Pseudoholomorphic curves with boundary: Can you count them? Can you really?

Members' seminar, Institute for Advanced Study
November 2019

Outline

- Gromov-Witten (GW) theory
- Open Gromov-Witten (OGW) theory

- WDVV equation in GW theory
- Open WDVV equation in OGW theory

- Quantum product in GW theory
- Relative quantum product in OGW theory

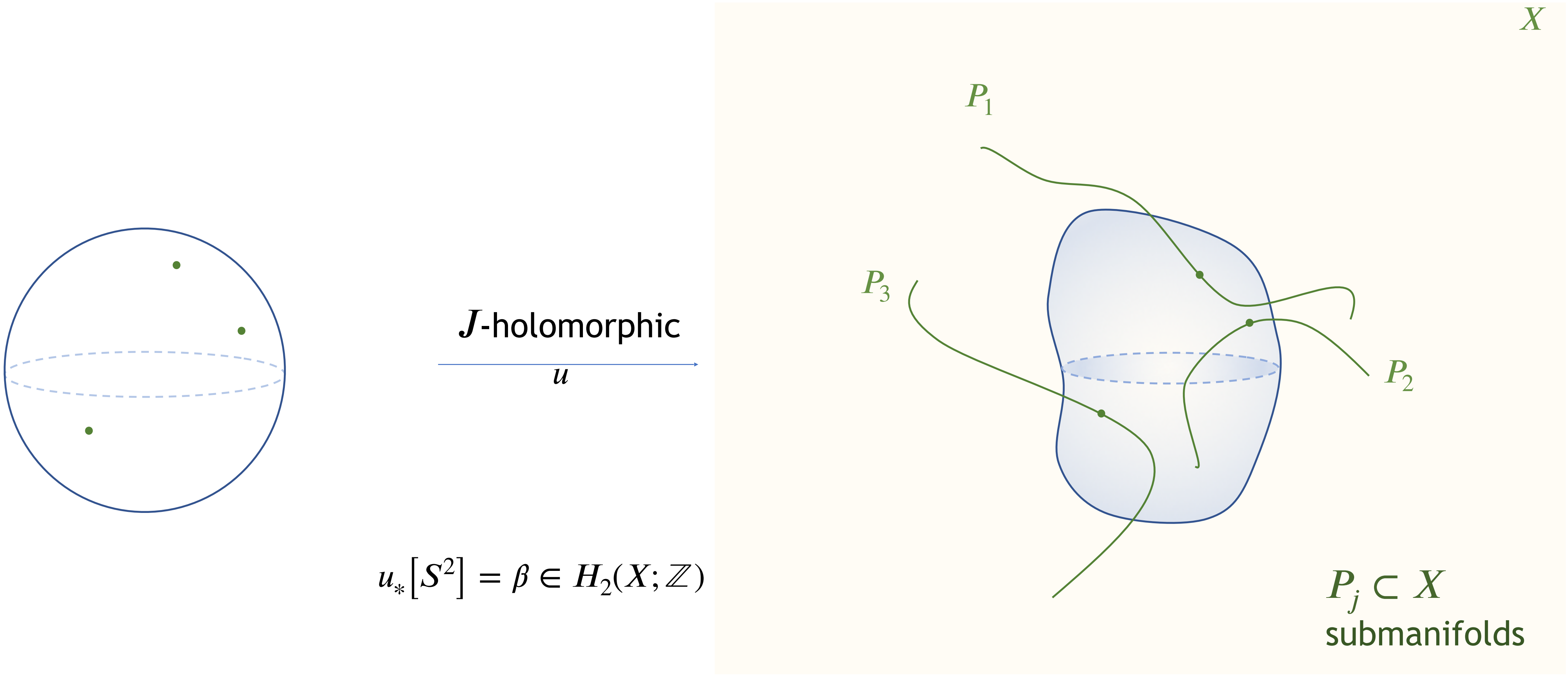
Gromov-Witten theory $(g = 0)$

Setting: (X, ω, J) symplectic manifold with almost complex structure

$X = X^{2n}$, ω 2-form such that $\omega^{\wedge n}$ is a volume form

$J \in End(TX)$, $J^2 = -\mathrm{Id}$, "ω-tame"

Example: $\left(\mathbb{C}P^n,\ \omega_{FS},\ J_0\right)$



<u>Question</u>: $\# u = ?$

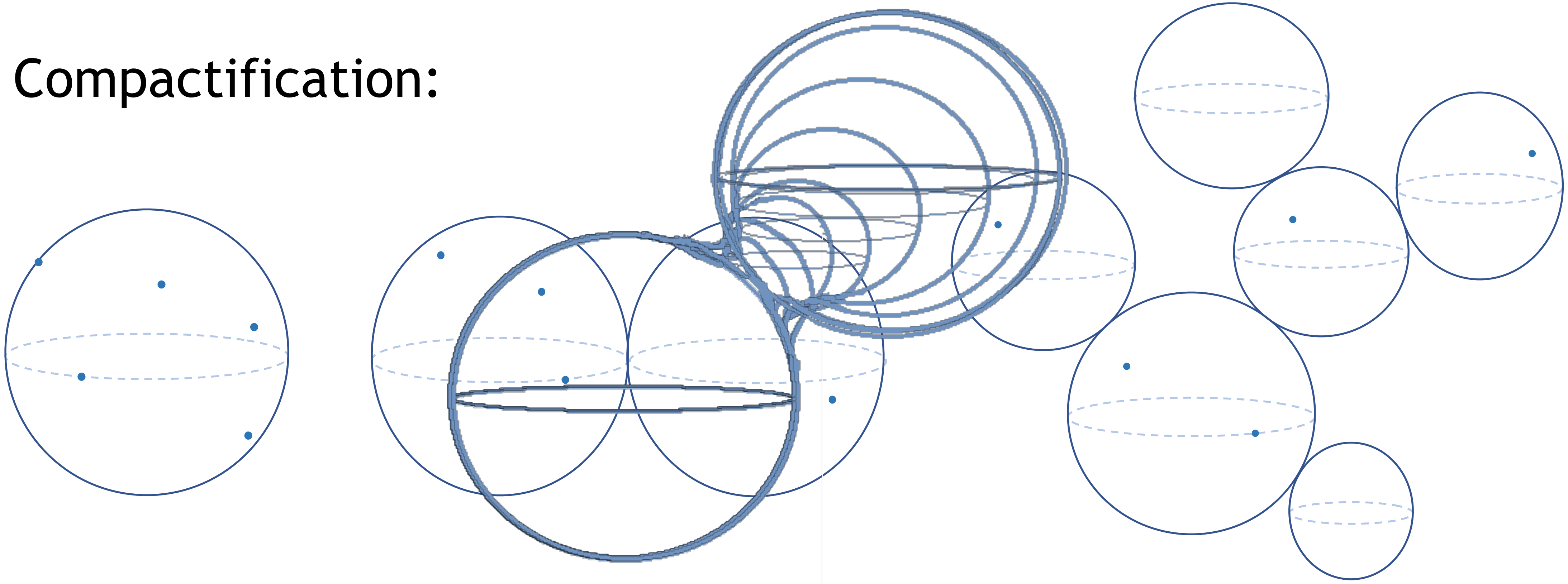
a) *Define invariantly of choices (e.g., representative of* $[P_j] \in H_*(X)$)

b) *Compute values*

The moduli space of sphere maps

$$\overline{\mathscr{M}}_l(\beta) = \overline{\left\{ \left(u : S^2 \xrightarrow{J-hol.} X,\ w_1, \ldots, w_l \right) : \begin{array}{c} [u] = \beta \in H_2(X;\mathbb{Z}) \\ w_j \in S^2,\ w_i \neq w_j \end{array} \right\} / \sim}$$

Compactification:



Rephrasing the problem

Count elements of $\overline{\mathscr{M}}_l(\beta)$ such that the marked points are mapped to given constraints.

