Decomposition theorem for semisimple algebraic holonomic D-modules

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Introduction

$\mathscr{D}$-modules

Let X be a complex manifold. Let $\mathscr{D}_X$ denote the sheaf of holomorphic differential operators on X. It is a sheaf of non-commutative algebras. In this talk, a $\mathscr{D}_X$-module is a left module over $\mathscr{D}_X$.

Example

- $\mathscr{D}_X$, more generally $\mathscr{D}_X/\mathscr{I}$ for left ideal $\mathscr{I}$.
- Holomorphic vector bundle with integrable connection (E, ∇). (It is called flat bundle in this talk.) In particular, $(\mathscr{O}_X, d)$.

$\mathscr{D}$-module is system of linear partial differential equations. For instance,

$$\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{D}_X/\mathscr{D}_X P, \mathscr{O}_X\right) = \left\{f \in \mathscr{O}_X \,\middle|\, Pf = 0\right\}, \quad (P \in \mathscr{D}_X).$$

Holonomic $\mathscr{D}$-modules

Holonomic $\mathscr{D}$-modules are appropriate generalization of ordinary differential equations to the several variable case.

$\mathscr{D}_X$ **has a natural filtration** $F_j\mathscr{D}_X = \left\{ P \in \mathscr{D}_X \,\middle|\, \mathrm{ord}(P) \leq j \right\}$**.**
Here, $\mathrm{ord}(P) = \left\{ \sum j_k \,\middle|\, a_{j_1,\dots,j_n} \neq 0 \right\}$ **when** $P = \sum a_{j_1,\dots,j_n} \prod_{k=1}^n \partial_k^{j_k}$**.**

$$F_j\mathscr{D}_X \cdot F_m\mathscr{D}_X \subset F_{j+m}\mathscr{D}_X, \quad \mathrm{Gr}^F \mathscr{D}_X = \bigoplus_{j=0}^{\infty} F_j\mathscr{D}_X / F_{j-1}\mathscr{D}_X \simeq \mathrm{Sym}^\bullet \Theta_X.$$

A coherent $\mathscr{D}_X$-module M locally has a filtration $F_\bullet M$ by coherent $\mathscr{O}_X$-submodules.

- $F_j\mathscr{D}_X \cdot F_k M \subset F_{j+k}M$.
- $F_j M = 0 \ (j << 0)$, $\bigcup_{j\in\mathbb{Z}} F_j M = M$.
- $\bigoplus F_j M$ **is finitely generated over** $\bigoplus F_j\mathscr{D}_X$.

$\mathrm{Gr}^F(M) = \bigoplus F_j(M)/F_{j-1}(M)$ **induces a coherent $\mathscr{O}_{T^*X}$-module.**
The support $\mathrm{Ch}(M)$ is called the characteristic variety of M ($\dim \mathrm{Ch}(M) \geq \dim X$).

Definition M **is called** *holonomic* **if** $\dim \mathrm{Ch}(M) = \dim X$.

Examples

Example 1 $\mathrm{Ch}(\mathscr{O}_X)$ **is the** 0**-section of** T^*X**. Hence,** $\mathscr{O}_X$ **is holonomic. More generally, any flat bundle is holonomic.**

Example 2 $\mathrm{Ch}(\mathscr{D}_X) = T^*X$**. Hence,** $\mathscr{D}_X$ **is not holonomic unless** $\dim X = 0$**.**

Example 3

Suppose $X = \mathbb{C}$**. Take** $P = z(z-1)\partial_z^2 + 1$**. Let** $\pi : T^*X \longrightarrow X$ **be the projection. Then,** $\mathrm{Ch}\big(\mathscr{D}_X/\mathscr{D}_X P\big) \subset T^*X$ **is the union of the** 0**-section and** $\pi^{-1}(0) \cup \pi^{-1}(1)$**. Because** $\dim \mathrm{Ch}(\mathscr{D}_X/\mathscr{D}_X P) = \dim X$**, the** $\mathscr{D}$**-module is holonomic.**

More generally, any ordinary differential equation gives a holonomic $\mathscr{D}_X$**-module.**

Functoriality of holonomic $\mathscr{D}$-modules

For any proper morphism of complex manifolds $f : X \longrightarrow Y$, we have the push-forward

$$f_\dagger : D^b_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow D^b_{\mathrm{hol}}(\mathscr{D}_Y).$$

Let $f^i_\dagger(M) := \mathscr{H}^i(f_\dagger(M))$.

