Lévy Matrices

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Question

How are the eigenvectors of a random matrix distributed?

Eigenvectors of Random Matrices

- *Gaussian Orthogonal Ensemble (GOE)*: An $N \times N$ matrix $\mathbf{W}$; entries $w_{ij} = w_{ji}$ independent, centered Gaussians with $\mathrm{Var}\, h_{ij} = N^{-1}$ for $i \neq j$, $\mathrm{Var}\, h_{ii} = 2N^{-1}$
- Distribution of $\mathbf{W}$ is invariant under orthogonal transformations.
- Then, by rotational symmetry, its normalized eigenvectors are uniformly distributed on the unit sphere $\mathbb{S}^{N-1}$.

Eigenvectors of Random Matrices

- Let $\mathbf{u}$ be an eigenvector of the GOE.
- Denote eigenvector coordinates $\mathbf{u} = (\mathbf{u}(1), \mathbf{u}(2), \ldots, \mathbf{u}(N))$. We normalize by $\|\mathbf{u}\|_2 = 1$.
- The first coordinate of the uniform measure on the sphere $\mathbb{S}^{N-1}$ is asymptotically Gaussian as $N \to \infty$ after rescaling by $\sqrt{N}$.
- So, the rescaled eigenvector coordinates are asymptotically standard Gaussian: for any $m \in \mathbb{N}$, $\sqrt{N}\mathbf{u}(m) \to \mathcal{N}(0,1)$ in distribution.

Eigenvectors of Random Matrices

Question

What happens for other entry distributions?

- Same is true for symmetric random matrices with finite variance entries (Bourgade–Yau, 2017)

Question

What happens when the entry distributions do not have a variance?

Heavy-Tailed Random Matrices

- We consider symmetric power law distributions: $\mathbb{P}\left[|X| > t\right] \sim t^{-\alpha}$.
- We consider $\alpha \in (0, 2)$.
- Infinite variance.
- For $\alpha < 1$, infinite mean!

Heavy-Tailed Random Matrices

- We consider a particular class of power law random variables: *Lévy distributions*.
- Fix $\alpha \in (0,2)$. Let X be a centered α-stable law:

$$\mathbb{E}\left[e^{\mathrm{i}tX}\right] = \exp\left(-C|t|^{\alpha}\right),$$

  with $C = \pi^{1/\alpha}\left(2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)\right)^{-1/\alpha} > 0$.
- *Lévy Matrix*: An $N \times N$ matrix $\mathbf{H} = \{h_{ij}\}$, where $h_{ij} = h_{ji}$ are independent with $h_{ij} \sim N^{-1/\alpha}X$.
- Scaling chosen so spectrum is well behaved as $N \to \infty$.
- Motivated by applications to physics, finance, neural networks.

Eigenvectors of Heavy-Tailed Random Matrices

Theorem (Aggarwal–L.–Marcinek, 2020)

Let $\mathbf{u}$ *denote the eigenvector corresponding to the median eigenvalue. For any* $m \in \mathbb{N}$, $\sqrt{N}\mathbf{u}(m)$ *converges in moments to*

$$\sqrt{\frac{\pi}{\Gamma\left(1+\frac{2}{\alpha}\right)}} \times \frac{1}{\sqrt{S}} \times \mathcal{N},$$

where $\mathcal{N}$ *is a standard Gaussian and* S *is an independent, positive random variable with Laplace transform* $\mathbb{E}e^{-tS} = \exp\left(-t^{\alpha/2}\right)$.

Eigenvectors of Heavy-Tailed Random Matrices

- Eigenvector entries are non-Gaussian for small eigenvalues; one-parameter family of distributions, determined by the location of the corresponding eigenvalue.
- Nearby eigenvectors correlated for small eigenvalues, unlike GOE eigenvectors.
- For $\alpha < 1$, radically different behavior suspected for large eigenvalues. (Bordenave–Guionnet, 2013: proved for $\alpha < 2/3$).