Can be expressed as an integral:

$$GW_l^\beta(\gamma_1, \ldots, \gamma_l) = \int_{\overline{\mathscr{M}}_l(\beta)} ev_1^*\gamma_1 \wedge \cdots \wedge ev_l^*\gamma_l \,.$$

$$\gamma_j = PD[P_j], \quad ev_j : \overline{\mathscr{M}}_l(\beta) \to X$$

Facts of life

- GW invariants are defined by the above integral if the space $\overline{\mathscr{M}}_l(\beta)$ is "nice"

- GW are generally hard to compute

- In some cases, can compute GW invariants by the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation

Open Gromov-Witten theory $(g = 0)$

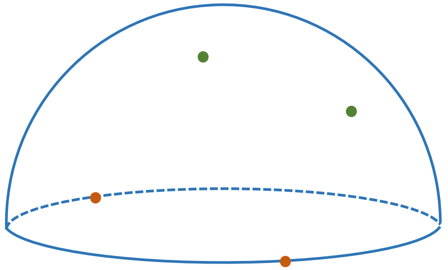
Setting: (X, ω, J) symplectic manifold with almost complex structure

$L \subset X$ a Lagrangian submanifold

$(\dim L = \frac{1}{2}\dim X,\ \omega|_L = 0)$

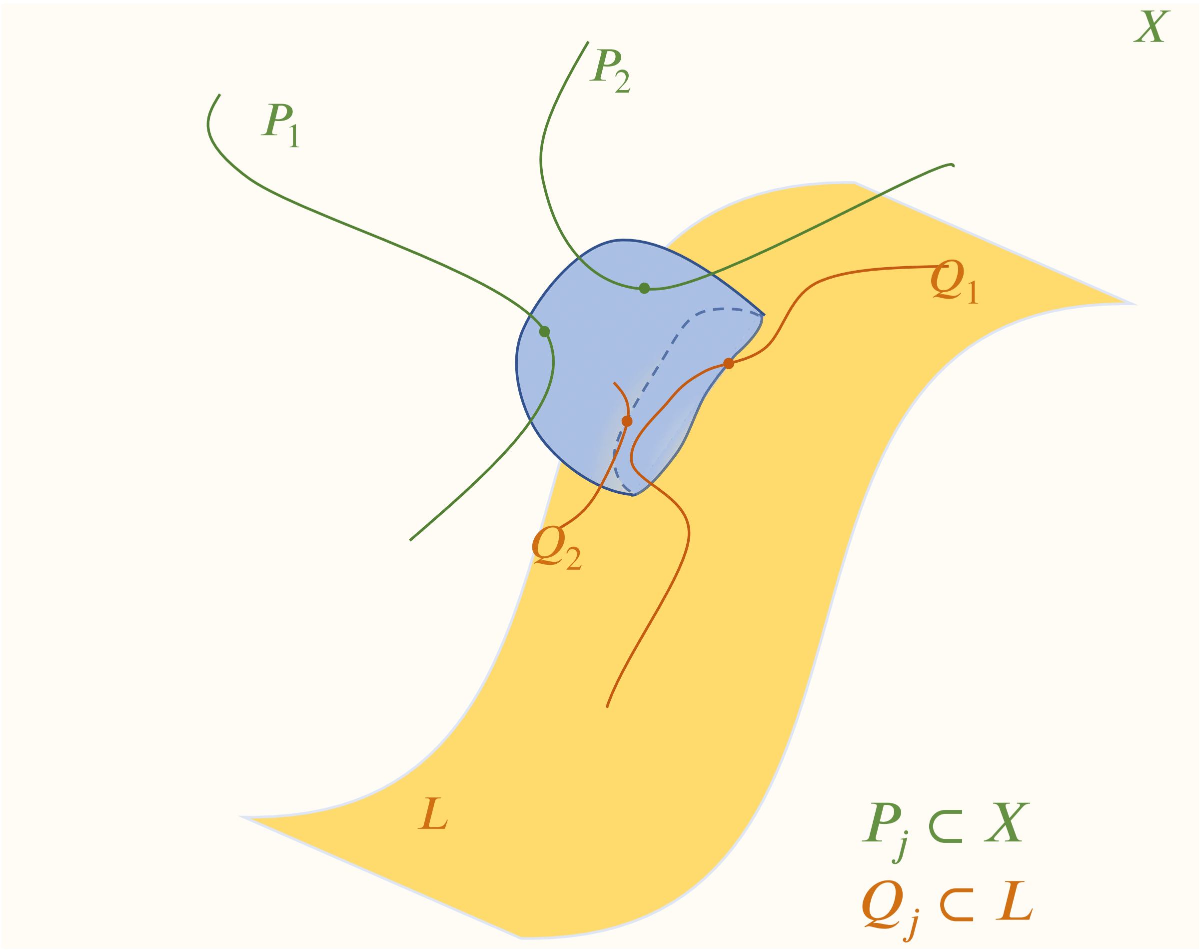
Example: $(X,\ L, \omega,\ J) = \left(\mathbb{C}P^n,\ \mathbb{R}P^n, \omega_{FS},\ J_0\right)$

Assumption: L oriented, relatively spin



J-holomorphic

$\xrightarrow{\quad u \quad}$



$$u_*\left[D^2, \partial D^2\right] = \beta \in H_2(X, L; \mathbb{Z})$$

$P_j \subset X$

$Q_j \subset L$

submanifolds

<u>Question</u>: # u =?

a) Define invariantly of choices (e.g., representative of

$$\left[P_j\right] \in H_*(X),$$

$$\left[Q_j\right] \in H_*(L)\ )$$

b) Compute values

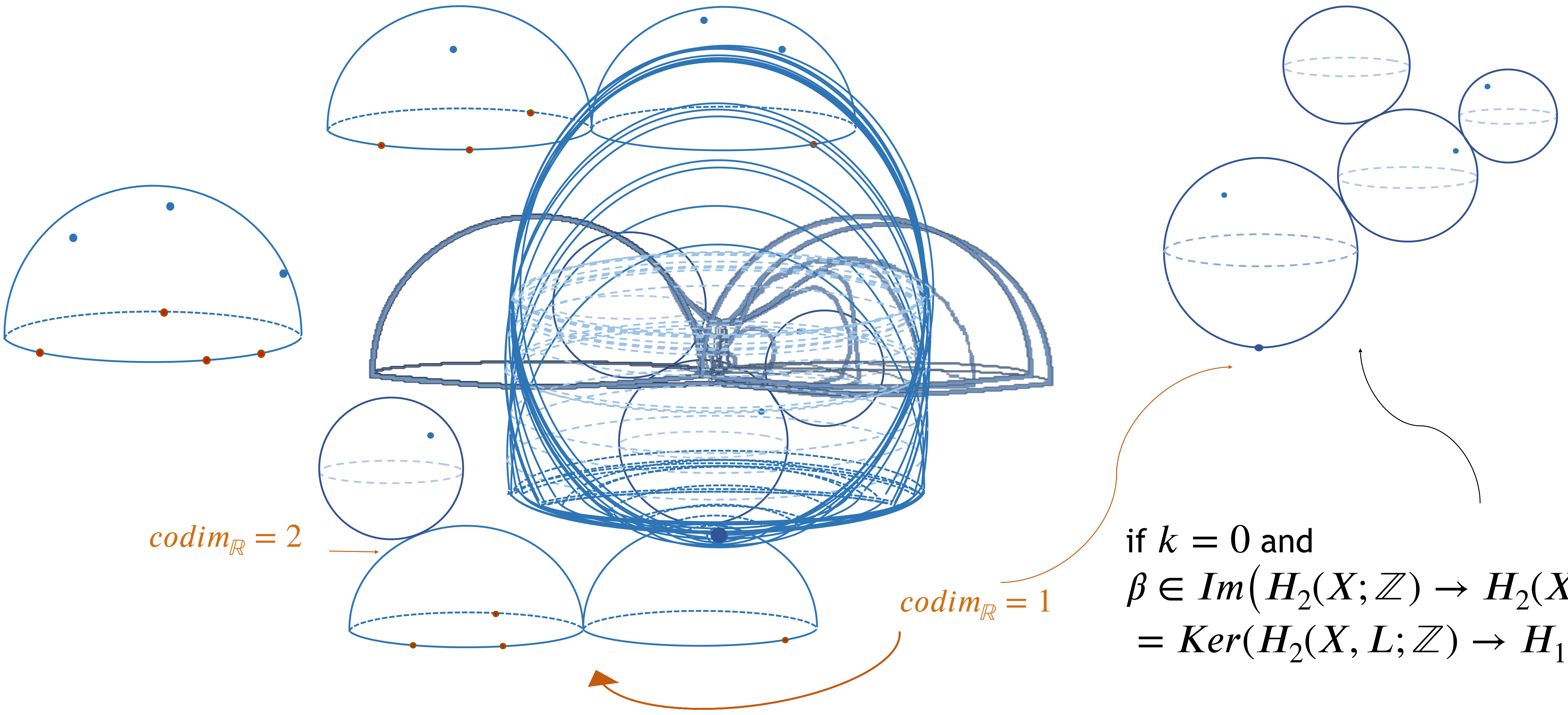
Rephrasing the problem

$$\overline{\mathscr{M}}_{k,l}(\beta) = \overline{\left\{ \begin{pmatrix} u:(D,\partial D) \xrightarrow{J-hol.} (X,L), \\ z_1,\ldots,z_k,\ w_1,\ldots,w_l \end{pmatrix} : \begin{matrix} [u] = \beta \in H_2(X,L;\mathbb{Z}) \\ z_i \in \partial D,\ w_j \in D \end{matrix} \right\}}/\sim$$

Want to count elements of $\overline{\mathscr{M}}_{k,l}(\beta)$ such that the marked points are mapped to given constraints:

$$OGW^{\beta}_{k,l}(\alpha_1,\ldots,\ \alpha_k;\gamma_1\ ,\ldots\gamma_l) \overset{?}{=} \int_{\overline{\mathscr{M}}_{k,l}(\beta)} evb_1^*\alpha_1 \wedge \cdots \wedge evb_k^*\alpha_k \wedge evi_1^*\gamma_1 \wedge \cdots \wedge evi_l^*\gamma_l\,.$$

