Geometry and Arithmetic of Crystallographic Packings

Alex Kontorovich

Rutgers / IAS

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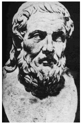

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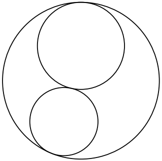

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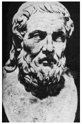

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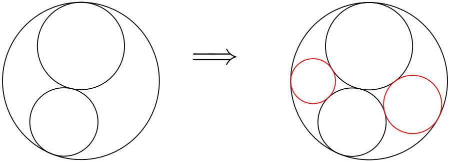

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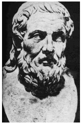

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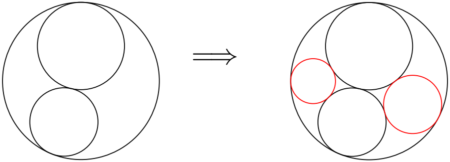

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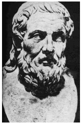

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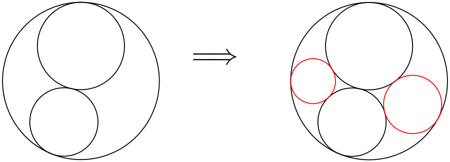

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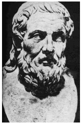

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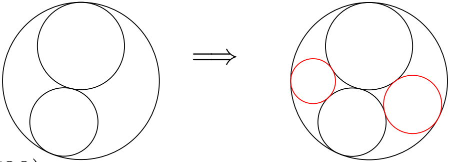

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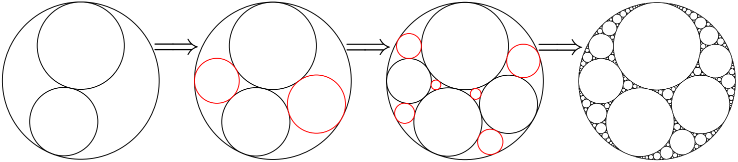

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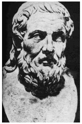

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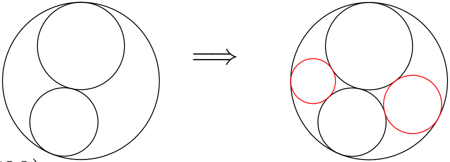

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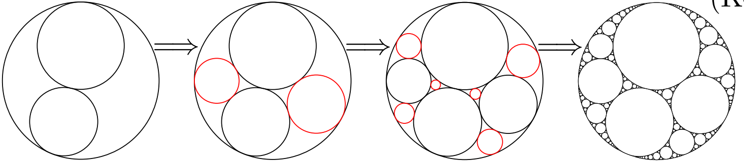

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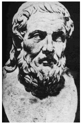

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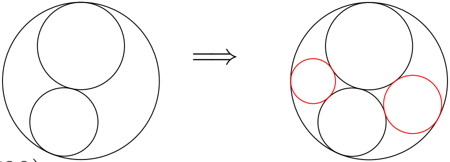

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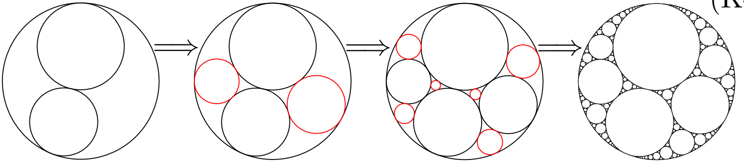

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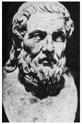

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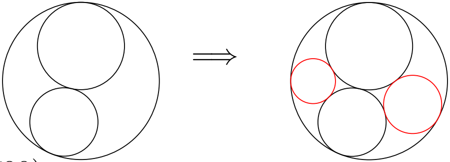

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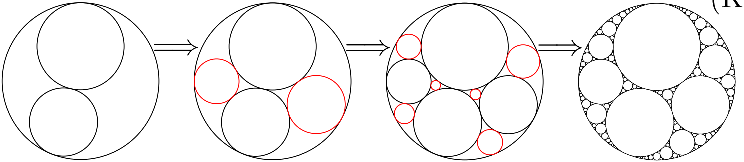

