Homotopy Theory at the Second Chromatic Level

Irina Bobkova

Institute for Advanced Study

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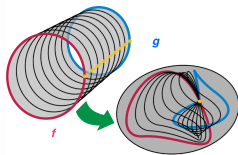
Homotopy Groups

Definition

- The n-th homotopy group of X is

$$\pi_n X := Map(S^n, X)/f \sim g.$$

- f and g are homotopic, $f \sim g$, if there is a continuous deformation from f to g:

- For $f, g \in \pi_n X$ we can define $f + g \in \pi_n X$.

Homotopy Groups of Spheres

The most basic case: $X = S^m$.

Question

How do we compute the groups

$$\pi_n S^m = Map(S^n, S^m)/\sim?$$

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Answer

It is very hard.

Homotopy Groups of Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2{}^2$	$\mathbb{Z}_{12}{\times}\mathbb{Z}_2$	$\mathbb{Z}_{84}{\times}\mathbb{Z}_2{}^2$	$\mathbb{Z}_2{}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2{}^2$	$\mathbb{Z}_{12}{\times}\mathbb{Z}_2$	$\mathbb{Z}_{84}{\times}\mathbb{Z}_2{}^2$	$\mathbb{Z}_2{}^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}{\times}\mathbb{Z}_{12}$	$\mathbb{Z}_2{}^2$	$\mathbb{Z}_2{}^2$	$\mathbb{Z}_{24}{\times}\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2{}^3$	$\mathbb{Z}_{120}{\times}\mathbb{Z}_{12}{\times}\mathbb{Z}_2$	$\mathbb{Z}_{84}{\times}\mathbb{Z}_2{}^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2{}^3$	$\mathbb{Z}_{72}{\times}\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}{\times}\mathbb{Z}_2$	$\mathbb{Z}_2{}^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2{}^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}{\times}\mathbb{Z}_{120}$

Homotopy Groups of Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2{}^2$	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2{}^2$	$Z_2{}^2$
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2{}^2$	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2{}^2$	$Z_2{}^2$
S^4	0	0	0	Z	Z_2	Z_2	$Z{\times}Z_{12}$	$Z_2{}^2$	$Z_2{}^2$	$Z_{24}{\times}Z_3$	Z_{15}	Z_2	$Z_2{}^3$	$Z_{120}{\times}Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2{}^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	$Z_2{}^3$	$Z_{72}{\times}Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}{\times}Z_2$	$Z_2{}^3$
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	$Z_2{}^3$
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z{\times}Z_{120}$

- $\pi_n S^m = Map(S^n, S^m) = 0$ for $n < m$.

Homotopy Groups of Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

- $\pi_n S^m = Map(S^n, S^m) = 0$ for $n < m$.
- Groups stabilize along the diagonals:
  $\pi_{i+n} S^n = Map(S^{i+n}, S^n)$ does not depend on n for large n.

Homotopy Groups of Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12} \times Z_2$	$Z_{84} \times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12} \times Z_2$	$Z_{84} \times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120} \times Z_{12} \times Z_2$	$Z_{84} \times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72} \times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24} \times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z \times Z_{120}$

- $\pi_n S^m = Map(S^n, S^m) = 0$ for $n < m$.
- Groups stabilize along the diagonals:
  $\pi_{i+n} S^n = Map(S^{i+n}, S^n)$ does not depend on n for large n.
- Stable homotopy groups: $\pi_i^s = \lim_n \pi_{i+n} S^n$

- The sphere spectrum $\mathcal{S}$ captures the stable homotopy groups of spheres

$$\pi_i \mathcal{S} = \pi_i^s = \lim_n \pi_{i+n} S^n$$

- $\pi_i \mathcal{S}$ are very hard to compute.