Compactification of $\overline{\mathscr{M}}_{k,l}(\beta)$



if $k = 0$ and

$\beta \in Im\big(H_2(X;\mathbb{Z}) \to H_2(X, L;$

$= Ker(H_2(X, L;\mathbb{Z}) \to H_1(L;$

Invariance problem

$$OGW_{k,l}^{\beta}\left(\alpha_1, \ldots, \alpha_k; \gamma_1, \ldots \gamma_l\right) \stackrel{?}{=} \int_{\overline{\mathcal{M}}_{k,l}(\beta)} evb_1^*\alpha_1 \wedge \cdots \wedge evb_k^*\alpha_k \wedge evi_1^*\gamma_1 \wedge \cdots \wedge evi_l^*\gamma_l$$

Issue: $\partial\overline{\mathcal{M}}_{k,l}\left(\beta\right) \neq \emptyset \implies$ a priori value not invariant

Definition – some previous results

OGW are defined when

- S^1 acts on (X, L) (Liu, 2004)

- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X = 2, 3$, real interior constraints, point boundary constraints *(Cho, Solomon, 2006)*

- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X$ odd, no boundary constraints *(Georgieva, 2013)*

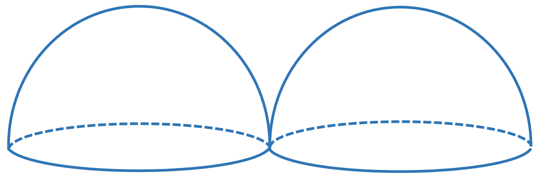
- (X, ω) is a Calabi-Yau threefold *(Joyce 2006; Fukaya, 2011; Ekholm 2017)*

Open WDVV – some previous results

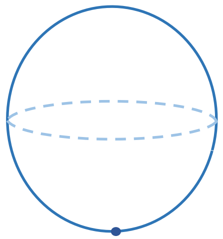
OGW **satisfy a WDVV-like equation when**
(X, L, ω, J) **is a real symplectic manifold with**

- $\dim_{\mathbb{C}} X = 2$, real interior constraints, point boundary constraints *(Solomon, 2007; Horev-Solomon, 2012, Chen 2019)*
- $\dim_{\mathbb{C}} X$ odd, no boundary constraints *(Georgieva-Zinger, 2013)*
- $\dim_{\mathbb{C}} X = 3$, real interior constraints, point boundary constraints, finite group symmetry *(Chen-Zinger, 2019)*

Our approach *(Joint with Jake Solomon)*



Cancel out
boundary bubbling
by
using a bounding
chain



Cancel out boundary
degeneration by
incorporating spheres

Invariance – Part I

A bounding chain is a special type of boundary constraint.

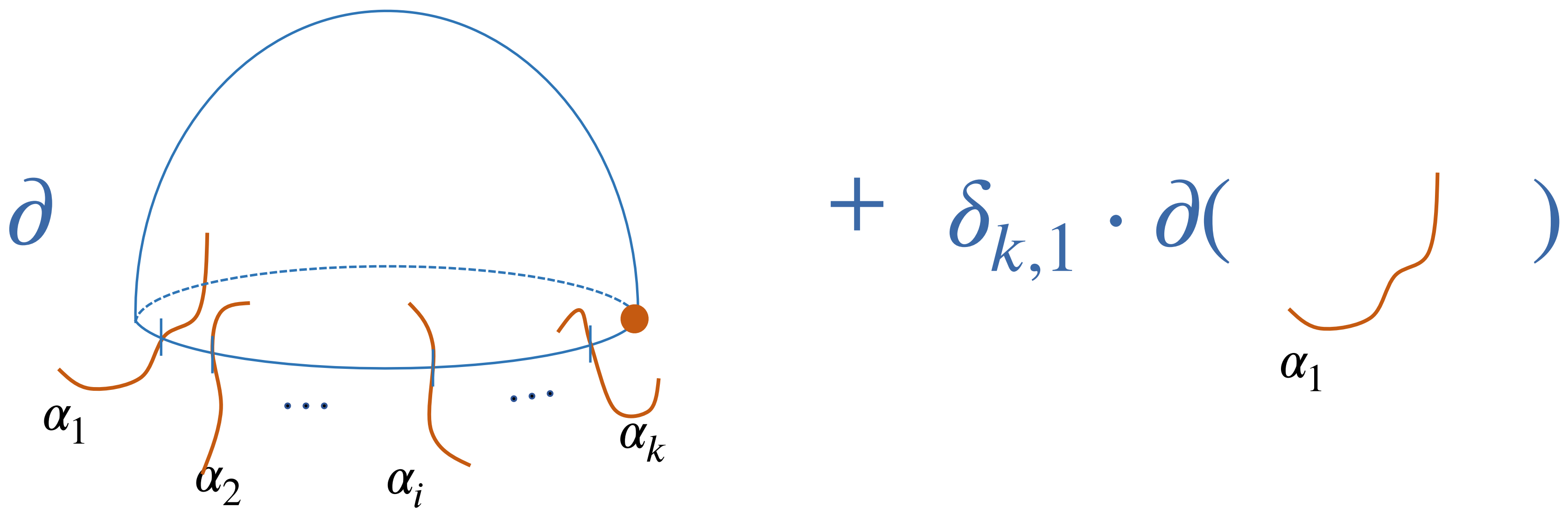
Geometrically, it keeps track of disk bubbling.

Algebraically, a bounding chain b is a solution of the Maurer-Cartan equation:

$$\sum_{k \geq 0} \mathfrak{m}_k(b^{\otimes k}) = c \cdot 1$$

$$\mathfrak{m}_k : A^*(L;R)^{\otimes k} \longrightarrow A^*(L;R)$$

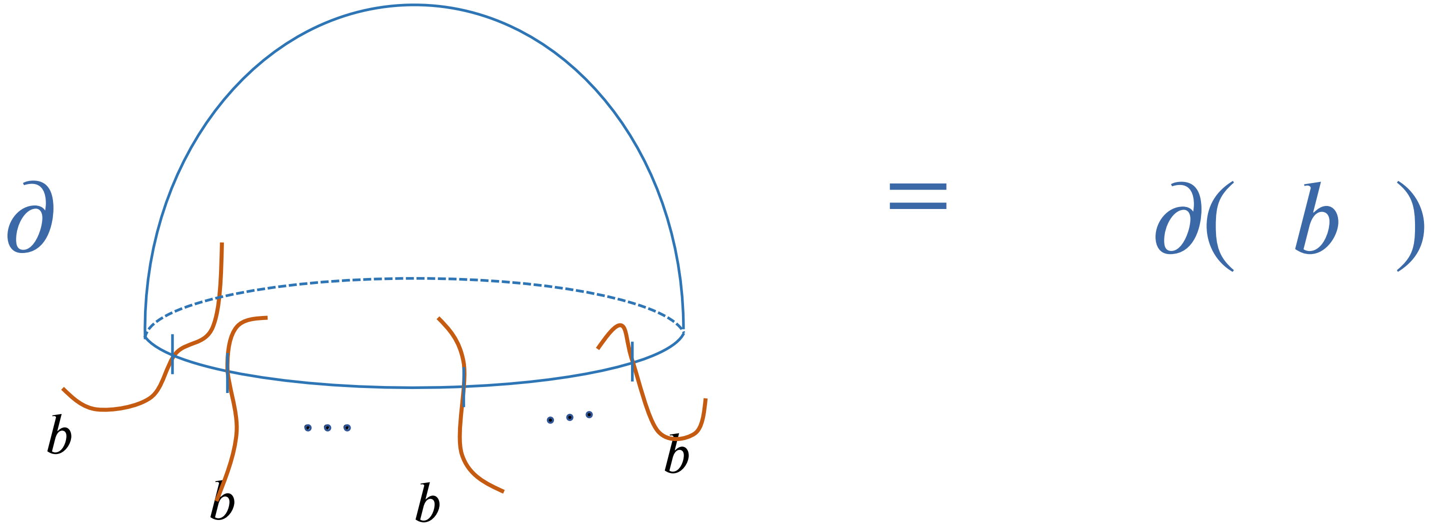
$$\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k) = (evb_0)_*\left(\wedge_{j=1}^{k} evb_j^* \alpha_j \right) + \delta_{k,1} \cdot d\alpha_1$$



The (strong) Maurer-Cartan equation

The strong MC equation of the A_∞ algebra $\left(A^*(L;R), \{\mathfrak{m}_k\}_{k=0}^{\infty}\right)$:

$$\sum_{k\geq 0} \mathfrak{m}_k(b^{\otimes k}) = 0$$



Idea: $\Omega = \left\langle \mathfrak{m}_k\left(b^{\otimes k}\right), b \right\rangle + \textit{corrections}.$

$\langle \eta, \zeta \rangle := \int_L \eta \wedge \zeta$

the Poincaré pairing

Theorem: $\exists\ \ \Omega = \Omega(b)$, a generating function of OGW invariants, invariant under gauge equivalence.

Remark: In particular, Ω is invariant under change of constraint within cohomology class.

