Structure theorems for intertwining operators

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Introduction and Summary

In this lecture we will focus on Yajima's L^p theory for wave operators, especially in $\mathbb{R}^3$, and the structure result that emerged from it.

- Recall definition of wave operators
- Iterated resolvent identity, and it's dual, Duhamel expansion
- Write wave operator as a (formal) infinite series involving the free Schrödinger evolution. Terminating the series is expensive, as it involves the unknown evolution.
- Beceanu's Wiener algebra formalism is a summation method for summing a divergent series.
- Solve an inversion problem in a suitable (somewhat complicated) algebra of operators. The invertibility condition guaranteed by spectral theory and zero energy condition.
- Restriction theory for the Fourier transform (Stein-Tomas), Strichartz estimates, crucial for the argument
- Open problem: redo in scaling invariant norm

Wave operators

Let V real-valued potential in $\mathbb{R}^d$, bounded, sufficiently decaying, $H := -\Delta + V$, $H_0 := -\Delta$. Define

$$W_\pm := \lim_{t \to \mp\infty} e^{itH} e^{-itH_0}$$

Exists in the strong L^2-sense: $d \geq 3$, $f \in L^1 \cap L^2(\mathbb{R}^d)$, $V \in L^2$:

$$W_\pm f = f \mp i \int_0^\infty e^{itH} V e^{-itH_0} f \, dt$$

$$\int_1^\infty \left\| e^{itH} V e^{-itH_0} f \right\|_2 dt \leq \int_1^\infty \|V\|_2 \|e^{-itH_0} f\|_\infty \, dt$$

$$\lesssim \|V\|_2 \int_1^\infty t^{-\frac{d}{2}} \|f\|_1 \, dt < \infty$$

Unitarity of evolution, density of $L^1 \cap L^2(\mathbb{R}^d)$ in L^2 shows limit exists for all $f \in L^2$ and $W_\pm$ are isometries.

Intertwining property of wave operators

$f(H)W_\pm = W_\pm f(H_0)$, or

$$f(H)P = f(H)W_\pm W_\pm^* = W_\pm f(H_0)W_\pm^*,$$

with P orthogonal projection onto $\mathrm{Ran}(W_\pm)$. Easy to see: $\mathrm{Ran}(W_\pm) \perp L^2_{pp}$ (eigenfunctions of H). Asymptotic completeness: $\mathrm{Ran}(W_\pm) = L^2_{ac}(\mathbb{R}^d)$, $L^2_{sc} = \{0\}$.
Iterated resolvent identity:

$$R(\lambda) = (H - (\lambda^2 + i0))^{-1} = R_0(\lambda) + R_0(\lambda)VR(\lambda) =$$
$$= ... = R_0(\lambda) + R_0(\lambda)VR_0(\lambda) + R_0(\lambda)VR_0(\lambda)VR_0(\lambda) + \dots$$

If V short range, small: $R(\lambda)$ inherits the limiting absorption principle. Split $V = |V|^{\frac{1}{2}} \mathrm{sign}(V)|V|^{\frac{1}{2}} = |V|^{\frac{1}{2}} U$.
Large V: $R(\lambda) = R_0(\lambda) + R_0(\lambda)|V|^{\frac{1}{2}}(I - UR_0(\lambda)|V|^{\frac{1}{2}})^{-1}UR_0(\lambda)$.

Yajima's L^p theory for the intertwining operator

In the 1990s Kenji Yajima showed that $W_\pm : L^p(\mathbb{R}^d) \to \mathbb{R}^d(\mathbb{R}^d)$, $1 \le p \le \infty$, $d \ge 3$, and $1 < p < \infty$, $d = 1, 2$. He needed to assume enough decay (and regularity in $d \ge 4$), and no zero energy eigenvalue/resonance. In dim=3 he needed $|V(x)| \le \langle x \rangle^{-5-\varepsilon}$. If zero energy singular, then $3/2 < p < 3$, $|V(x)| \le \langle x \rangle^{-6-\varepsilon}$.