It is not the same as the push-forward of $\mathscr{O}_X$-modules or $\mathbb{C}_X$-modules. For instance, if Y is a point,

$$f^i_\dagger(M) = H^{i+\dim X}\big(X, \Omega^\bullet_X \otimes M\big).$$

We have 6-operations for algebraic holonomic $\mathscr{D}$-modules.
Let $f : X \longrightarrow Y$ be algebraic morphism of algebraic manifolds.

$$f_*, f_! : D^b_{\mathrm{hol}}(\mathscr{D}^{\mathrm{alg}}_X) \longrightarrow D^b_{\mathrm{hol}}(\mathscr{D}^{\mathrm{alg}}_Y)$$

$$f^*, f^! : D^b_{\mathrm{hol}}(\mathscr{D}^{\mathrm{alg}}_Y) \longrightarrow D^b_{\mathrm{hol}}(\mathscr{D}^{\mathrm{alg}}_Y)$$

We also have $\boxtimes$ and $\mathscr{H}om$, duality, nearby and vanishing cycle functors.

Riemann-Hilbert correspondence

Let $\Omega_X^\bullet$ denote the de Rham complex of X. For any $\mathscr{D}_X$-module M, we obtain the complex of sheaves

$$\mathrm{DR}_X(M) := \Omega_X^\bullet \otimes M[\dim X].$$

Kashiwara **If M is holonomic, $\mathrm{DR}_X(M)$ is cohomologically $\mathbb{C}$-constructible. Moreover, $\mathrm{DR}_X(M)$ satisfies the middle perversity condition. We obtain**

$$\mathrm{DR}_X : D^b_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow D^b_{\mathbb{C}\text{-}c}(\mathbb{C}_X).$$

Riemann-Hilbert correspondence (Kashiwara, Mebkhout)

We have a full subcategory of regular holonomic $\mathscr{D}$-modules $D^b_{rh}(\mathscr{D}_X) \subset D^b_{\mathrm{hol}}(\mathscr{D}_X)$, and DR_X gives an equivalence

$$D^b_{rh}(\mathscr{D}_X) \longrightarrow D^b_{\mathbb{C}\text{-}c}(\mathbb{C}_X),$$

compatible with other functors.

Decomposition theorem (geometric origin)

Theorem (Beilinson-Bernstein-Deligne-Gabber) **Let X and Y be complex projective manifolds, and let $f : X \longrightarrow Y$ be a morphism. Let M be a regular holonomic $\mathscr{D}$-module on X of semisimple geometric origin. Then,**

$$f_\dagger(M) \simeq \bigoplus f_\dagger^i(M)[-i]$$

in $D^b_{rh}(\mathscr{D}_Y)$.

- **BBDG worked for perverse sheaves, which is translated to regular holonomic $\mathscr{D}$-modules.**
- **BBDG used the technique of modulo p-reductions, and reduced to the theorem for perverse sheaves with Frobenius action on algebraic varieties over finite fields.**
- **Morihiko Saito generalized the decomposition theorem to the context of regular holonomic $\mathscr{D}$-modules underlying polarizable pure Hodge modules.**
- **de Cataldo and Migliorini gave an alternative proof using the Hodge theory in a completely different way.**

Decomposition theorem (semisimple holonomic $\mathscr{D}$-modules)

Theorem **Let X and Y be complex projective manifolds. Let $f : X \longrightarrow Y$ be a morphism. Let M be a semisimple holonomic $\mathscr{D}_X$-module. Then,**

$$f_\dagger M \simeq \bigoplus f_\dagger^i(M)[-i]$$

in $D^b_{\mathrm{hol}}(\mathscr{D}_X)$, and each $f_\dagger^i(M)$ is semisimple. (conjectured by Kashiwara)

Arithmetic geometry (semisimple perverse sheaves)

Drinfeld, Gaitsgory, Böckle-Khare

Generalized Hodge theory (semisimple holonomic $\mathscr{D}$-modules)

Sabbah, M.

What to do for the generalization?

We need to enhance holonomic $\mathscr{D}$-modules with some "structure".

- Beilinson-Bernstein-Deligne-Gabber used *Frobenius actions*.
- Morihiko Saito and de Cataldo-Migliorini used *Hodge structures*.

Morihiko Saito introduced and established the theory of *mixed Hodge modules*, i.e., regular holonomic $\mathscr{D}$-modules enhanced with mixed Hodge structure.
He proved the Hard Lefschetz Theorem for regular holonomic $\mathscr{D}$-modules underlying polarizable pure Hodge modules, which implies the decomposition theorem for projective morphisms.

