Twisted integral orbit parametrizations

Aaron Pollack

Institute for Advanced Study

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Disclaimer

All statements are to be understood as essentially true (i.e., true once the statement is modified slightly).

Arithmetic invariant theory

This talk is about **arithmetic invariant theory**.

Suppose:

- G is a linear algebraic group (e.g., GL_2, GL_6)
- V is a finite-dimensional rational representation of G (e.g. $\mathrm{Sym}^2(V_2)$, $\wedge^3 V_6$)

Definition

Arithmetic invariant theory is the study of

- Parametrizing the orbits $G(\mathbf{Q})$ on $V(\mathbf{Q})$ or
- Parametrizing the orbits $G(\mathbf{Z})$ on $V(\mathbf{Z})$ if G, V have structures over the integers.

Frequently, the orbits tend to be parametrized by interesting arithmetic objects.

Binary quadratic forms

Example

$G = \mathrm{GL}_2,\ V = \mathrm{Sym}^2(V_2) \otimes \det^{-1}.$

$\mathrm{GL}_2(\mathbf{Q})$ acts on $\mathrm{Sym}^2(\mathbf{Q}^2) = 2 \times 2$ symmetric matrices:

$$g \cdot \left(\begin{array}{cc} a & \frac{b}{2} \\ \frac{b}{2} & c \end{array}\right) = \det(g)^{-1} g \left(\begin{array}{cc} a & \frac{b}{2} \\ \frac{b}{2} & c \end{array}\right) g^t.$$

Make integral:

- $G(\mathbf{Z}) = \mathrm{GL}_2(\mathbf{Z}) \subseteq \mathrm{GL}_2(\mathbf{Q})$
- $V_{\mathbf{Z}} = \left\{\left(\begin{array}{cc} a & \frac{b}{2} \\ \frac{b}{2} & c \end{array}\right) : a, b, c \in \mathbf{Z}\right\}$, lattice inside $\mathrm{Sym}^2(\mathbf{R}^2)$
- Equivalently: $V_{\mathbf{Z}} = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbf{Z}\}$

$$f((x,y)) = ax^2 + bxy + cy^2 \mapsto (g \cdot f)((x,y)) = \det(g)^{-1} f((x,y)g).$$

Orbits on binary quadratic forms over $\mathbf{Q}$

If $f(x,y) = ax^2 + bxy + cy^2$, then

$$\mathrm{Disc}(\mathrm{f}) := b^2 - 4ac = -4\det\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

is an invariant of the orbit: I.e.,

$$\mathrm{Disc}(f) = \mathrm{Disc}(g \cdot f) \text{ for } g \in \mathrm{GL}_2\,.$$

Orbits over Q (with $\mathrm{Disc}(s) \neq 0$):

- If $s_1, s_2 \in \mathrm{Sym}^2(\mathbf{Q}^2)$ and $\mathrm{Disc}(s_1) = \mathrm{Disc}(s_2) \neq 0$, then $\exists g \in \mathrm{GL}_2(\mathbf{Q})$ with $\det(g)^{-1} g s_1 {}^t g = s_2$.
- Orbits parametrized by Disc, or quadratic étale algebras over $\mathbf{Q}$, $\mathrm{GL}_2(\mathbf{Q})s \mapsto \mathbf{Q}(\sqrt{\mathrm{Disc}(s)})$.

Orbits on binary quadratic forms over $\mathbf{Z}$

For $D \in \mathbf{Z}$, set

$$V_{\mathbf{Z}}^{D} = \{f(x,y) : \mathrm{Disc}(f) = D\} = \left\{ s = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} : -4\det(s) = D \right\}.$$

D must be a square modulo 4 for this set to be nonempty; i.e., $D \equiv 0, 1$ modulo 4.