Special case

Theorem: If $H^*(L;\mathbb{R}) \simeq H^*(S^n;\mathbb{R})$, then

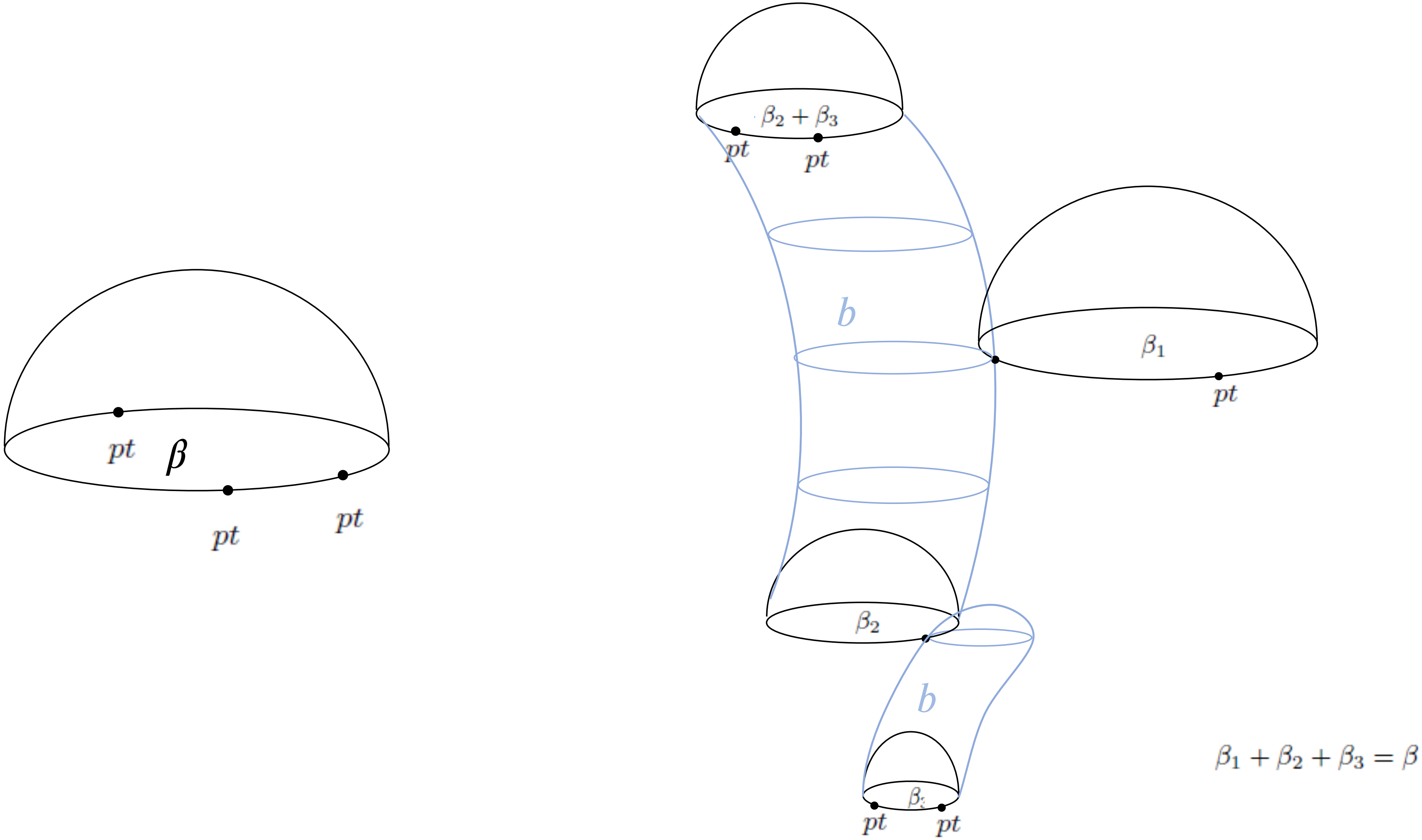
$$b \sim b' \Leftrightarrow \int_L b = \int_L b'.$$

Significance:

Every bounding chain b has the form

b "=" $a \cdot pt + (non\ closed\ corrections)$

So, OGW invariants count configurations of disks with point boundary constraints.



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$\beta_2 + \beta_3$
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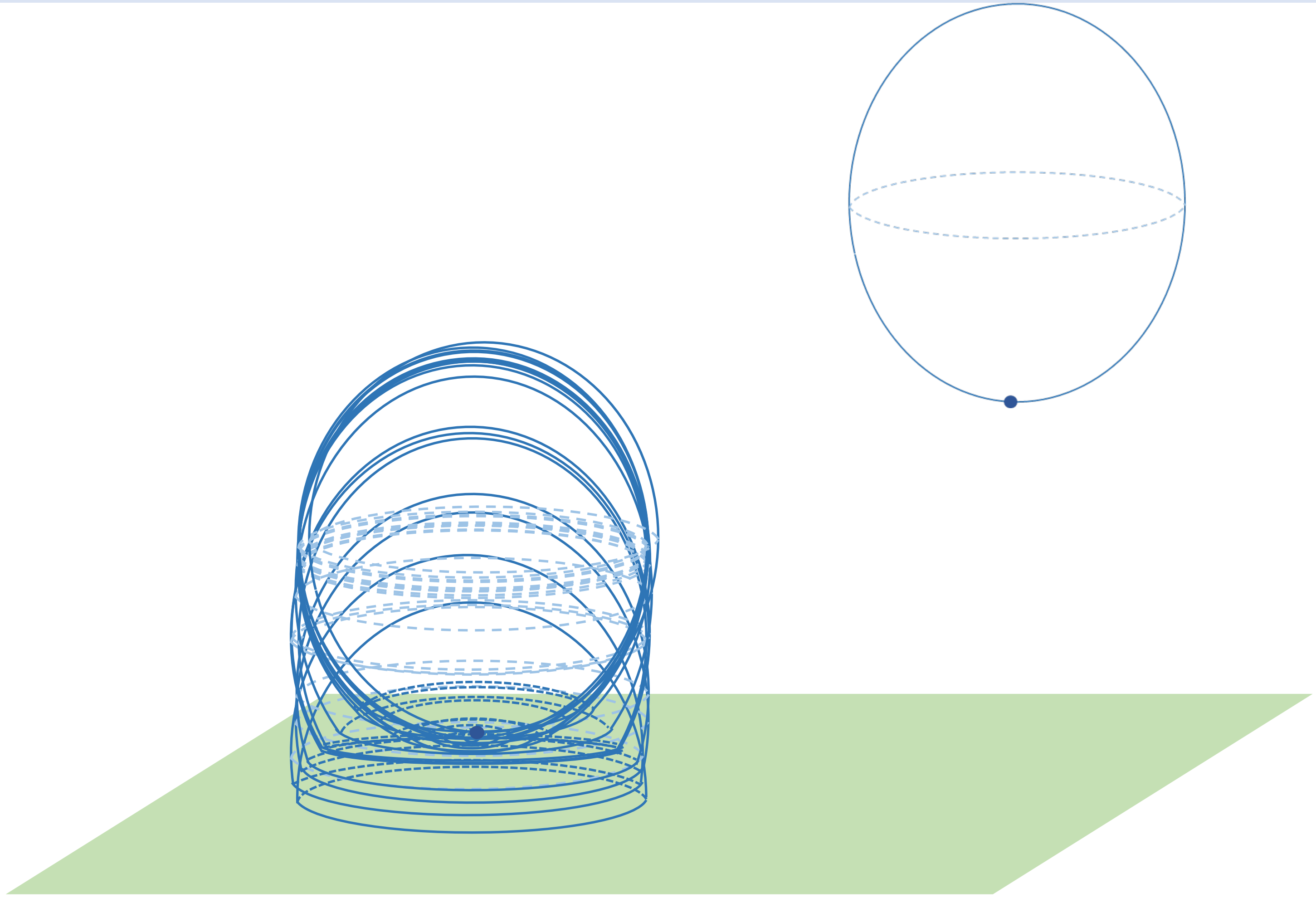
$\beta_1 + \beta_2 + \beta_3 = \beta$

Invariance – Part II



The mapping cone complex

$$i : A^*(X;Q) \to R[-n]\,, \quad i(\eta) = \int_L \eta\,,$$

$Q \leq R$ coefficient rings

$$Cone(i) = A^*(X;Q) \oplus R[-n-1]$$

chains in X

data from L (e.g., Ω)

Strategy:

1. Find an invariant in $H^*(Cone(i))$
2. "Project" to second component

Remark: The plain projection is not a chain map. So, correct by contributions from first component.

Theorem: $\exists\ \bar{\Omega} = \bar{\Omega}(b)$, a generating function of

$O\bar{G}W$, invariant under gauge equivalence.