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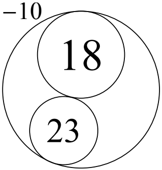

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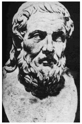

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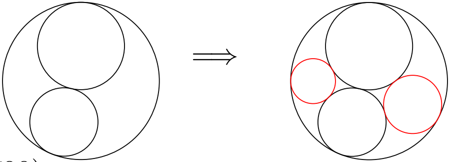

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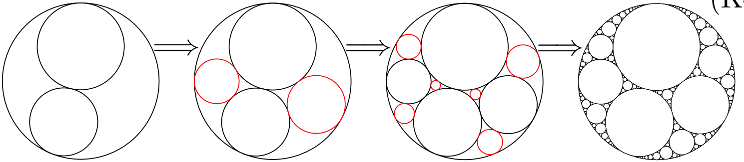

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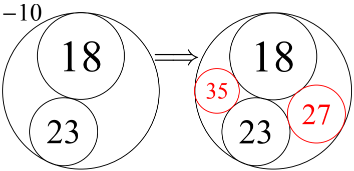

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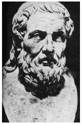

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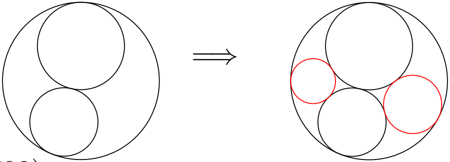

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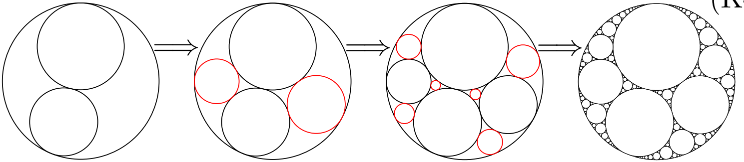

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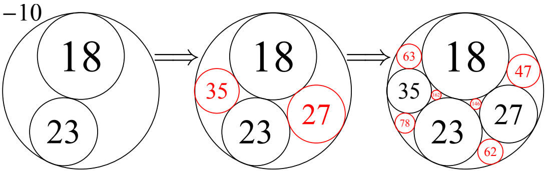

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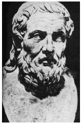

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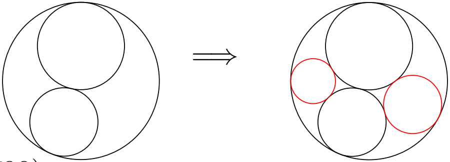

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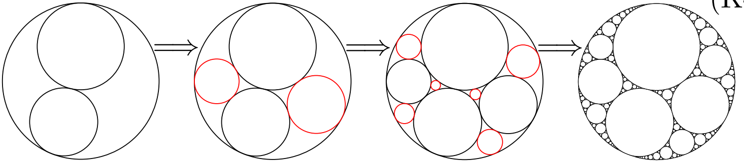

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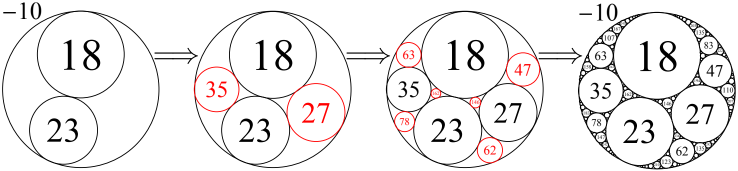

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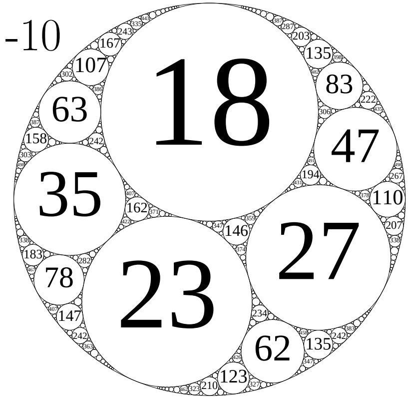