Corollary: dispersive estimates for $e^{it\Phi(H)} P_c(H)$ from those for $e^{it\Phi(H_0)}$ via

$$e^{it\Phi(H)} P_c(H) = W e^{it\Phi(H_0)} W^*$$

Importance of 0 energy condition implied by this, too. For example, in dim=3

$$\left\| e^{itH} f \right\|_\infty \le \|W\|_{\infty\to\infty} \|W\|_{1\to 1}\ C t^{-\frac{3}{2}} \|f\|_1, \qquad f \perp \text{bound states}$$

Possible issues: (i) strong assumptions on potential (ii) in some nonlinear applications 0 energy singularities do arise.

Yajima's proof, expansion of the wave operators

Iterate Duhamel (Fourier transform of iterated resolvent identity) with $f \in L^2$:

$$Wf = f + W_1 f + \ldots + W_n f + \ldots,$$

$$W_1 f = i \int_{t>0} e^{-it\Delta} V e^{it\Delta} f \, dt, \ \ldots$$

$$W_n f = i^n \int_{t>s_1>\ldots>s_{n-1}>0} e^{-i(t-s_1)\Delta} V e^{-i(s_1-s_2)\Delta} V \ldots$$
$$e^{-is_{n-1}\Delta} V e^{it\Delta} f \, dt \, ds_1 \ldots ds_{n-1}$$

Keel-Tao Strichartz endpoint (in $\mathbb{R}^3$)

$$\|e^{itH_0} f\|_{L^2_t L^{6,2}_x} \lesssim \|f\|_{L^2}$$

$$\left\| \int_{\mathbb{R}} e^{-isH_0} F(s) \, ds \right\|_{L^2_x} \lesssim \|F\|_{L^2_t L^{6/5,2}_x},$$

$V : L^{6,2}_x(\mathbb{R}^3) \to L^{6/5,2}_x(\mathbb{R}^3)$, $V \in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$
Dyson series converges in L^2 if $\|V\|_{3/2} \ll 1$.

Representations of the summands W_n

Taking Fourier transforms on the previous slides yield, for V, f, g Schwartz functions, $\varepsilon > 0$:

$$\langle W_n^\varepsilon f, g\rangle =$$

$$\frac{(-1)^n}{(2\pi)^3}\int_{\mathbb{R}^{3(n+1)}} \frac{\prod_{\ell=1}^n \widehat{V}(\xi_\ell - \xi_{\ell-1})\, d\xi_1 \ldots\, d\xi_{n-1}}{\prod_{\ell=1}^n (|\eta+\xi_\ell|^2 - |\eta|^2 + i\varepsilon)} \widehat{f}(\eta)\overline{\widehat{g}}(\eta+\xi_n)\, d\eta\, d\xi_n$$

$$\langle W_{1+}^\varepsilon f, g\rangle = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} \frac{\widehat{V}(\xi)}{|\eta+\xi|^2 - |\eta|^2 + i\varepsilon}\widehat{f}(\eta)\overline{\widehat{g}}(\eta+\xi)\, d\eta\, d\xi$$

$$= \int_{\mathbb{R}^6} K_1^\varepsilon(x, x-y) f(y)\, dy\, \overline{g}(x)\, dx$$

$$K_1^\varepsilon(x,z) = c|z|^{-2}\int_0^\infty e^{-is\hat{z}\cdot(x-z/2)}\,\widehat{V}(-s\hat{z})e^{-\varepsilon\frac{|z|}{2s}}\, s\, ds, \quad \hat{z} = z/|z|$$

$$K_1(x,z) = c|z|^{-2}L(|z| - 2x\cdot\hat{z}, \hat{z}), \;\; L(r,\omega) = \int_0^\infty \widehat{V}(-s\hat{z})e^{i\frac{rs}{2}}\, s\, ds$$

The structure of W_1 in $\mathbb{R}^3$

$S_\omega x := x - 2(\omega \cdot x)\omega$ reflection about plane $\omega^\perp$.