Ω vs. $\bar{\Omega}$:

For $k \neq 0$ or $\beta \notin Im(H_2(X;\mathbb{Z}) \rightarrow H_2(X,L;\mathbb{Z}))$,

$$OGW_{\beta,k}(\eta_1,\ldots,\eta_l) = O\bar{G}W_{\beta,k}(\eta_1,\ldots,\eta_l)\,.$$

Otherwise, $OGW_{\beta,k}(\eta_1,\ldots,\eta_l) = 0\,.$

The WDVV equation

$\gamma_j \in H^*(X;\mathbb{R})$ basis; $\quad g_{ij} = \int_X \gamma_i \wedge \gamma_j,\ \left(g^{ij}\right) = \left(g_{ij}\right)^{-1}$

t_j = formal variables associated with γ_j

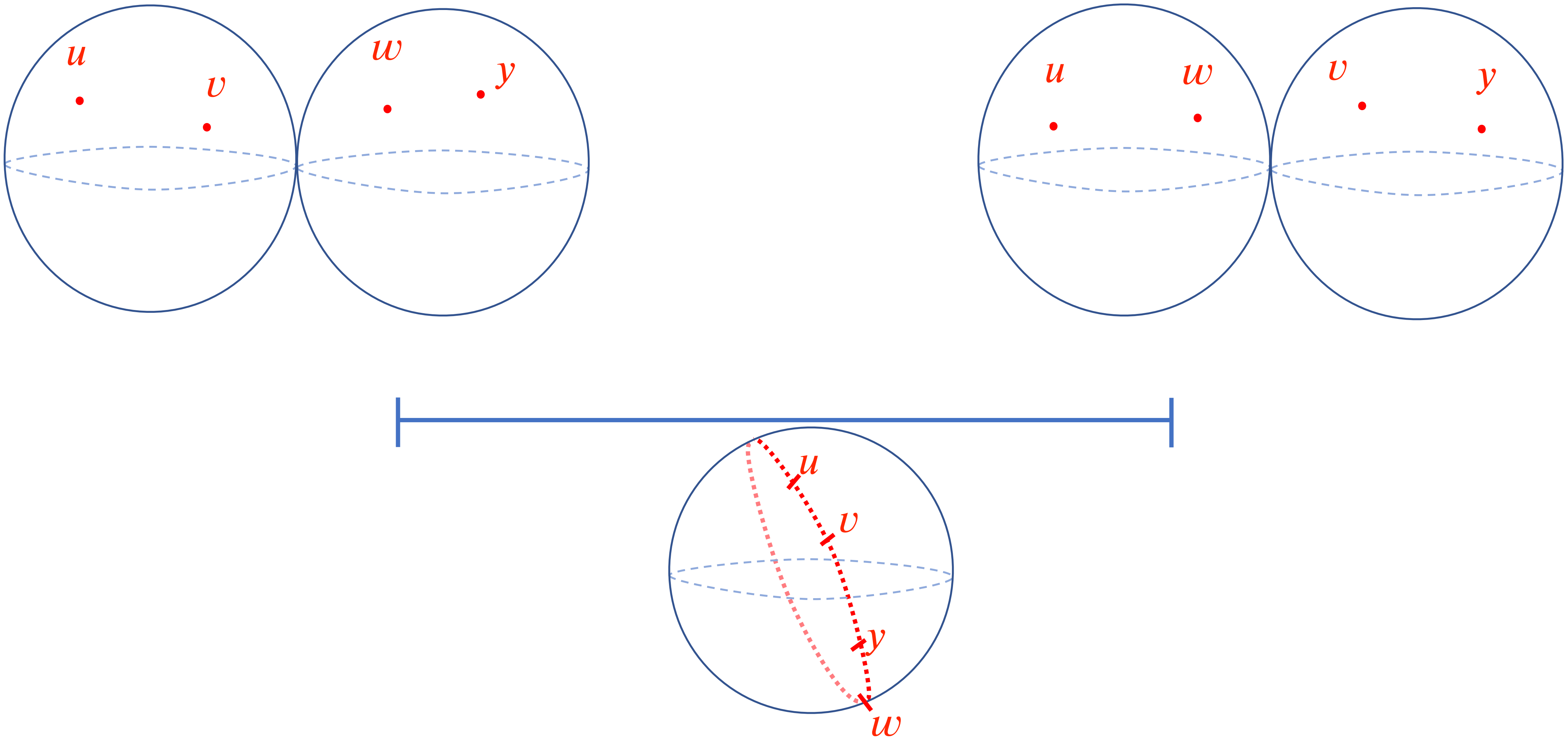
Φ

WDVV equation:

$$\partial_u \partial_v \partial_i \Phi\, g^{ij} \partial_j \partial_w \partial_y \Phi = \partial_u \partial_w \partial_i \Phi\, g^{ij}\, \partial_j \partial_v \partial_y \Phi$$

$$\forall u, v, w, y \in \left\{ t_j \right\}$$

$$\partial_u\partial_v\partial_i\Phi\, g^{ij}\partial_j\partial_w\partial_y\Phi = \partial_u\partial_w\partial_i\Phi\, g^{ij}\,\partial_j\partial_v\partial_y\Phi$$



Kontsevich (1994)

degree = d	No. of degree-d curves in through 3d-1 points
1	1
2	1
3	12
4	620
5	87,304
6	26,312,976
7	14,616,808,192

Open WDVV *(Joint with J. Solomon)*

$t_0, \ldots, t_N$ = formal variables associated with γ_j

s = formal variable associated with the bounding chain point part

Φ = generating function for GW, $\bar{\Omega}$ = generating function for $O\bar{G}W$

c = the Maurer-Cartan constant

Theorem:

$$\partial_v \partial_w \partial_i \Phi \, g^{ij} \partial_j \partial_u \bar{\Omega} - \partial_v \partial_w \bar{\Omega} \cdot \partial_u c =$$

$$= \partial_u \partial_w \partial_i \Phi \, g^{ij} \partial_j \partial_v \bar{\Omega} - \partial_u \partial_w \bar{\Omega} \cdot \partial_v c$$

$\forall u, v \in \{s, t_0, \ldots, t_N\}$, $w \in \{t_0, \ldots, t_N\}$.

- If $[L] = 0 \in H_n(X;\mathbb{R})$, then

$$\partial_v\partial_w\partial_i\Phi\ g^{ij}\partial_j\partial_u\bar{\Omega}\ - \partial_v\partial_w\bar{\Omega}\cdot\partial_u\partial_s\bar{\Omega} =$$

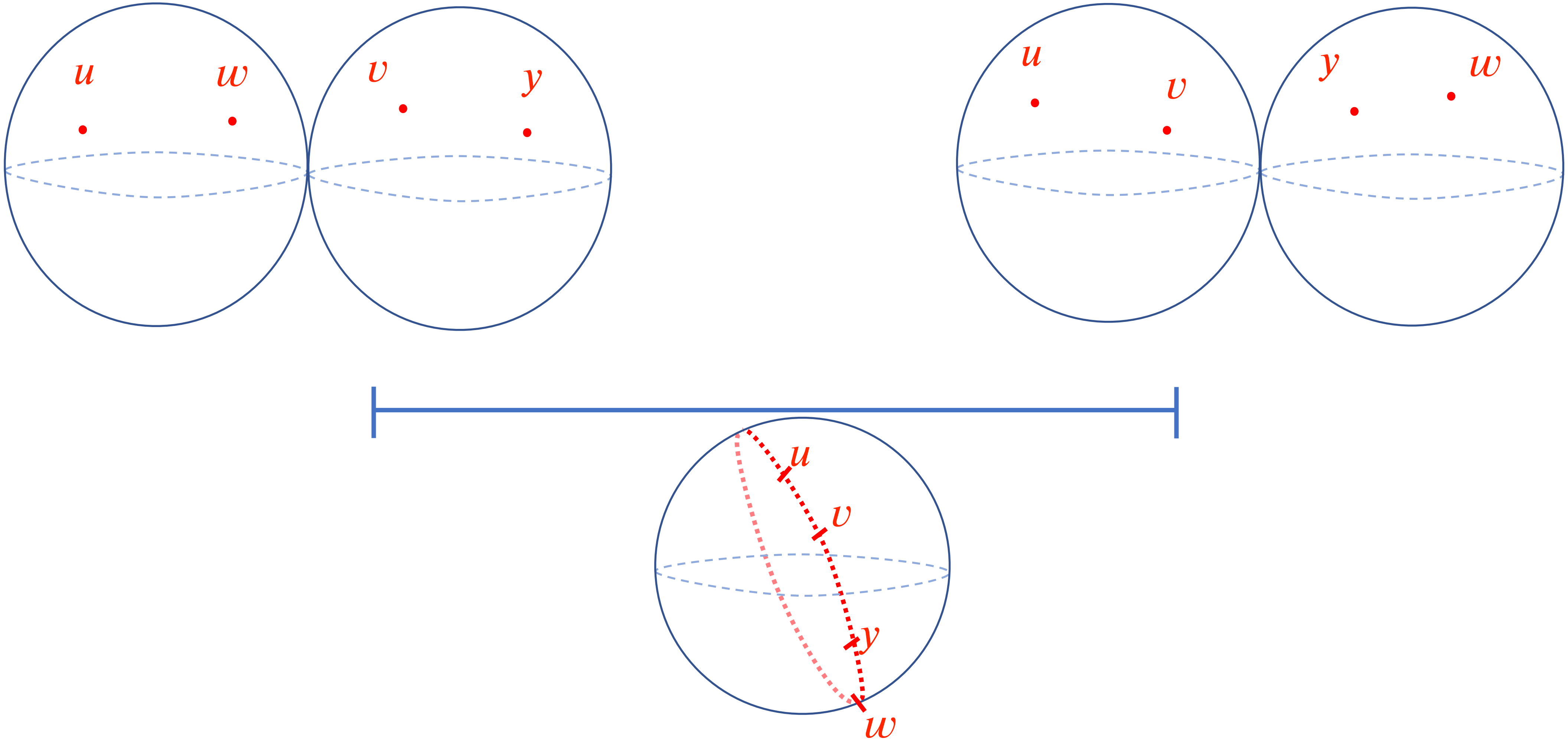
$$= \ \partial_u\partial_w\partial_i\Phi\ g^{ij}\partial_j\partial_v\bar{\Omega}\ - \partial_u\partial_w\bar{\Omega}\cdot\partial_v\partial_s\bar{\Omega}$$