For holonomic $\mathscr{D}$-modules, we use a generalized Hodge structure, called *twistor structure*, introduced by Simpson.

Twistor structure

- *Twistor structure* **is a holomorphic vector bundle on $\mathbb{P}^1$.**
- *Mixed twistor structure* (V,W) **is a twistor structure V with an increasing filtration W such that**
  - $W_j(V)=0\ (j<<0)$, $W_j(V)=V\ (j>>0)$
  - $\mathrm{Gr}_j^W(V)=W_j(V)/W_{j-1}(V)$ **is a direct sum of $\mathscr{O}_{\mathbb{P}^1}(j)$.**
- (V,W) **is a "structure" on the vector space $V_{|1}\ (1\in\mathbb{P}^1)$.**

Hodge $\Longrightarrow$ Twistor

Simpson **complex Hodge structure = $\mathbb{C}^*$-equivariant twistor structure**

Let $(H;F,G)$ be a complex Hodge structure, i.e., H is a $\mathbb{C}$-vector space, and F,G are decreasing filtrations.

$$\xi(H;F) = \sum_{j\in\mathbb{Z}} F^j \lambda^{-j}$$

is a free $\mathbb{C}[\lambda]$-module, and hence a vector bundle on $\operatorname{Spec}\mathbb{C}[\lambda]$.

$$\xi(H;G) = \sum_{j\in\mathbb{Z}} G^j \lambda^{j}$$

is a free $\mathbb{C}[\lambda^{-1}]$-module, and hence a vector bundle on $\operatorname{Spec}\mathbb{C}[\lambda^{-1}]$. By gluing them, we obtain an algebraic vector bundle on $\mathbb{P}^1$.

Simpson's Meta theorem

Any concepts and theorems concerned with Hodge structures should have their generalization to the context of twistor structures.

What to do for the generalization?

- **Establish the theory of twistor version of Hodge modules, i.e., holonomic $\mathscr{D}$-modules enhanced with twistor structure.**
- **Relate the** *semisimplicity* **condition and the existence of** *polarizable pure twistor structure* **for algebraic holonomic $\mathscr{D}$-modules.**

For *flat bundles* **and** *smooth projective morphisms***, both two issues were established by the previous works of Simpson and Corlette.**

When Simpson introduced the concept of twistor structure, he also introduced the concept of *polarized variation of pure twistor structure***, as a twistor version of polarized variation of pure Hodge structure. Moreover, he established the equivalence between polarized variation of twistor structure and** *harmonic bundles* **(up to weights).**

Harmonic bundles

Let $(E,\overline{\partial}_E)$ be a holomorphic vector bundle on a complex manifold X.
Let θ be a Higgs field (holomorphic section of $\mathrm{End}(E)\otimes\Omega^1_X$, $\theta\wedge\theta=0$).
Let h be a Hermitian metric of E. We obtain

- Chern connection $\nabla_h=\overline{\partial}_E+\partial_{E,h}$

  (the unitary connection determined by $\overline{\partial}_E$ and h)
- the adjoint $\theta^\dagger_h$ of θ with respect to h
- the connection $\mathbb{D}^1:=\nabla_h+\theta+\theta^\dagger_h$.

Definition If $\mathbb{D}^1$ is integrable ($\mathbb{D}^1\circ\mathbb{D}^1=0$), then h is called *pluri-harmonic metric*, and $(E,\overline{\partial}_E,\theta,h)$ is called *harmonic bundle*.

If $\dim X=1$, it is equivalent to *the Hitchin equation* $[\overline{\partial}_E,\partial_E]+[\theta,\theta^\dagger_h]=0$.

Harmonic bundles and polarized variations of pure twistor structure

Proposition (Simpson)

Harmonic bundle $\longleftrightarrow$ **Polarized variation of pure twistor structure (weight 0)**

Let $(E,\overline{\partial}_E,\theta,h)$ **be a harmonic bundle on** X **(**$\Longrightarrow \partial_{E,h},\theta_h^{\dagger}$**)**

- **We have the family of integrable connections**

$$\nabla^{\lambda} := \overline{\partial}_E + \lambda\theta^{\dagger} + \partial_E + \lambda^{-1}\theta \quad (\lambda \in \mathbb{C}^*).$$

- $\mathbb{D}^{\lambda} := \overline{\partial}_E + \lambda\theta^{\dagger} + \lambda\partial_E + \theta$ **is extended to the family for** $\lambda \in \mathbb{C}$.
  $(E,\mathbb{D}^{\lambda})$ $(\lambda \in \mathbb{C})$: *family of λ-flat bundles*
- $\mathbb{D}^{\dagger\lambda^{-1}} := \partial_E + \lambda^{-1}\theta + \lambda^{-1}\overline{\partial}_E + \theta^{\dagger}$ **is extended to the family for** $\lambda \in \mathbb{C}^* \cup \{\infty\}$.