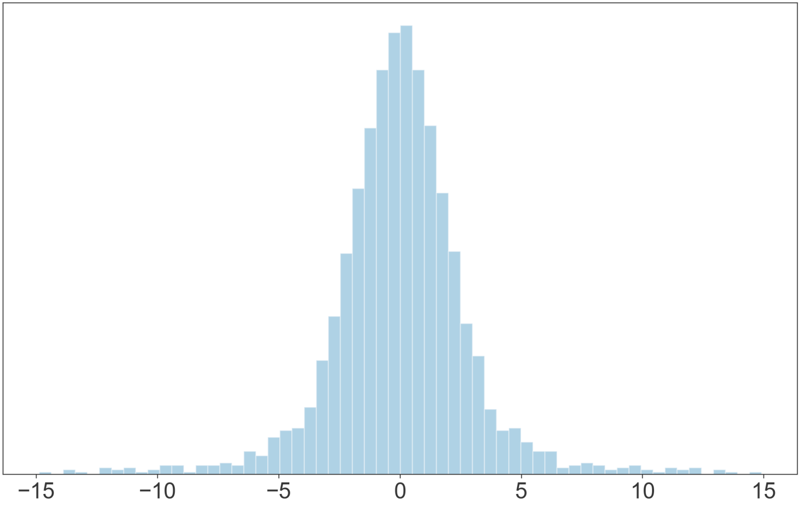
Lévy Matrices

Question

How are the eigenvalues distributed?

- Normalization $N^{-1/\alpha}$ ensures that most eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are of order one.
- Global spectral distribution converges to a heavy-tailed deterministic measure $\rho_\alpha(x)$ (Ben Arous–Guionnet, 2008).

Lévy Matrices



Basic Notions

Definition: An eigenvector $\mathbf{v} = (v_1, v_2, \ldots, v_N)$ is of $\mathbf{H}$ with $\|\mathbf{v}\|_2 = 1$ is *completely delocalized* if $\max_{i \le N} |v_i| < N^{\varepsilon - 1/2}$ for large N.

This corresponds to eigenvector mass being "spread out." True for GOE.

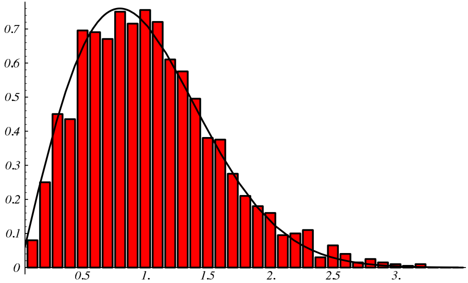
Basic Notions

Definition: *Local eigenvalue statistics* are statistics of a finite number of eigenvalues, for example rescaled gaps $N(\lambda_i - \lambda_{i+1})$.

GOE eigenvalues are highly correlated and appear to repel each other.

Uncorrelated eigenvalues are said to display "Poisson statistics."

Figure: Distribution of a GOE eigenvalue gap in the bulk of the spectrum.



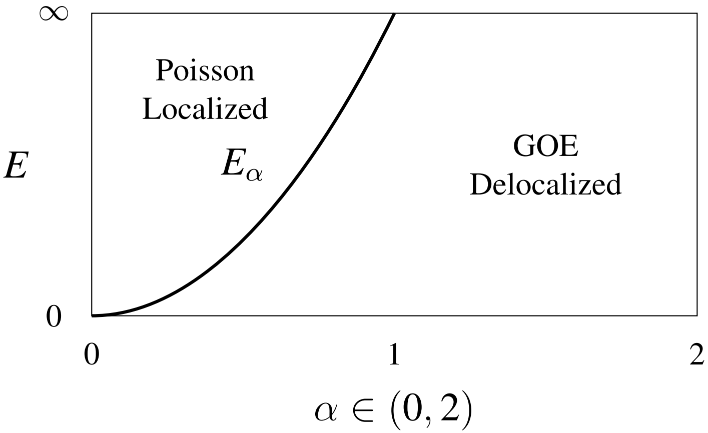
Predictions (Non-rigorous)

Tarquini–Biroli–Tarzia (2016):

1. If $\alpha \in [1, 2)$, GOE local statistics and complete eigenvector delocalization.
2. If $\alpha \in (0, 1)$, then there exists a *mobility edge* $E_\alpha > 0$:
   - If $|\lambda| < E_\alpha$, GOE local statistics and complete eigenvector delocalization,
   - If $|\lambda| > E_\alpha$, Poisson local statistics and complete eigenvector localization,
   - Explicit formula for E_α.

