Integral virtual fundamental chains

via finer virtual structures on moduli spaces

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3. (i) $H : M \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ give vector field X_H by $dH =: \omega(X_H, \cdot)$.
   (ii) Morse theory of $A_H([x, \hat{x}]) := -\int_D \hat{x}^*\omega - \int_0^1 H(x(t), t)dt$ gives Floer homology.
   (iii) Differential counts perturbed J-cylinders, or (J, H)-cylinders $(du - X_H \otimes dt)^{0,1} = 0$.

Global invariants via moduli spaces, geom. transversality

Global invariants, GW invariant & Floer homology, are defined:

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(5) Divide domain-reparam sym (topology/cptness). X orbifold-based.

Outside of special cases of semi-positivity, what can we do?

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As of Oct 5, 2017, the output from the above is a virtual fund. chain/cycle over $\mathbb{Q}$, which is a regular replacement of bad compact zero set $\mathbf{f}^{-1}(0)$ or $\bigsqcup_I s_I^{-1}(0)/\Upsilon$. Υ is the patching identification.
Because the symmetry and transversality don't get along $\Rightarrow$
Need symmetric $\mathbb{Q}$-weighted branchwise $\pitchfork$ multi-sections, or equiv.

Integral virtual fundamental chains (joint w/ Guangbo Xu)

Example: 2-sphere S w/ an orbifold point $\{z\} := S^{\mathbb{Z}_3}$ of symm. $\mathbb{Z}_3$.

1. It fits as a special case of the previous polyfold/Kuranishi theories, and it is already regular and no room in the 0-bundle to do much else (like perturbation). Euler char. rational.
2. But we know it has no hole, whose information can be captured, and indeed, $S \backslash \{z\}$ is a pseudocyle, and we count 2 for this "virtual" (pseudo-)cycle.
3. The subspace of points with nontrivial stabilizers being of codimension 2 (in the base) is not enough: We still need to regularize, and face imcompatibility of symm. & transversality.
4. Another aspect is that the normal bundle $N_{S^{\mathbb{Z}_3}} S$ has a complex structure in this example, which generalizes.

A good coordinate system GCS (of 2 charts)

Any Kuranishi-type theory, you can get a finitely many charts covering moduli X, a good coordinate system. Consider 2 charts (4 generalizes).
$C_I := (s_I : U_I \to E_I, \underline{\psi_I} : s_I^{-1}(0)/\Gamma_I \to X; \Gamma_I\}$ and one for J.

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(2) (Enough) open part $C_I|_{U_{JI}}$ of C_I embeds in C_J, intertwining all the data. So $s_I|_{U_{JI}}$ sits in as part of s_J. Images denoted by $\check{\cdot}$.

(2) U_J can have larger dimension. The extra direction is 'cancelled out' by ds_J matching normals of $\check{U}_{JI}$ in U_J with fiber normals of $\check{E}_I$ in E_J", a canonical condition at zeros $s_J|_{\check{U}_{JI}}$.

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One way to go up dimension is by improving (2) into condition over W_{JI}, a tubular neighborhood projecting onto $\check{U}_{JI}$ (remembering the fiber linear structure, containing no extra zeros), and extend $\check{E}_J$ to $\tilde{E}_{JI}$ over W_{JI}, and asking $s_J|_{W_{JI}}/\tilde{E}_{JI} \pitchfork 0$. A $\pitchfork$ perturbation of s_I is immediately lifted to a $\pitchfork$ peturbation over W_{JI}. After things are in the same setting in E_J, one can use relative tranversality.

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Theorem (Y., Feb 2014)

Can do it globally on GCS via $\exists$ (! up to refinem't) of level-1 str.

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* *"Floer chain"- and GW-moduli spaces for general symplectic mflds have group-normal complex str. $\mathcal{J}$, ! up to quasi-iso/cob.*
* *Using the stabilizer-free moduli spaces $\mathring{\tilde{X}}^{[\{Id\}]}$ with $L = \{Id\}$, the $\mathbb{Z}$-VFC, Floer homology and GW are well-defined over $\mathbb{Z}$.*

Some further directions

Other part worth exploring:

1. How to make use of $\mathring{\tilde{M}}^{[L]}$ for other groups $[L]$ with $L \neq \{Id\}$. Maybe use pseudocycle stratification to relate to homotopy quotient defined using universal family-dependent J in certain situations.
2. Since multiple branch-covers can be stabilizer-free, so integral GW (defined for all, not just for CY3) is not GV. What is the geometric meaning on the curve counting side?
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5. Use forgetful functor (Y.), one gets a polyfold version, but one should be able to directly with some simplification (soon).
6. Find alternative condition (to group-normal complex) applicable to other moduli spaces of geometric PDE.

Thank you!