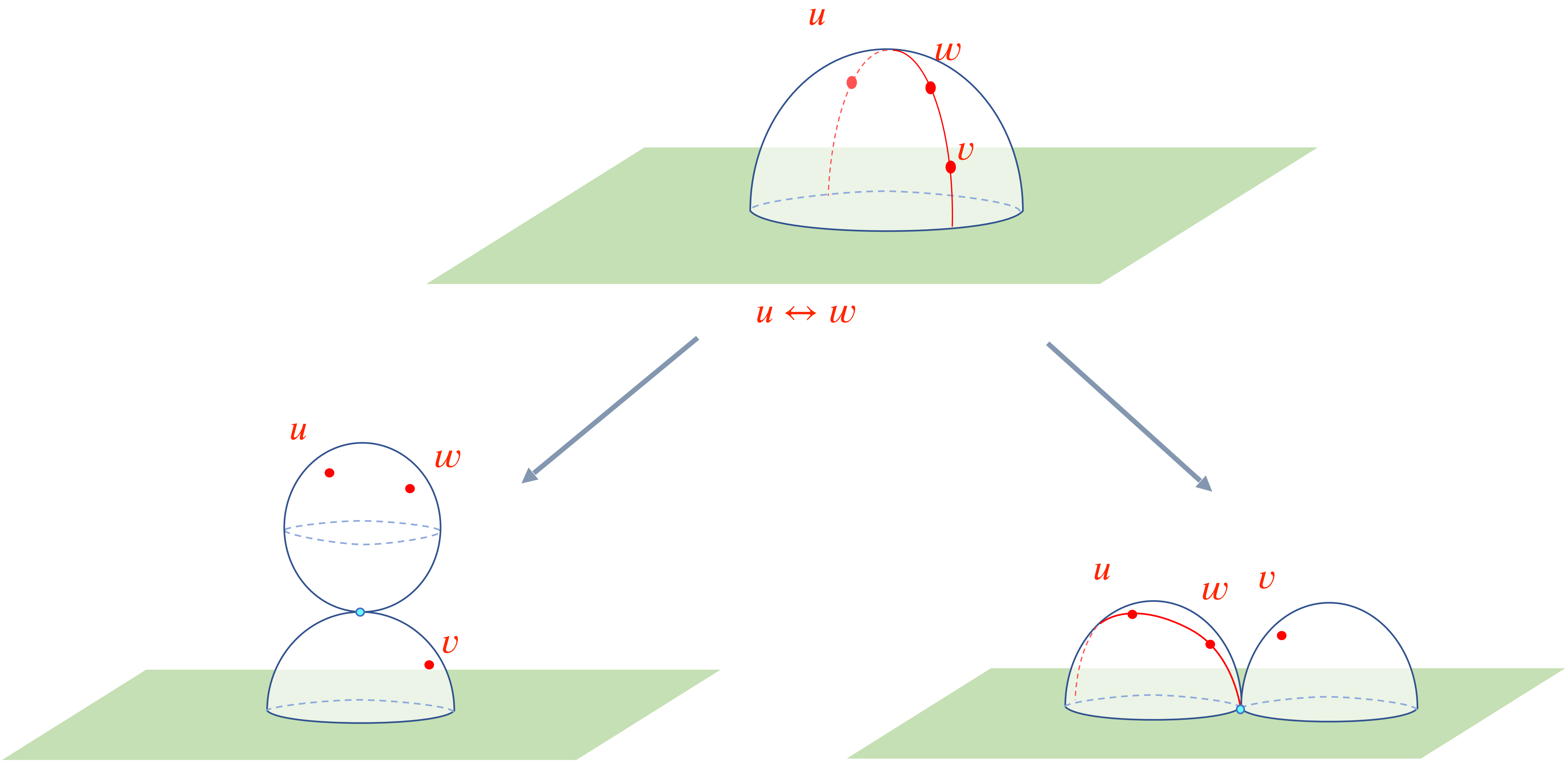
- If $[L] \neq 0 \in H_n(X;\mathbb{R})$, then

$$\partial_v\partial_w\partial_i\Phi\ g^{ij}\partial_j\partial_u\Omega\ - \partial_v\partial_w\Omega\cdot\partial_u\partial_s\Omega =$$

$$= \ \partial_u\partial_w\partial_i\Phi\ g^{ij}\partial_j\partial_v\Omega - \partial_u\partial_w\Omega\cdot\partial_v\partial_s\Omega$$

$$\partial_u\partial_v\partial_i\Phi\, g^{ij}\partial_j\partial_w\partial_y\Phi = \partial_u\partial_w\partial_i\Phi\, g^{ij}\, \partial_j\partial_v\partial_y\Phi$$





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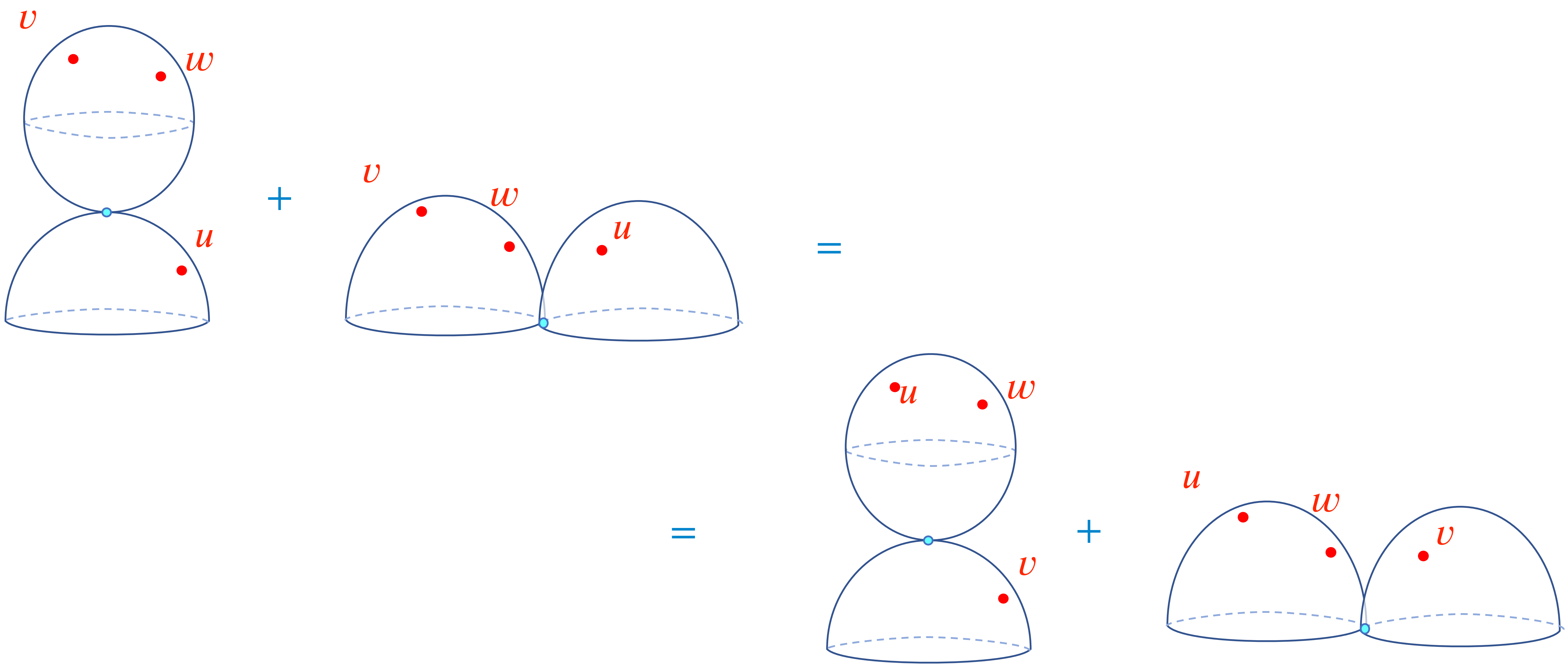
$$\partial_v\partial_w\partial_i\Phi\, g^{ij}\partial_j\partial_u\bar{\Omega} - \partial_v\partial_w\bar{\Omega}\cdot\partial_u\partial_s\bar{\Omega} =$$

$$= \partial_u\partial_w\partial_i\Phi\, g^{ij}\partial_j\partial_v\bar{\Omega} - \partial_u\partial_w\bar{\Omega}\cdot\partial_v\partial_s\bar{\Omega}$$



Special case

<u>Theorem</u>:

Let $(X, L, \omega, J) = \left(\mathbb{C}P^n, \mathbb{R}P^n, \omega_{FS}, J_0\right)$.