We obtain *something* **on** $\mathbb{P}^1 \times X$ *(Variation of Twistor structure)*.

Functoriality of harmonic bundles for smooth projective morphisms

Simpson

Many important theorems for polarized variation of Hodge structure can be generalized to the context of harmonic bundles.

Let $f : X \longrightarrow Y$ **be a smooth (i.e., submersive) projective morphism.**
Let $(E, \overline{\partial}_E, \theta, h)$ **be a harmonic bundle on** X**.**
We take a relatively ample line bundle L **of** X**.**

Theorem (Simpson)

- $c_1(L)^j : f_\dagger^{-j}(E, \mathbb{D}^1) \simeq f_\dagger^j(E, \mathbb{D}^1)$ $(j \geq 0)$.
- **We have the naturally induced harmonic bundles** $f_\dagger^j(E, \overline{\partial}_E, \theta, h)$ **over** $f_\dagger^j(E, \mathbb{D}^1)$**.**

  In other words, the push-forward of flat bundles by smooth projective morphisms is enriched to the push-forward of harmonic bundles.

Corlette-Simpson correspondence

$$\begin{pmatrix} (E, \overline{\partial}_E, \theta) \\ \textbf{Higgs bundle} \end{pmatrix} \Longleftarrow \begin{pmatrix} (E, \overline{\partial}_E, \theta, h) \\ \textbf{harmonic bundle} \end{pmatrix} \Longrightarrow \begin{pmatrix} (E, \mathbb{D}^1) \\ \textbf{flat bundle} \end{pmatrix}$$

Theorem (Corlette, Simpson)

If X is smooth projective, the following objects are equivalent.

$$\begin{pmatrix} (E, \overline{\partial}_E, \theta) \\ \textbf{Higgs bundle} \\ \textbf{polystable} \\ \mathrm{ch}_*(E) = \mathrm{rank}\, E \end{pmatrix} \Longleftrightarrow \begin{pmatrix} (E, \overline{\partial}_E, \theta, h) \\ \textbf{harmonic bundle} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} (E, \mathbb{D}^1) \\ \textbf{flat bundle} \\ \textbf{semisimple} \end{pmatrix}$$

(The case $\dim X = 1$, $\mathrm{rank}\, E = 2$ due to Hitchin, Donaldson, Diederich-Ohsawa.)

Decomposition theorem in the case where everything is smooth

For semisimple flat bundles and a smooth morphism of smooth projective manifolds, we obtain the decomposition theorem from

- **Functoriality and Hard Lefschetz Theorem for harmonic bundles by the push-forward via smooth projective morphisms.**
- **Semisimple flat bundles $\Longleftrightarrow$ Harmonic bundles**

We would like to generalize this story to the singular case.

- **Establish the theory of twistor version of Hodge modules, i.e., holonomic $\mathscr{D}$-modules enhanced with twistor structure.**
- **Relate the** *semisimplicity* **condition and the existence of** *polarizable pure twistor structure* **for algebraic holonomic $\mathscr{D}$-modules.**

Main Theorems in the theory of twistor $\mathscr{D}$-modules

Theorem (Sabbah, M)

- Pure case: **push-forward by projective morphism (hard Lefschetz Theorem)**
- Mixed case: **6-operations on the derived category of algebraic mixed twistor $\mathscr{D}$-modules.**

Theorem

- Pure case: **Polarizable pure twistor $\mathscr{D}$-modules are the "minimal extension" of variations of polarized pure twistor structure.**
- Mixed case: **Mixed twistor $\mathscr{D}$-modules are locally obtained as gluing of admissible variation of mixed twistor structure.**

The most essential ideas were given by *Saito* in the context of *Hodge modules*.

The local theory of wild harmonic bundles is fundamental.

Hard Lefschetz Theorem (Polarizable pure twistor $\mathscr{D}$-module)

Let $f : X \longrightarrow Y$ be any projective morphism with a relatively ample line bundle L.
Let $\mathscr{T}$ be a polarizable pure twistor $\mathscr{D}$-module of weight w on X.
Let M be the underlying holonomic $\mathscr{D}_X$-module.