- After over 80 years, $\pi_i \mathcal{S}$ is known only up to $i \approx 80$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2{}^2$	$Z_{12} \times Z_2$	$Z_{84} \times Z_2{}^2$	$Z_2{}^2$
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2{}^2$	$Z_{12} \times Z_2$	$Z_{84} \times Z_2{}^2$	$Z_2{}^2$
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2{}^2$	$Z_2{}^2$	$Z_{24} \times Z_3$	Z_{15}	Z_2	$Z_2{}^3$	$Z_{120} \times Z_{12} \times Z_2$	$Z_{84} \times Z_2{}^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	$Z_2{}^3$	$Z_{72} \times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24} \times Z_2$	$Z_2{}^3$
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S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z \times Z_{120}$

Stable Homotopy Groups

i	0	1	2	3	4	5	6	7	8	9	10	11	12
π_i^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{504}$	0

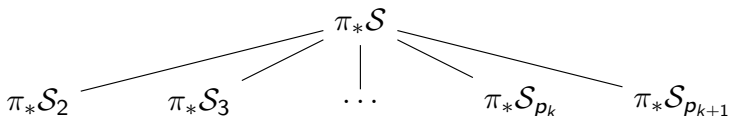| 13	14	15	16	17	18	19
$\mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{480} \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_{264} \times \mathbb{Z}_2$

One prime at a time

Theorem (Serre)

$$\pi_n \mathcal{S} = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ \text{Finite abelian group}, & \text{if } n > 0. \end{cases}$$
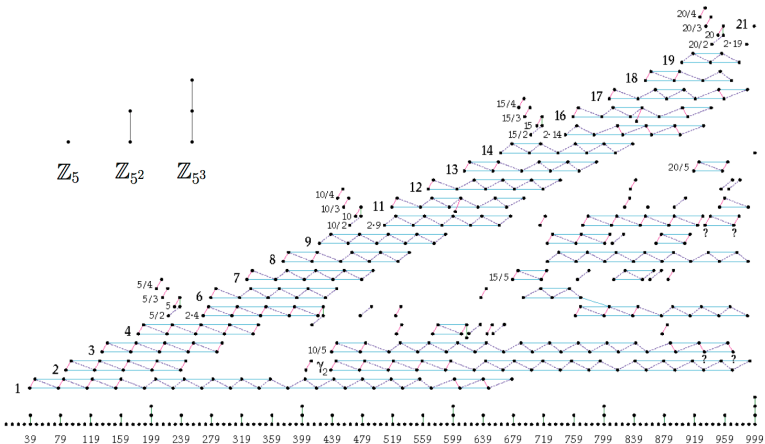
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$\pi_* \mathcal{S}_p$ represents the p-primary part of $\pi_* \mathcal{S}$.

Computation by Ravenel
Illustration by Hatcher
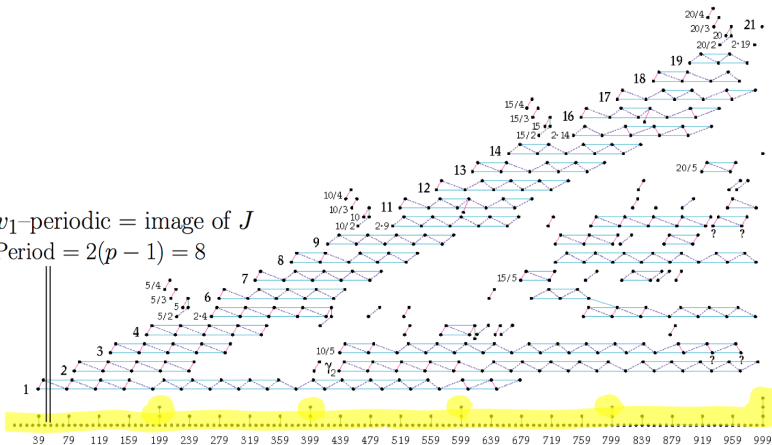Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$

First chromatic level

Computation by Ravenel
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$\pi_* S_{(5)}$