Then all $O\bar{G}W$ invariants are computable via recursions produced by OWDVV + general properties of $\bar{\Omega}$ (open Gromov-Witten axioms, wall crossing).

$$(X, L) = (\mathbb{C}P^n, \mathbb{R}P^n)$$

Initial condition:

$$OGW^n_{1,2} = 2$$

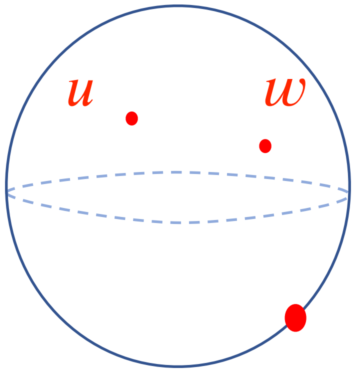
dim = n	degree = d	No. of boundary points = k	Resulting invariant
3	3	6	-2
	5	10	90
	7	14	-29,178
	9	18	35,513,586
5	5	8	-2
	9	14	1974
	13	20	-42,781,410
	17	26	7,024,726,794,150
7	7	10	-2
	13	18	35,498
	19	26	-40,083,246,650
	25	34	680,022,893,749,060,370
9	9	12	-2
	17	22	587,334
	25	32	-31,424,766,229,890
	33	42	49,920,592,599,715,322,910,150
15	29	34	2,247,512,778

Back to the general case

Quantum product

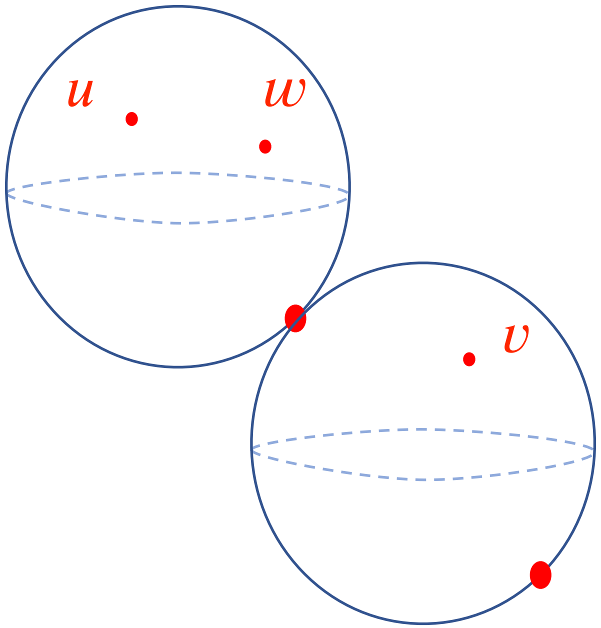
$$* \ : H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

$u * w$ = the locus of points on spheres that pass through u, w

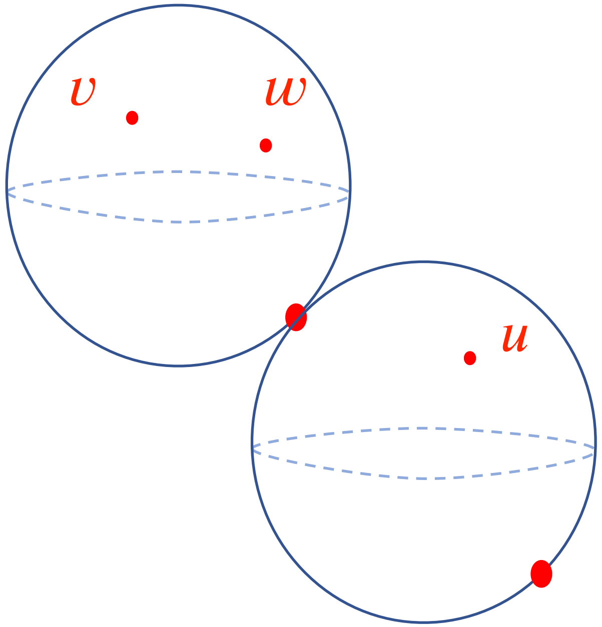


Associativity

$(u * w) * v$



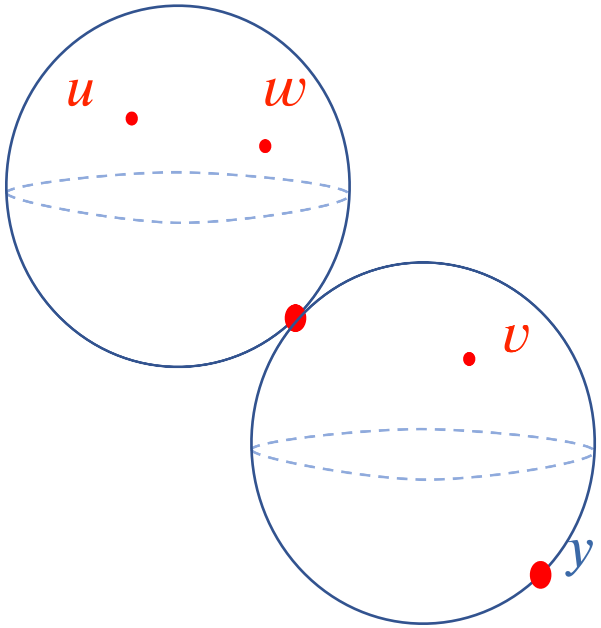
$u * (w * v)$



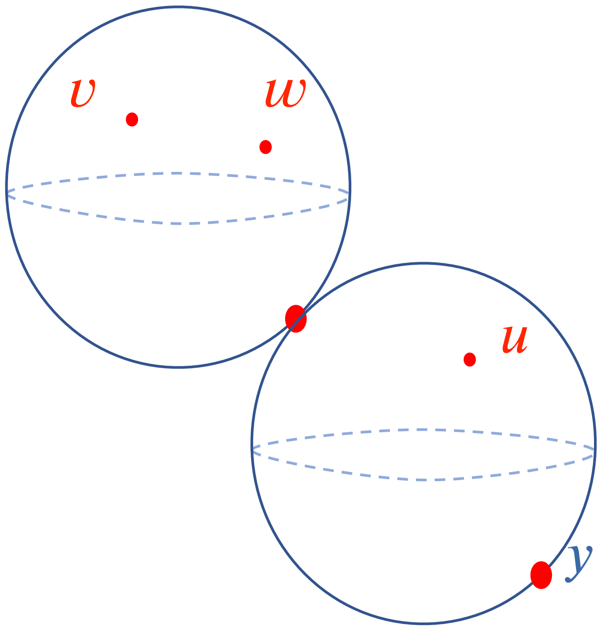
$$\partial_u\partial_v\partial_i\Phi\ g^{ij}\partial_j\partial_w\partial_y\Phi \;=\; \partial_u\partial_w\partial_i\Phi\ g^{ij}\ \partial_j\partial_v\partial_y\Phi$$

Associativity

$(u * w) * v$



$u * (w * v)$



<u>Corollary</u>: WDVV is equivalent to the associativity of $*$.

Relative quantum product *(Joint with J. Solomon)*

$$מ : H^*(X,L) \otimes H^*(X,L) \longrightarrow H^*(X,L)$$

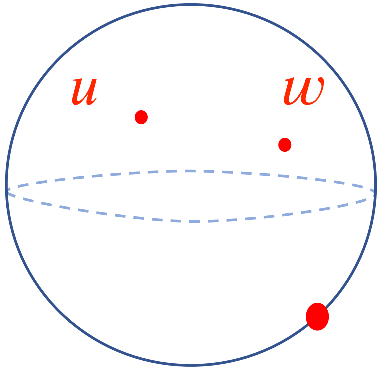
mem

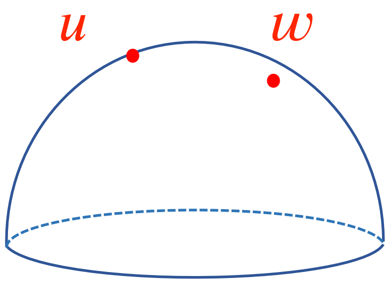
For $i_{\mathbb{R}}: A^*(X;\mathbb{R}) \to \mathbb{R}$, $i_{\mathbb{R}}(\eta) = \int_L \eta$, we have

$Cone(i_{\mathbb{R}}) = A^*(X) \oplus \mathbb{R}$, $H^*(X,L) \simeq H^*\left(Cone(i_{\mathbb{R}})\right)$.

$$מ : H^*\left(Cone(i)\right) \otimes H^*\left(Cone(i)\right) \longrightarrow H^*(Cone(i))$$