Earlier predictions of Cizeau–Bouchaud (1994) were slightly different. For example, a "mixed phase" for $\alpha \in (1, 2)$.

Predictions (Tarquini–Biroli–Tarzia, Non-rigorous)



Mobility Edge

Mobility edge E_α predicted by the following equation.

$$K_\alpha^2(s_\alpha^2 - s_{1/2}^2)|l(E_\alpha)|^2 - 2s_\alpha K_\alpha[\mathrm{Re}\ l(E_\alpha)] + 1 = 0, \quad s_\alpha = \sin(\pi\alpha/2)$$

$$K_\alpha = \frac{\alpha}{2}\Gamma(1/2 - \alpha/2)^2, \quad l(E) = \frac{1}{\pi}\int_0^\infty k^{\alpha-1}\left[\widehat{L}_{\alpha/2}^{C(E),\beta(E)}(k)\right] e^{ikE}\, dk$$

- $C(E)$ and $\beta(E)$ are parameters that may be determined explicitly from a self-consistent equation.
- $L_{\alpha/2}^{C,\beta}$ is the probability distribution of a general stable law with skewness parameter β and scale parameter C.
- Solution only for $\alpha \in (0, 1)$, diverges like $(1-\alpha)^{-1}$ as $\alpha \to 1$.

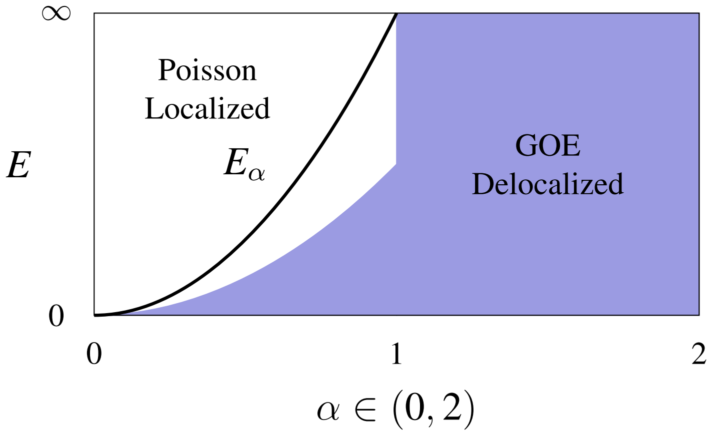
Rigorous Results

Theorem (Aggarwal–L.–Yau, 2018)

1. *If* $\alpha \in (1, 2)$ *then all eigenvectors of* $\mathbf{H}$ *are completely delocalized, and local statistics are GOE.*
2. *For almost all* $\alpha \in (0, 2)$, *there exists a* $c_\alpha > 0$ *such that any eigenvector of* $\mathbf{H}$ *with eigenvalue* $\lambda \in (-c_\alpha, c_\alpha)$ *is completely delocalized, and local statistics around any* $E \in (-c_\alpha, c_\alpha)$ *are GOE*

- Part 1 shows there is no mobility edge for $\alpha \in (1, 2)$, validating predictions of Tarquini–Biroli–Tarzia
- Part 2 establishes the existence of a GOE/delocalization regime at small energies when $\alpha < 1$.
- Builds on investigations of Bordenave–Guionnet (2013, 2017) of the delocalized phase.

Rigorous Results



Purple shading indicates the scope of our results.

Future Work

- Understand the localized phase.
- Understand the mobility edge.