$$\begin{aligned}(W_1 f)(x) &= \int_0^\infty \int_{\mathbb{S}^2} L(r - 2\omega \cdot x, \omega) f(x - r\omega)\, dr d\omega \\ &= \int_{\mathbb{S}^2} \int_{\mathbb{R}} \mathbb{1}_{[r > -2\omega \cdot x]} L(r, \omega) f(S_\omega x - r\omega)\, dr d\omega \\ &= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_1(x, dy, \omega) f(S_\omega x - y)\, d\omega\end{aligned}$$

Therefore, with $\mathcal{H}^1_{\ell_\omega}$ Hausdorff measure on line along ω

$$g_1(x, dy, \omega) := \mathbb{1}_{[(y+2x)\cdot\omega > 0]} L(y \cdot \omega, \omega)\, \mathcal{H}^1_{\ell_\omega}(dy)$$

$$\int_{\mathbb{S}^2} \|g_1(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x}\, d\omega \le \int_{\mathbb{S}^2} \int_{\mathbb{R}} |L(r, \omega)|\, dr d\omega =: \|L\|$$

$$\|W_1 f\|_p \le \|L\| \|f\|_p$$

Bounding L

Define

$$\|f\|_{B^\beta} := \|\mathbb{1}_{[|x|\le 1]} f\|_2 + \sum_{j=0}^{\infty} 2^{j\beta} \big\|\mathbb{1}_{[2^j \le |x| \le 2^{j+1}]} f\big\|_2 < \infty$$

Then $\dot{B}^{\frac{1}{2}} \hookrightarrow L^{\frac{3}{2},1}(\mathbb{R}^3)$, $\dot{B}^1 \hookrightarrow L^{\frac{6}{5},1}(\mathbb{R}^3)$, and

$$\|L(r,\omega)\|_{L^2_{r,\omega}} \lesssim \|V\|_{L^2}$$
$$\|L(r,\omega)\|_{L^1_{r,\omega}} \lesssim \sum_{k\in\mathbb{Z}} 2^{k/2} \|\mathbb{1}_{[2^k,2^{k+1}]}(|r|) L(r,\omega)\|_{L^2_{r,\omega}} \lesssim \|V\|_{\dot{B}^{\frac{1}{2}}} \lesssim \|V\|_{B^{\frac{1}{2}}}$$

Yajima showed for small potentials that $\|V\|_{B^{1+\varepsilon}} \ll 1$ implies

$$\|W_n f\|_p \le C^n \|V\|^n_{B^{1+\varepsilon}} \|f\|_p$$

which can be summed. For large potentials he incurred significant losses by terminating the expansion through the last term which contains perturbed evolution.

Structure Theorem I

Theorem (Beceanu-S. 16)

$V \in B^{1+}$ real-valued, zero energy regular for $H = -\Delta + V$. There exists $g(x, dy, \omega) \in L^1_\omega \mathcal{M}_y L^\infty_x$ with

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x} \, d\omega < \infty$$

$$(W_+ f)(x) = f(x) + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, dy, \omega) f(S_\omega x - y) \, d\omega.$$

X Banach space of measurable functions on $\mathbb{R}^3$, invariant under translations and reflections, Schwartz functions are dense (or dense in Y with $X = Y^*$). Assume $\|\mathbb{1}_H f\|_X \leq A\|f\|_X$ for all half spaces $H \subset \mathbb{R}^3$ and $f \in X$ with some uniform constant A. Then

$$\|W_+ f\|_X \leq AC(V)\|f\|_X \qquad \forall\, f \in X$$

where $C(V)$ is a constant depending on V alone.

Structure Theorem II

Theorem (Beceanu-S. 16)

$V \in B^{1+2\gamma}$, $0 < \gamma$, with 0 energy hypothesis. Then

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L_x^\infty}\, d\omega \le C_0 (1 + \|V\|_{B^{1+2\gamma}})^{38+\frac{105}{\gamma}} (1 + M_0)^{4+\frac{3}{\gamma}}$$

$$\sup_{\eta \in \mathbb{R}^3} \sup_{\varepsilon > 0} \left\| \left(I + R_0(|\eta|^2 \pm i\varepsilon)V\right)^{-1} \right\|_{\infty \to \infty} =: M_0 < \infty$$

C_0 absolute constant.