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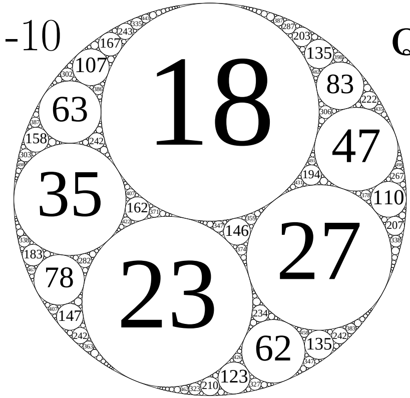

Q: (GLMWY 2003)

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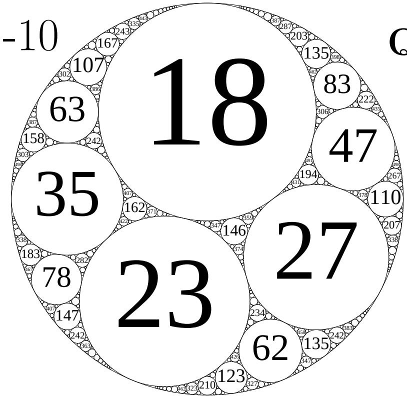

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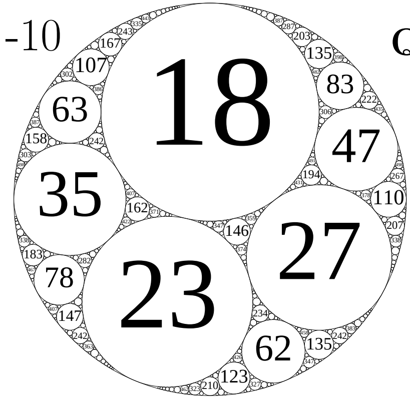

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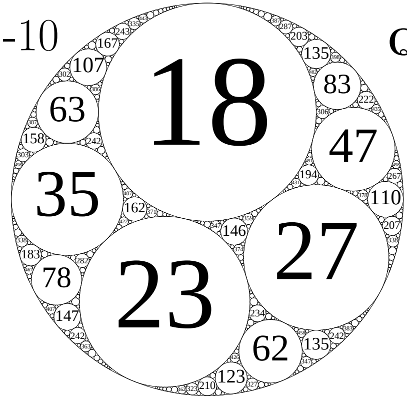

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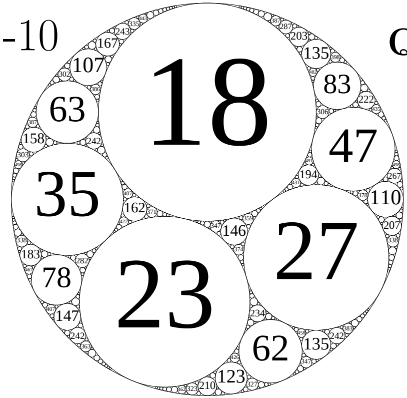

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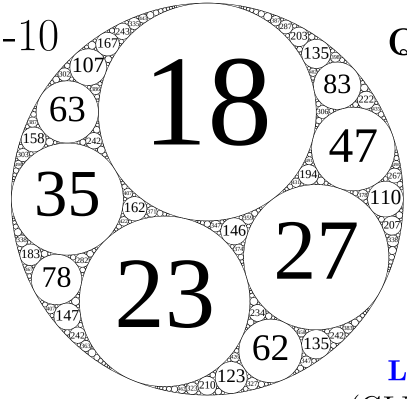