Theorem (Saito, Sabbah, M)

- **We have a naturally induced polarizable pure twistor $\mathscr{D}$-module $f_\dagger^i \mathscr{T}$ of weight $w+i$ over $f_\dagger^i M$.**

  In other words, the push-forward of holonomic $\mathscr{D}$-modules by projective morphisms is enriched to the push-forward of polarizable pure twistor $\mathscr{D}$-modules.

- $c_1(L)^j : f_\dagger^{-j} M \longrightarrow f_\dagger^j M$ **are** *isomorphisms* **for** $j \geq 0$**.**

Characterization of semisimplicity

Theorem On any projective manifold,

$$\left(\begin{array}{c} \text{semisimple} \\ \text{holonomic } \mathscr{D}\text{-modules} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \sqrt{-1}\mathbb{R} \text{ polarizable pure} \\ \text{twistor } \mathscr{D}\text{-module (weight } w) \end{array} \right)$$

We need the correspondences of

- $\sqrt{-1}\mathbb{R}$ polarizable pure twistor $\mathscr{D}$-modules
- $\sqrt{-1}\mathbb{R}$ *wild harmonic bundles*
- semisimple meromorphic flat bundles

We need

- Local theory of wild harmonic bundles around singularity.
- Global theory of wild harmonic bundles on projective manifolds. (Generalization of Corlette-Simpson correspondence)
- Study of irregular singularity of meromorphic flat bundles.

Wild harmonic bundles

Let X be a complex manifold with a normal crossing hypersurface D.
Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X \setminus D$. The Higgs bundle $(E, \overline{\partial}_E, \theta)$ induces a coherent sheaf on the cotangent bundle $T^*(X \setminus D)$.
The support Σ_θ of the coherent sheaf is called the spectral variety.

Definition
$(E, \overline{\partial}_E, \theta, h)$ *tame* $\overset{\text{def}}{\Longleftrightarrow}$ **The closure of Σ_θ in $T^*X(\log D)$ is proper over X.**
$(E, \overline{\partial}_E, \theta, h)$ *wild* $\overset{\text{def}}{\Longleftrightarrow}$ **The closure of Σ_θ in $T^*X(\log D) \otimes \mathscr{O}(ND)$ is proper over X for some natural number N (locally given).**

Example
Let $f \in \mathscr{O}_X(*D)$. A harmonic bundle $E = \mathscr{O}_{X \setminus D}\, e$ with the Higgs field $\theta = df$ and the metric $h(e,e) = 1$ is a *wild* harmonic bundle on (X, D).

Example
Polarized variation of complex Hodge structure gives a *tame* harmonic bundle.

For the analysis of wild harmonic bundles, we should impose a "non-degeneracy" condition on Σ_θ along D, which is not restrictive.

Local theory

The asymptotic behaviour of polarized variation of Hodge structure:
(Cattani, Griffiths, Kaplan, Kashiwara, Kawai, Schmid)

- **Extension of PVHS on $X \setminus D$ to a meromorphic object on (X, D) (The Hodge filtration of the Deligne extension)**
- **Limit polarized mixed Hodge structure (“Positivity” at any points of D)**

They are generalized to the context of harmonic bundles.

The asymptotic behaviour of harmonic bundles:

Prolongation **Extension of wild harmonic bundle to a meromorphic object on (X, D)**

Reduction **Reductions from wild harmonic bundles to polarized mixed twistor structures**

Prolongation

Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on (X, D).

- $(E, \mathbb{D}^{\lambda})$ $(\lambda \in \mathbb{C})$ **are prolonged to** *meromorphic λ-flat bundles* **on** (X, D)**.**
- $(E, \mathbb{D}^{\dagger \lambda^{-1}})$ $(\lambda^{-1} \in \mathbb{C})$ **are prolonged to** *meromorphic λ^{-1}-flat bundles* **on** $(X^{\dagger}, D^{\dagger})$**.**
- **Because their** *Stokes structures* **and** *KMS-structures* **are the same for** $\lambda \in \mathbb{C}^{*}$**, they give an object on** $\mathbb{P}^1 \times X$ **(***meromorphic variation of twistor structure***).**

- *Stokes structure* **is a family of filtrations induced by the growth order of the** $\mathbb{D}^{\lambda}$**-flat sections (**$\mathbb{D}^{\dagger \lambda^{-1}}$**-flat sections).**
- *KMS-structure (Kashiwara-Malgrange-Sabbah-Simpson-structure)* **is a family of filtrations and the decompositions on D induced by the parabolic structure and the residue of $\mathbb{D}^{\lambda}$ (the residue of $\mathbb{D}^{\dagger \lambda^{-1}}$).**

Reduction

Taking the associated graded objects with respect to *the Stokes structure*, *the KMS-structure* and *the weight filtration*, we obtain the following sequence of the reductions.