- 0 energy regular means that $\left\| \left(I + (-\Delta)^{-1} V\right)^{-1} \right\|_{\infty \to \infty} < \infty$.
- would be desirable to bound M_0 through this and size of V is some sense. Control of M_0 is not effective. See Rodnianski-Tao 2015, effective limiting absorption principles.
- Fall short by $\frac{1}{2}$ of scaling invariant class $\dot{B}^{\frac{1}{2}}$. First theorem also works in B^1, but lose quantitative control there.

Wiener algebra and inversion

We cannot sum the Dyson series. Instead we use Beceanu's operator-valued Wiener formalism. Recall classical Wiener theorem:

Proposition

Let $f \in L^1(\mathbb{R}^d)$. There exists $g \in L^1(\mathbb{R}^d)$ with

$$(1+\hat{f})(1+\hat{g}) = 1 \quad \text{on} \quad \mathbb{R}^d \tag{1}$$

iff $1+\hat{f} \neq 0$ everywhere. Equivalently, there exists $g \in L^1(\mathbb{R}^d)$ so that

$$(\delta_0 + f) * (\delta_0 + g) = \delta_0 \tag{2}$$

iff $1+\hat{f} \neq 0$ everywhere on $\mathbb{R}^d$. The function g is unique.

Two critical features (compactness as in Arzela-Ascoli):

- uniform L^1-modulus of continuity under translation.
- vanishing at ∞ in L^1 sense.

Classical Wiener Theorem proof

χ Schwartz function, $\hat{\chi}(\xi) = 1$, $|\xi| \le 1$, $\hat{\chi}$ of compact support.
Take $L \gg 1$ so that

$$\|f_1\|_1 < \frac{1}{2}, \quad f_1 := f - L^d \chi(L\cdot) * f$$

Then by a series expansion can write
$(\delta + f_1)^{-1} = \delta - f_1 + f_1 * f_1 - \ldots$. Thus find $g_1 \in L^1$ with

$$(1 + \hat{f}_1(\xi))(1 + \hat{g}_1(\xi)) = (1 + \hat{f}(\xi))(1 + \hat{g}_1(\xi)) = 1, \quad |\xi| \gg 1$$

Need to find solution $g_2 \in L^1$ for $|\xi| \ll 1$, then combined by partition of unity on Fourier side (convolution by Schwartz functions in the original variable).
Fix $\xi_0 \in \mathbb{R}^d$, $z_0 := (1 + \hat{f}(\xi_0))^{-1}$. We want to find $g \in L^1$ s.t.

$$\left[1 + z_0(\hat{f}(\xi) - \hat{f}(\xi_0))\right]\left[1 + \hat{g}(\xi)\right] = z_0, \quad |\xi - \xi_0| \ll 1$$

This is enough, by covering $|\xi| \le 100L$ by finitely many such intervals, again partition of unity.

Classical Wiener Theorem, proof continued

For any $1 > \varepsilon > 0$,

$$\omega_{\varepsilon,\xi_0}(x) := e^{ix\cdot\xi_0}\varepsilon^d\chi(\varepsilon x), \quad \widehat{\omega_{\varepsilon,\xi_0}}(\xi) = \widehat{\chi}(\varepsilon^{-1}(\xi - \xi_0))$$