מ $((w,0),(u,a))$ $=$

$= \Big($ 

, 

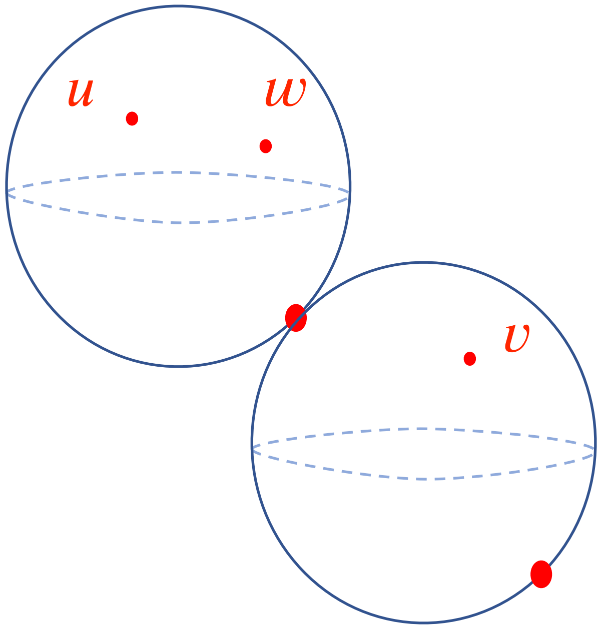
$+ \quad \partial_w c \cdot a \Big)$

Associativity

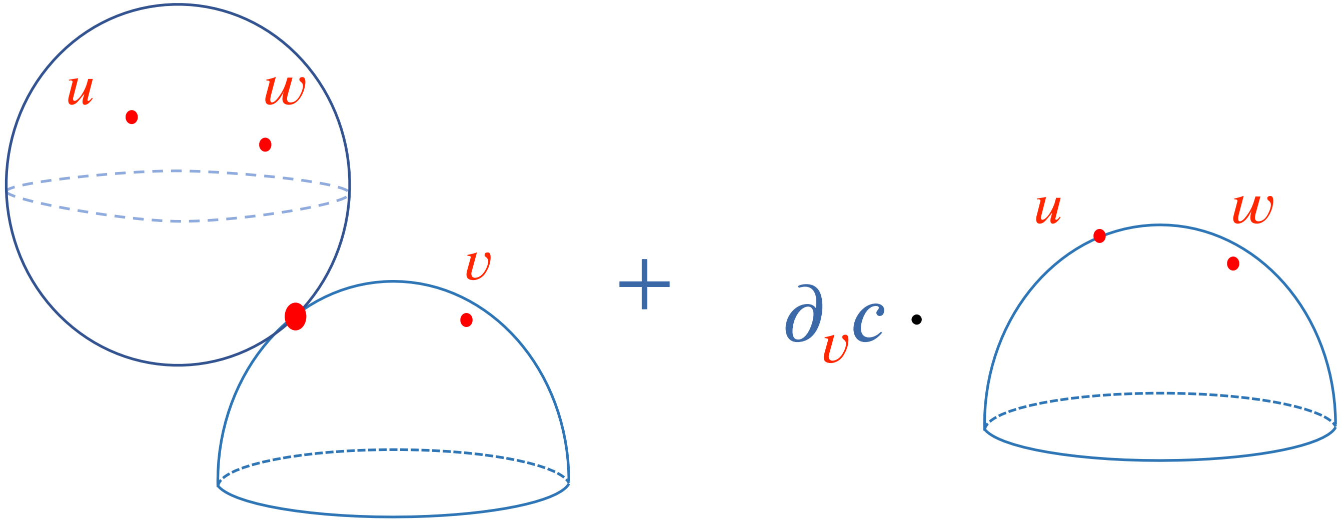
$u, v, w \in H^*(X)$

$$מ(v\ ,מ\ (w, u))$$

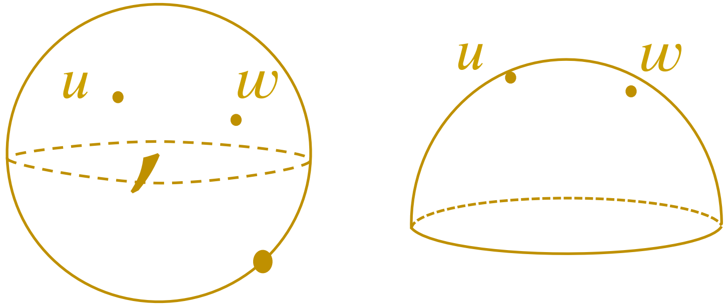
First component:



מ$(v$, מ$(w, u))$ – second component:



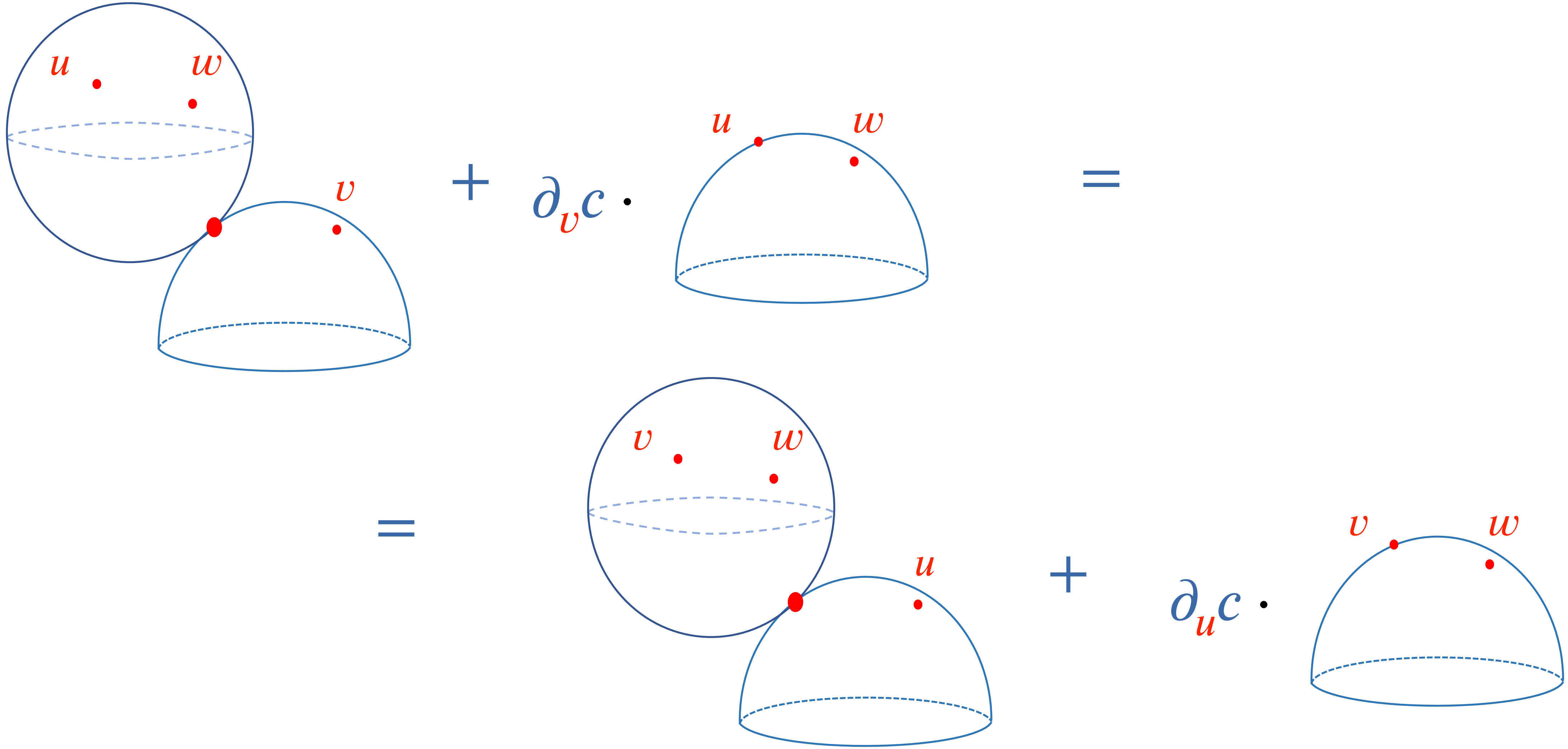
מ $(w, (u, a)) = ($



$+ \; \partial_w c \cdot a)$

$$\partial_v\partial_w\partial_i\Phi\; g^{ij}\partial_j\partial_u\bar{\Omega} \;-\; \partial_u\partial_v\bar{\Omega}\cdot\partial_u c =$$

$$= \partial_v\partial_w\partial_i\Phi\; g^{ij}\partial_j\partial_u\bar{\Omega} \;-\; \partial_u\partial_v\bar{\Omega}\cdot\partial_u c$$



Theorem:

The open WDVV is equivalent to the associativity of מ.

So...

Genus zero curves with boundary: Can you count them?

- **Pretty much.**

Can you really?

- **Sometimes...**

Thank you