillings

For every Liouville domain W, we can assign it with two Floer type theories $SH^*(W), SH^*_+(W)$, so that they fit into a long exact sequence,

$$\ldots \to H^*(W) \to SH^*(W) \to SH^*_+(W) \to H^{*+1}(W) \to \ldots.$$

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Theorem (Z 19')

Let Y be ADC, then $\delta_\partial : SH^*_+(W) \to H^{*+1}(W) \to H^{*+1}(Y)$ is independent of the filling W as long as $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective.

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A quick corollary is that if Y is flexibly fillable, then $H^*(W) \to H^*(Y)$ is independent of filling.

Generalizations

- Since $1 \in \operatorname{Im} \delta_\partial$ is equivalent to $SH_*(W) = 0$, therefore for ADC contact manifolds, the vanishing of symplectic cohomology is independent of filling.

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- The vanishing of symplectic cohomology is the first symplectic property measuring the complexity of symplectic domains in a whole hierarchy. The next one is the existence of dilation, i.e. $\exists x \in SH^1(W)$ such that $\Delta(x) = 1$. In general, we can consider the S^1-equivariant symplectic cohomology, we say W carries a k-dilation, if 1 is killed in the $k+1$-th page of the spectral sequence from the u-filtration.

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Theorem (Z 19')

There are structure maps related to the existence of k-dilation. When Y is ADC, and all of them are independent of the filling W as long as $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective.

Applications

- An instant corollary of the previous theorem is that the existence of k-dilation is independent of filling, if the boundary is ADC.

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- If Y is ADC, then whether $\operatorname{Im}\delta_\partial$ contains an element of degree $> \frac{\dim Y+1}{2}$ is an obstruction to Weinstein fillability. The obstruction is symplectic in natural, in particular, there are infinitely many $4k+3$ contact manifolds that are exactly fillable, almost Weinstein fillable, but not Weinstein fillable.

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- The k-dilation can be used to define a cobordism obstruction for ADC manifolds.

Further questions

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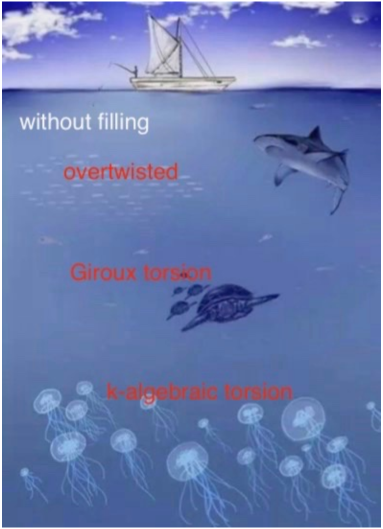
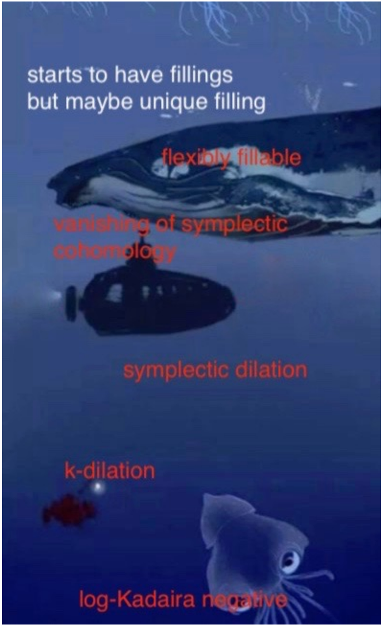
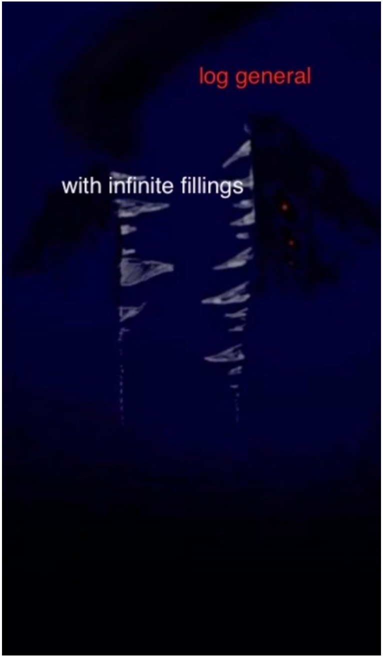
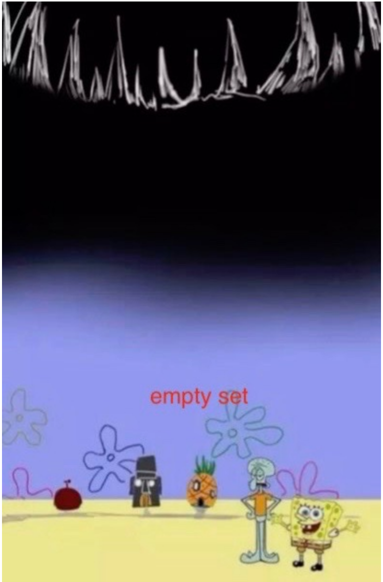
2. The existence of k-dilation implies uniruledness, it is unlikely that uniruledness will imply the existence of k-dilation for some k for affine varieties. Is there a symplectic characterization of uniruledness for affine varieties?
3. The k-dilation gives a rough classification of symplectic domains with log Kodaira dimension $-\infty$, what about other log Kodaira dimension?


without filling
overtwisted
Giroux torsion
k-algebraic torsion


starts to have fillings
but maybe unique filling
flexibly fillable
vanishing of symplectic
cohomology
symplectic dilation
k-dilation
log-Kadaira negative


quasi dilation
qusi cyclic dilation
Log Calabi-Yau
positive log Kodaira


log general
with infinite fillings


empty set

Thank you!