Orbits over $\mathbf{Z}$:

1. quadratic rings S_D over $\mathbf{Z}$, $\mathrm{GL}_2(\mathbf{Z})s \mapsto S_D := \mathbf{Z}[x]/(x^2 - Dx + \frac{D^2-D}{4})$, $D = \mathrm{Disc}(s)$;
2. plus elements in $\mathrm{Cl}(S_D)$: explicit module for S_D.

Bhargava's HCL

- There is no *a priori* reason to believe parametrizing orbits $G(\mathbf{Z})$ on $V(\mathbf{Z})$ would yield interesting results
- However, Bhargava found many examples of such interesting parametrizations

Example (Bhargava)

1. $\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$ acting on $\mathbf{Z}^2 \otimes \mathrm{Sym}^2(\mathbf{Z}^3)$
2. $\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$ acting on $\mathbf{Z}^2 \otimes \mathbf{Z}^3 \otimes \mathbf{Z}^3$
3. $\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_6(\mathbf{Z})$ acting on $\mathbf{Z}^2 \otimes \wedge^2(\mathbf{Z}^6)$.

- Orbits parametrize **cubic** rings T over $\mathbf{Z}$ (e.g. $\mathbf{Z}[7^{1/3}]$), plus finer data
- For example, in case 2, orbits parametrize (T, I_1, I_2) with $I_1, I_2 \in \mathrm{Cl}(T)$, $I_1 I_2 = 1$.

Twisted orbit problems

Work in arithmetic invariant theory over $\mathbf{Z}$, i.e., parametrizations $G(\mathbf{Z})$ on $V(\mathbf{Z})$, has occurred when the group G is *split*, e.g.

Split

- $G(\mathbf{Z}) = \mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z})$
- $V(\mathbf{Z}) = \mathbf{Z}^2 \otimes \mathbf{Z}^3 \otimes \mathbf{Z}^3 = \mathbf{Z}^2 \otimes M_3(\mathbf{Z})$.

Problem

What about "twisted", or non-split cases?

Twisted

- Let S be a quadratic ring over $\mathbf{Z}$,

$$\mathrm{SL}_3(\mathbf{Z}) \times \mathrm{SL}_3(\mathbf{Z}) = \mathrm{SL}_3(\mathbf{Z} \times \mathbf{Z}) \rightsquigarrow \mathrm{SL}_3(S).$$

- $\sigma : S \to S$, $\sigma(a + b\sqrt{D}) = a - b\sqrt{D}$, $a, b \in \mathbf{Z}$,

$$M_3(\mathbf{Z}) \rightsquigarrow H_3(S) = \{h \in M_3(S) : h = {}^t h^\sigma\}$$

- $\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(S)$ acts on $\mathbf{Z}^2 \otimes H_3(S)$

Sample theorem

Theorem (P.)

The orbits of $\mathrm{GL}_2(\mathbf{Z}) \times \mathrm{SL}_3(S)$ *on* $\mathbf{Z}^2 \otimes H_3(S)$ *parametrize triples* (T, I, β) *up to equivalence where*

- T *is a cubic ring*
- I *is a* $T \otimes_{\mathbf{Z}} S$*-fractional ideal*
- $\beta \in (T \otimes_{\mathbf{Z}} \mathbf{Q})^{\times}$
- *the norm of* I *in* $T \otimes \mathbf{Q}$ *is principal, generated by* β

Key idea:

- Relate orbits $\mathcal{O}$ in $\mathbf{Z}^2 \otimes H_3(S)$ that are "big" (technically: in the open orbit for $(\mathrm{GL}_2 \times \mathrm{GL}_3(S))(\mathbf{Q})$)
- To orbits $\widetilde{\mathcal{O}}$ in $H_3(S) \otimes T$ that are "small" (technically: for which the stabilizer of the line spanned by an element in the orbit is a parabolic subgroup of $\mathrm{GL}_3(T \otimes S)(\mathbf{Q})$.)
- The "lifted" orbits $\widetilde{\mathcal{O}}$ are easier to understand from the point of view of AIT, which is why this helps you

Example

Using lifted orbits $\widetilde{\mathcal{O}}$ for binary quadratic forms

Recall: If $s_1, s_2 \in \mathrm{Sym}^2(\mathbf{Q}^2)$, and $\mathrm{Disc}(s_1) = \mathrm{Disc}(s_2) \neq 0$, then $\exists g \in \mathrm{GL}_2(\mathbf{Q})$ with $\det(g)^{-1} g s_1 {}^t g = s_2$.