$p = 5$

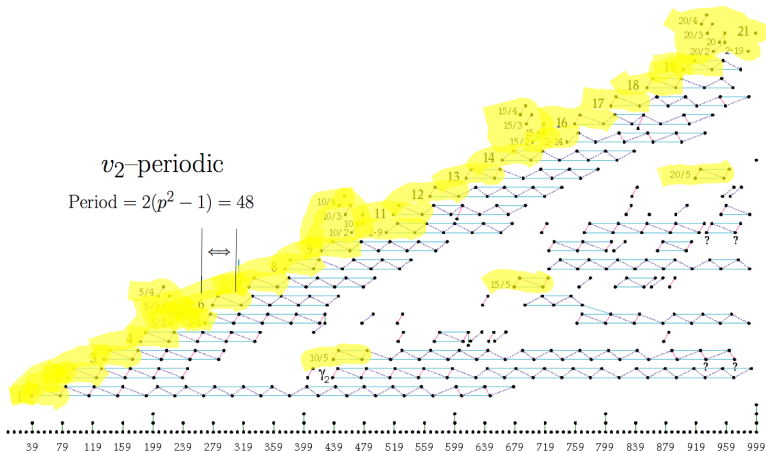v_1–periodic = image of J
Period = $2(p-1) = 8$

Second chromatic level

Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

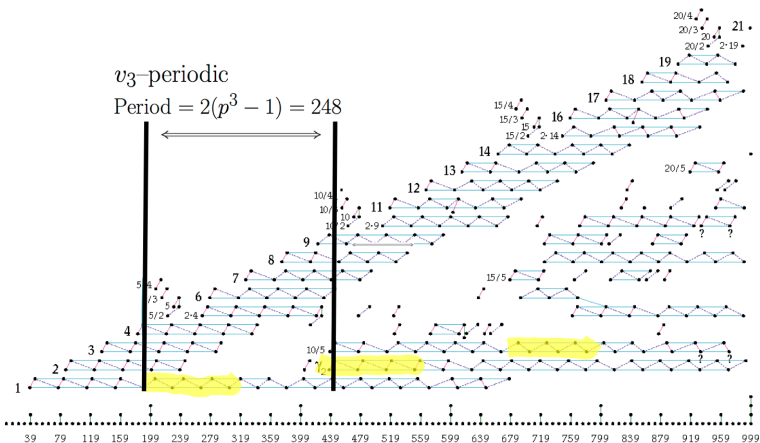$p = 5$

v_2–periodic

Period $= 2(p^2 - 1) = 48$

Third chromatic level

Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$

v_3–periodic
Period $= 2(p^3 - 1) = 248$

Chromatic Filtration

$$\pi_* \mathcal{S}$$

$\pi_* \mathcal{S}_2 \quad \pi_* \mathcal{S}_3 \quad \ldots \quad \pi_* \mathcal{S}_{p_k} \quad \pi_* \mathcal{S}_{p_{k+1}}$

Chromatic Filtration

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$\pi_* \mathcal{S}_2 \quad \pi_* \mathcal{S}_3 \quad \cdots \quad \pi_* \mathcal{S}_{p_k} \quad \pi_* \mathcal{S}_{p_{k+1}}$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$\pi_* \mathcal{S}_{2,n} \quad \pi_* \mathcal{S}_{3,n} \quad \cdots \quad \pi_* \mathcal{S}_{p_k,n} \quad \pi_* \mathcal{S}_{p_{k+1},n}$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$\pi_* \mathcal{S}_{2,2} \quad \pi_* \mathcal{S}_{3,2} \quad \cdots \quad \pi_* \mathcal{S}_{p_k,2} \quad \pi_* \mathcal{S}_{p_{k+1},2}$

$\pi_* \mathcal{S}_{2,1} \quad \pi_* \mathcal{S}_{3,1} \quad \cdots \quad \pi_* \mathcal{S}_{p_k,1} \quad \pi_* \mathcal{S}_{p_{k+1},1}$

$n = 1$
Adams, Mahowald, Miller, . . .