Then

$$\sup_{\xi_0\in\mathbb{R}^d} \left\| f * \omega_{\varepsilon,\xi_0} - \hat{f}(\xi_0)\omega_{\varepsilon,\xi_0} \right\|_1 \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Indeed,

$$f * \omega_{\varepsilon,\xi_0}(x) - \hat{f}(\xi_0)\omega_{\varepsilon,\xi_0}(x)$$
$$= \int_{\mathbb{R}^d} f(y)e^{i(x-y)\cdot\xi_0}\varepsilon^d[\chi(\varepsilon(x-y)) - \chi(\varepsilon x)]\,dy$$

and send $\varepsilon \to 0$ by Lebesgue dominated convergence:

$$\left\| f * \omega_{\varepsilon,\xi_0}(x) - \hat{f}(\xi_0)\omega_{\varepsilon,\xi_0}(x) \right\|_{L^1_x}$$
$$= \int_{\mathbb{R}^d} |f(y)| \|\varepsilon^d[\chi(\varepsilon(x-y)) - \chi(\varepsilon x)]\|_{L^1_x}\,dy$$
$$= \int_{\mathbb{R}^d} |f(y)| \|\chi(\cdot - \varepsilon y) - \chi(\cdot)\|_{L^1_x}\,dy$$

Classical Wiener Theorem, proof continued

Hence, we may solve

$$\left[1 + z_0(\hat{f}(\xi) - \hat{f}(\xi_0))\right]\left[1 + \hat{g}(\xi)\right] = z_0, \quad |\xi - \xi_0| \ll 1$$

by a series because for ξ near ξ_0 we have

$$z_0(\hat{f}(\xi) - \hat{f}(\xi_0)) = z_0\,\mathcal{F}\left[f * \omega_{\varepsilon,\xi_0} - \hat{f}(\xi_0)\omega_{\varepsilon,\xi_0}\right]$$

and we can take $|\varepsilon| \ll 1$ so that

$$|z_0|\left\|f * \omega_{\varepsilon,\xi_0} - \hat{f}(\xi_0)\omega_{\varepsilon,\xi_0}\right\|_1 < \frac{1}{2}$$

Two main ingredients: (i) vanishing at infinity in L^1 sense. (ii) uniform L^1 modulus of continuity.
This is the proof scheme we use in the operator setting, i.e., when functions take values in certain spaces of operators. Dividing by complex numbers gets replaced by inversion of operators.

An operator-valued version

X Banach space, $\mathcal{W}_X$ algebra of bounded linear maps $T : X \to L^1(\mathbb{R}; X)$ with convolution

$$S * T(\rho)f = \int_{\mathbb{R}} S(\rho - \sigma)T(\sigma)f \, d\sigma$$

Adjoin unit, denote larger algebra $\widetilde{\mathcal{W}}_X$. Fourier transform satisfies

$$\sup_\lambda \| \hat{T}(\lambda)\|_{\mathcal{B}(X)} \leq \| T\|_{\mathcal{W}_X}$$

Theorem (Beceanu 2009, Beceanu-Goldberg 2010)

Suppose $T \in \mathcal{W}_X$ satisfies

1. $\lim_{\delta \to 0} \| T(\rho) - T(\rho - \delta)\|_{\mathcal{W}_X} = 0.$
2. $\lim_{R \to \infty} \| T\chi_{|\rho| \geq R}\|_{\mathcal{W}_X} = 0.$

If $I + \hat{T}(\lambda)$ invertible in $\mathcal{B}(X)$ for all λ, then $\mathbf{1} + T$ possesses an inverse in $\widetilde{\mathcal{W}}_X$ of the form $\mathbf{1} + S$.

Wiener algebra and resolvents

Set $R_0^-(\lambda^2)(x) = (4\pi|x|)^{-1}e^{-i\lambda|x|}$, $\widehat{T^-}(\lambda) = VR_0^-(\lambda^2)$. Then

$$T^-(\rho)f(x) = (4\pi\rho)^{-1}V(x)\int_{|x-y|=\rho} f(y)\,dy \qquad (3)$$

and thus

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |T^-(\rho)f(x)|\,dx\,d\rho &\le \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x)|}{|x-y|}|f(y)|\,dy\,dx \\ &\le \frac{1}{4\pi}\|V\|_{\mathcal{K}}\|f\|_1.\end{aligned}$$

where $\|V\|_{\mathcal{K}} = \||x|^{-1} * |V|\|_\infty$. Algebra is $\mathcal{W}_{L^1}$, pointwise invertibility condition on Fourier side:

$$(I + VR_0^-(\lambda^2))^{-1} \in \mathcal{B}(L^1)$$

Spectral theory/zero energy assumption. Beceanu-Goldberg thus prove dispersive estimates for Schrödinger in $\mathbb{R}^3$ for $\|V\|_{\mathcal{K}} < \infty$.