Proof: Suppose

- $s = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$,
- $D = \mathrm{Disc}(s) = -4\det(s) = b^2 - 4ac$.

Set

$$\widetilde{s} = s + \frac{\sqrt{D}}{2}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} a & \frac{b+\sqrt{D}}{2} \\ \frac{b-\sqrt{D}}{2} & c \end{pmatrix}.$$

Then

- $\det(\widetilde{s}) = 0$, so $\widetilde{s}$ is a *rank one* matrix in $M_2(\mathbf{Q}(\sqrt{D}))$
- **Eigenvector property**: $s\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\widetilde{s} = \frac{\sqrt{D}}{2}\widetilde{s}$.

Proof, continued

Proof sketch

1. Suppose $s_1, s_2 \in \mathrm{Sym}^2(\mathbf{Q}^2)$ with $\mathrm{Disc}(s_1) = \mathrm{Disc}(s_2)$.
2. Easy: Since $\widetilde{s_1}$ and $\widetilde{s_2}$ are rank one, there is $g \in \mathrm{GL}_2(\mathbf{Q}\sqrt{D})$ with $g\widetilde{s_1}\,{}^t g^\sigma = \widetilde{s_2}$.
3. $g = g_1 + \frac{\sqrt{D}}{2} g_2$, some $g_1, g_2 \in M_2(\mathbf{Q})$.
4. Set $h = g_1 + g_2 s \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in M_2(\mathbf{Q})$.
5. Immediate from Eigenvector property: $h\widetilde{s_1}\,{}^t h = \widetilde{s_2}$.
6. Thus $hs_1\,{}^t h = s_2$ and $\det(h) = 1$.

Thank you

Thank you for your attention!

From the point of view of automorphic forms...

Question

From the point of view of automorphic forms, why **might** you be interested in arithmetic invariant theory?

Most basic answer: Help you understand the Fourier expansions of automorphic forms.

For example, suppose

$$f(Z) = \sum_{T \in \mathrm{Sym}^2(\mathbf{Z}^3)^\vee} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

is a Siegel modular form of degree three (i.e., for GSp_6).

Then if $m \in \mathrm{SL}_3(\mathbf{Z})$, $a_f(T) = a_f(m \cdot T)$.

Orbits

$$\mathrm{GL}_3(\mathbf{Z}) \backslash \mathrm{Sym}^2(\mathbf{Z}^3)^\vee \leftrightarrow \textbf{Orders in quaternion algebras}$$

Fourier coefficients on GSp_6

If $\mathcal{O}$ is a quaternion order, and $T \in \mathrm{Sym}^2(\mathbf{Z}^3)^\vee$, $T \leftrightarrow \mathcal{O}$, define $a_f(\mathcal{O}) := a_f(T)$.

Theorem (Evdokimov + ϵ)

Suppose $f(Z)$ as above, weight k level one Hecke Eigenform. Suppose furthermore that B is a quaternion algebra, ramified at infinity, and B_0 is a maximal order in B. Then

$$\sum_{n \geq 1, \mathcal{O} \subseteq B_0} \frac{a_f(\mathbf{Z} + n\mathcal{O})}{n^s [B_0 : \mathcal{O}]^{s-k+3}} = a(B_0) \frac{L(\pi, Spin, s)}{\zeta(2s-3k+6)\zeta^{D_B}(2s-3k+8)}.$$

Here $\zeta^{D_B}(s)$ is the Riemann zeta function, with the Euler factors at primes dividing the discriminant of B removed, and the sum is over all positive integers n and quaternion orders $\mathcal{O}$ contained in B_0.