Chromatic Filtration

$\pi_* \mathcal{S}$

$\pi_* \mathcal{S}_2$ $\pi_* \mathcal{S}_3$ $\cdots$ $\pi_* \mathcal{S}_{p_k}$ $\pi_* \mathcal{S}_{p_{k+1}}$

$\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$
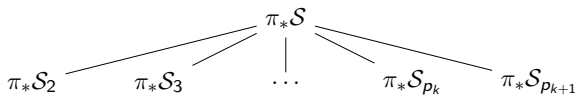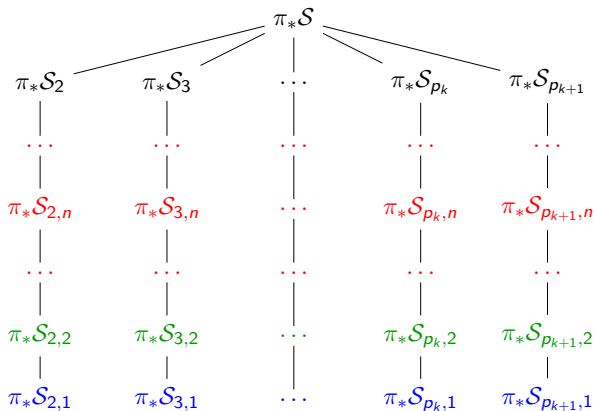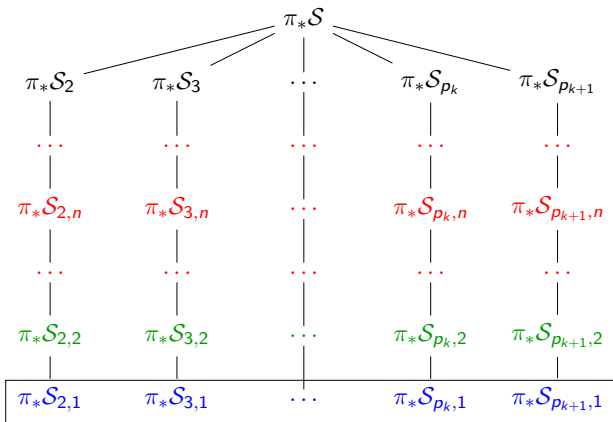
$\pi_* \mathcal{S}_{2,n}$ $\pi_* \mathcal{S}_{3,n}$ $\cdots$ $\pi_* \mathcal{S}_{p_k,n}$ $\pi_* \mathcal{S}_{p_{k+1},n}$
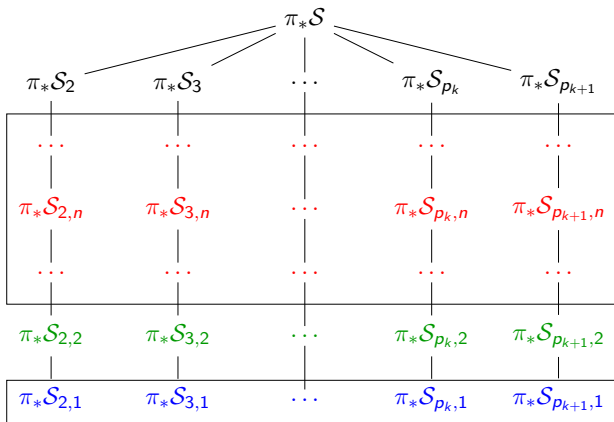
$n \geq 3$
???
Hopkins, Behrens, Lawson . . .

$\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$

$\pi_* \mathcal{S}_{2,2}$ $\pi_* \mathcal{S}_{3,2}$ $\cdots$ $\pi_* \mathcal{S}_{p_k,2}$ $\pi_* \mathcal{S}_{p_{k+1},2}$

$\pi_* \mathcal{S}_{2,1}$ $\pi_* \mathcal{S}_{3,1}$ $\cdots$ $\pi_* \mathcal{S}_{p_k,1}$ $\pi_* \mathcal{S}_{p_{k+1},1}$

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Chromatic Filtration

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Theorem (B.-Goerss)

There exists a resolution of spectra

$$\bar{\mathcal{S}}_{2,2} \to \mathcal{E}_1 \to \mathcal{E}_2 \to D\mathcal{E}_2 \to D\mathcal{E}_1$$

where the terms are well understood.

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1. Breaks $\bar{\mathcal{S}}_{2,2}$ into easier pieces.
2. Useful for applications.
3. Self-duality?

Thank you!