> *Theorem*
>
> **(good) wild** $\overset{\text{Stokes}}{\Longrightarrow}$ **tame** $\overset{\text{KMS}}{\Longrightarrow}$ **polarized mixed twistor structure**
>
> $\overset{\text{weight}}{\Longrightarrow}$ **polarized mixed Hodge structure**

After *Prolongation* and *Reduction*, the structure of (good) wild harmonic bundles around singularity is well understood.

- **The reduced object is *a useful approximation* of the original one.**
- **We can conclude that the meromorphic object has a nice structure along D.**

Global theory

Study wild harmonic bundles on $X \setminus D$, where X is *a complex projective manifold* **and D is a simple normal crossing hypersurface.**

- **Generalization of the Corlette-Simpson correspondences**
- **Generalization of the theorem of Corlette in the context of meromorphic flat bundles**

(They are well established after the work of many people including Biquard, Boalch, Jost, Jiayu Li, Mehta, Sabbah, Seshadri, Simpson, Steer, Wren, Zuo, and M)

Generalization of Corlette's theorem (meromorphic flat bundles)

Theorem **A meromorphic flat bundle** $(\mathscr{V},\nabla)$ **on** (X,D) **is** *semisimple* **if and only if it comes from** *a* $\sqrt{-1}\mathbb{R}$*-wild harmonic bundle*.

$\sqrt{-1}\mathbb{R}$*-wild* $\overset{\text{def}}{\Longleftrightarrow}$ **(the eigenvalues of the residues of the Higgs field)** $\in \sqrt{-1}\mathbb{R}$

A general framework in global analysis:

(i) **Take** *an appropriate metric* **of** $(\mathscr{E},\nabla)_{|X-D}$. **(Some finiteness condition.)**

(ii) **Deform it along** *a non-linear heat flow*.

(iii) **The limit of the flow should be a pluri-harmonic metric.**

To construct an appropriate metric in the step (i), we need to know the local structure of the meromorphic flat bundle.

We need a resolution of turning points (bad singularity) if $\dim X \geq 2$*.*

Irregular singularities

Regular singular (Generalization of Fuchs type differential system)

The higher dimensional case was well established *(Deligne)*

Irregular singular

The classical result in the one dimensional case was appropriately generalized to that in the higher dimensional case.

(Majima, Malgrange, Sabbah, André, Kedlaya, and M)

The local theory of meromorphic flat bundles on curves

(i) **Formal decomposition (on a ramified covering)**

(ii) **Asymptotic analysis and Stokes structure**

(iii) **Generalized Riemann-Hilbert correspondence**

In the higher dimensional case, we have one additional step before (i). Briefly, we need birational transform not only ramified covering.

Theorem (conjectured by Sabbah, proved by Kedlaya, M)

Suppose X is projective. For any meromorphic flat bundle $(\mathscr{V},\nabla)$ on (X,D), there exists a projective birational morphism $\varphi:(X',D')\longrightarrow(X,D)$ such that $\varphi^*(\mathscr{V},\nabla)$ has *no turning points*, **i.e., the formal completion at any point has a nice decomposition (on a ramified covering).**

Summary

On any smooth projective varieties,

$$\text{semisimple holonomic } \mathscr{D}\text{-modules} \iff \text{polarizable } \sqrt{-1}\mathbb{R}\text{-pure twistor } \mathscr{D}\text{-modules}$$

Let X, Y be smooth projective varieties, and let $f : X \longrightarrow Y$ be a morphism.

polarizable pure twistor $\mathscr{D}_X$-module $\overset{f^i_\dagger}{\Longrightarrow}$ polarizable pure twistor $\mathscr{D}_Y$-module

$\Downarrow$ $\Downarrow$

$\mathscr{D}_X$-module $\overset{f^i_\dagger}{\Longrightarrow}$ $\mathscr{D}_Y$-module

Moreover, for a relatively ample line bundle L,

$$c_1(L)^j : f_\dagger^{-j}\mathscr{T} \xrightarrow{\simeq} f_\dagger^j \mathscr{T} \otimes \mathbb{T}^S(j)$$

Then, we obtain the decomposition theorem for semisimple holonomic $\mathscr{D}$-modules.