Small versus large Kato norm

Rodnianski-S. 2000 show that if

$$\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,dy < 4\pi$$

then for V real valued one has dispersive estimate

$$\|e^{itH}f\|_\infty \le C|t|^{-\frac{3}{2}}\|f\|_1, \quad H=-\Delta+V$$

Strategy: write evolution via functional calculus, density of spectral measure is imaginary part of the resolvent, expand resolvent into an infinite Born series, derive time decay from oscillatory integrals.

Open problem was then to do something similar for large V of this type. Beceanu-Goldberg did that (assuming zero energy regular) by means of the Wiener algebra on the previous slide.

Algebra for intertwining operators

The formulas for W_n suggest using three-variable kernels. Set

$$Z := \{ T(x_0, x_1, y) \in \mathcal{S}'(\mathbb{R}^9) \mid \mathcal{F}_y T(x_0, x_1, \eta) \in L^\infty_\eta L^\infty_{x_1} L^1_{x_0} \}$$
$$\|T\|_Z := \sup_{\eta \in \mathbb{R}^3} \|\mathcal{F}_y T(x_0, x_1, \eta)\|_{L^\infty_{x_1} L^1_{x_0}}$$

Operation $\circledast$ on $T_1, T_2 \in Z$

$$(T_1 \circledast T_2)(x_0, x_2, y) = \mathcal{F}^{-1}_\eta \Big[\int_{\mathbb{R}^3} \mathcal{F}_y T_1(x_0, x_1, \eta) \mathcal{F}_y T_2(x_1, x_2, \eta) \, dx_1 \Big](y)$$

Seminormed space $V^{-1}B$ defined as

$$V^{-1}B = \{ f \text{ measurable} \mid V(x) f(x) \in B^\sigma \}$$

with the seminorm $\|f\|_{V^{-1}B} := \|Vf\|_{B^\sigma}$. Set $X_{x,y} := L^1_y V^{-1} B_x$. Then $L^1_y L^\infty_x$ dense in $X_{x,y}$.

$X_{x,y}$ and Y spaces

Let Y be the space (algebra under $\circledast$) of three-variable kernels

$$Y := \Big\{ T(x_0, x_1, y) \in Z \mid \forall f \in L^\infty$$

$$(fT)(x_1, y) := \int_{\mathbb{R}^3} f(x_0)\, T(x_0, x_1, y)\, dx_0 \in X_{x_1,y} \Big\},$$

with norm

$$\|T\|_Y := \|T\|_Z + \|T\|_{\mathcal{B}(V^{-1}B_{x_0}, X_{x_1,y})}$$

For $\mathfrak{X} \in L^1_y L^\infty_x$, define *contraction* of $T \in Y$ by $\mathfrak{X}$ to be

$$(\mathfrak{X}T)(x, y) := \int_{\mathbb{R}^6} \mathfrak{X}(x_0, y_0)\, T(x_0, x, y - y_0)\, dx_0\, dy_0.$$

Then $\mathfrak{X}T \in X_{x,y}$, $\|\mathfrak{X}T\|_X \leq \|T\|_Y \|\mathfrak{X}\|_X$. This turns Y into an algebra.

Reason behind these structures: define

$$\mathcal{F}_y T^{\varepsilon}_{1+}(x_0, x_1, \eta) = e^{-ix_1\eta} R_0(|\eta|^2 - i\varepsilon)(x_0, x_1) V(x_0)\, e^{ix_0\eta}$$
$$T^{\varepsilon}_{2+} = T^{\varepsilon}_{1+} \circledast T^{\varepsilon}_{1+}, \;\; T^{\varepsilon}_{3+} = T^{\varepsilon}_{2+} \circledast T^{\varepsilon}_{1+} \;\; \text{etc.}$$

Then

$$\langle W^{\varepsilon}_{n+} f, g\rangle = \frac{(-1)^n}{(2\pi)^3} \int_{\mathbb{R}^6} \mathcal{F}^{-1}_{x_0} \mathcal{F}_{x_n,y}\, T^{\varepsilon}_{n+}(0, \xi_n, \eta) \widehat{f}(\eta) \overline{\widehat{g}}(\eta + \xi_n)\, d\eta\, d\xi_n$$
$$= (-1)^n \int_{\mathbb{R}^9} \mathcal{F}^{-1}_{x_0} T^{\varepsilon}_{n+}(0, x, y) f(x-y) \overline{g}(x)\, dy\, dx.$$

as well as

$$\langle W^{\varepsilon}_{+} f, g\rangle$$
$$= \langle f, g\rangle - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} \mathcal{F}^{-1}_{x_0} \mathcal{F}_{x_1,y}\, T^{\varepsilon}_{+}(0, \xi_1, \eta) \widehat{f}(\eta) \overline{\widehat{g}}(\eta + \xi_1)\, d\eta\, d\xi_n$$
$$= \langle f, g\rangle - \int_{\mathbb{R}^9} \mathcal{F}^{-1}_{x_0} T^{\varepsilon}_{+}(0, x, y) f(x-y) \overline{g}(x)\, dy\, dx.$$

Key invertibility problem

Here

$$\mathcal{F}_y T_{\pm}^{\varepsilon}(x_0, x_1, \eta) := e^{ix_0\eta}\big(R_V(|\eta|^2 \mp i\varepsilon)V\big)(x_0, x_1)e^{-ix_1\eta}$$

$T_{1+}^{\varepsilon}, T_{+}^{\varepsilon} \in Z$ and resolvent identity reads as follows:

$$(I + T_{1+}^{\varepsilon}) \circledast (I - T_{+}^{\varepsilon}) = (I - T_{+}^{\varepsilon}) \circledast (I + T_{1+}^{\varepsilon}) = I$$

We need to invert this in the smaller algebra Y, otherwise too little control of wave operators.

If $I + T_{1+}^{\varepsilon}$ is invertible in Y, hence in Z, its inverse is $I - T_{+}^{\varepsilon}$ both in Z and in Y, hence we obtain that $T_{+}^{\varepsilon} \in Y$ uniformly in $\varepsilon > 0$.

Define Y with $\sigma \geq \frac{1}{2}$ fixed. Then

$$\sup_{\varepsilon>0} \| T^{\varepsilon}_{1+} \|_Y \lesssim \|V\|_{B^{\frac{1}{2}+\sigma}} \quad \text{whence by induction}$$

$$\sup_{\varepsilon>0} \| T^{\varepsilon}_{n+} \|_Y \leq C^n \|V\|^n_{B^{\frac{1}{2}+\sigma}} \quad \text{for all} \quad n \geq 1$$

and

$$(W_{n+}f)(x) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g^{\varepsilon}_n(x, dy, \omega) f(S_\omega x - y)\, d\omega$$

where for fixed $x \in \mathbb{R}^3$, $\omega \in \mathbb{S}^2$ the expression $g^{\varepsilon}_n(x, \cdot, \omega)$ is a measure satisfying

$$\sup_{\varepsilon>0} \int_{\mathbb{S}^2} \|g^{\varepsilon}_n(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x}\, d\omega \leq C^n \|V\|^n_{B^{\frac{1}{2}+\sigma}}$$

Recursive definition of the structure functions

Identifying operator W^{ε}_{n+} with its kernel one has

$$\begin{aligned} W^{\varepsilon}_{n+} &= (-1)^n \mathbb{1}_{\mathbb{R}^3} T^{\varepsilon}_{n+} = (-1)^n \mathbb{1}_{\mathbb{R}^3}(T^{\varepsilon}_{(n-1)+} \circledast T^{\varepsilon}_{1+}) \\ &= -((-1)^{n-1} \mathbb{1}_{\mathbb{R}^3} T^{\varepsilon}_{(n-1)+}) T^{\varepsilon}_{1+} = -W^{\varepsilon}_{(n-1)+} T^{\varepsilon}_{1+} \end{aligned}$$

Second line: contraction of a kernel in Y by an element of X.
Thus

$$\sup_{\varepsilon>0} \| W^{\varepsilon}_{n+} \|_X \leq \| \mathbb{1}_{\mathbb{R}^3} \|_{V^{-1}B} \sup_{\varepsilon>0} \| T^{\varepsilon}_{n+} \|_Y \leq C^n \| V \|^{n+1}_{B^{\frac{1}{2}+\sigma}} \tag{4}$$

and with $f^{\varepsilon}_{y'}(x') = W^{\varepsilon}_{(n-1)+}(x', y')$ we have

$$g^{\varepsilon}_n(x, dy, \omega) := \int_{\mathbb{R}^3} g^{\varepsilon}_{1, f^{\varepsilon}_{y'}}(x, d(y - S_{\omega} y'), \omega)\, dy'$$

and $g^{\varepsilon}_{1, f^{\varepsilon}_{y'}}$ is the structure function for the potential $f^{\varepsilon}_{y'} V$.

Wiener theorem in Y

Proposition

$V \in B^\sigma$ with $\frac{1}{2} \le \sigma < 1$, define Y with this σ, V. Suppose $S \in Y$ satisfies, for some $N \ge 1$

$$\lim_{\varepsilon \to 0} \|\varepsilon^{-3}\chi(\cdot/\varepsilon) * S^N - S^N\|_Y = 0$$
$$\lim_{L \to \infty} \|(1 - \hat{\chi}(y/L))S(y)\|_Y = 0$$

Assume $I + \hat{S}(\eta)$ has inverse in $\mathcal{B}(L^\infty)$ of the form $(I + \hat{S}(\eta))^{-1} = I + U(\eta)$, with $U(\eta) \in \mathcal{F}Y$ for all $\eta \in \mathbb{R}^3$, and uniformly so, i.e.,

$$\sup_{\eta \in \mathbb{R}^3} \|U(\eta)\|_{\mathcal{F}Y} < \infty$$

Finally, suppose $\eta \mapsto \hat{S}(\eta)$ is uniformly continuous $\mathbb{R}^3 \to \mathcal{B}(L^\infty)$. Then $I + S$ is invertible in Y under $\circledast$.

A scaling invariant condition

Schwartz V, set $\|V\| := \|L_V\|_{L^1_{t,\omega}}$. Recall

$$L_V(t,\omega) = \int_0^\infty \widehat{V}(-\tau\omega) e^{\frac{i}{2}t\tau}\, \tau\, d\tau$$

For any Schwartz function v in $\mathbb{R}^3$

$$\|v\|_B := \sup_{\Pi} \int_{-\infty}^{\infty} \|\delta_{\Pi(t)}\, v(x)\|\, dt$$

where Π is a 2-dimensional plane through the origin, and $\Pi(t) = \Pi + t\vec{N}$, $\vec{N}$ being the unit norm to Π. Then

$$\|v\|_B \lesssim \sup_{\omega\in\mathbb{S}^2} \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|\psi(2^{-k}x')v(x'+s\omega)\|_{\dot{H}^{\frac{1}{2}}(\omega^\perp)}\, ds$$

This is finite on Schwartz functions.

A scaling invariant theorem for small potentials

Theorem (Beceanu-S. 17)

There exists $c_0 > 0$ *so that for any real-valued* V *with* $\|V\|_B + \|V\|_{\dot{B}^{\frac{1}{2}}} \leq c_0$, *there exists* $g(x, y, \omega) \in L^1_\omega \mathcal{M}_y L^\infty_x$ *with*

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x} \, d\omega \lesssim c_0$$

such that for any $f \in L^2$ *one has the representation formula*

$$(W_+ f)(x) = f(x) + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, dy, \omega) f(S_\omega x - y) \, d\omega.$$

No theorem for large scaling invariant potentials yet. Requires redoing all the spectral theory in this new norm.