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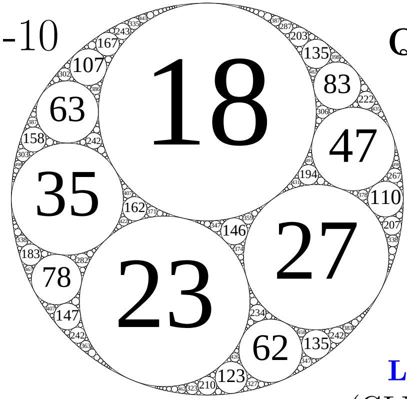

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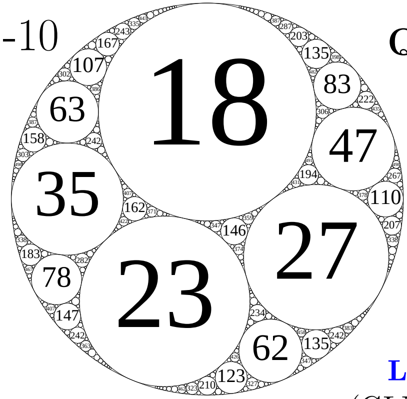

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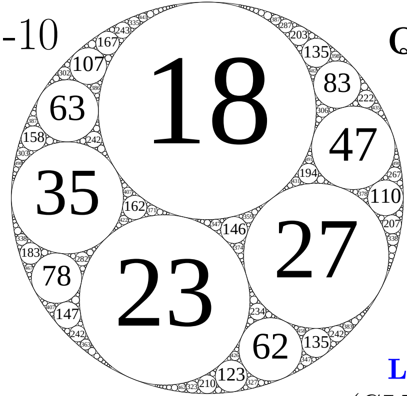

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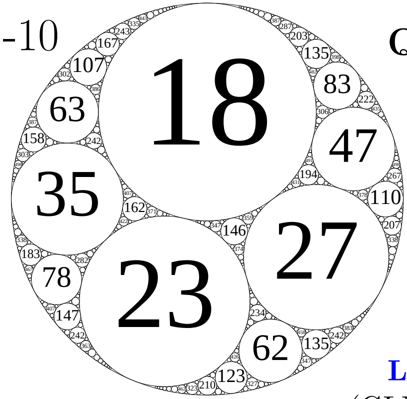

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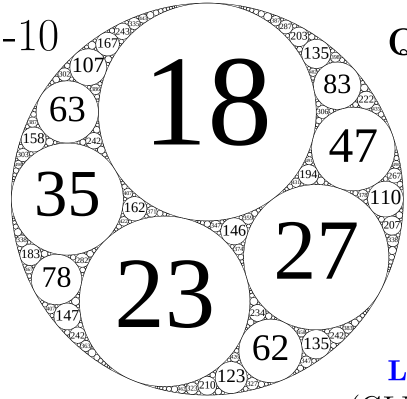

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(Survey in: K. "From Apollonius to Zaremba" *BAMS* 2013)

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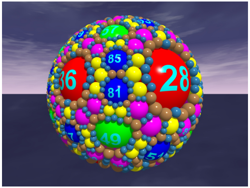

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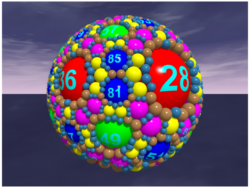

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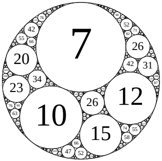

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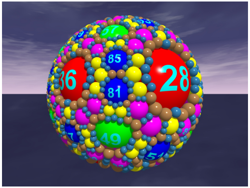

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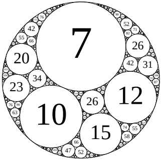

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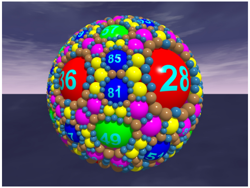

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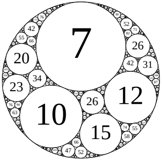

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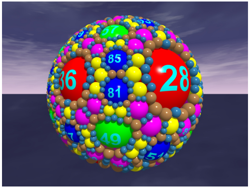

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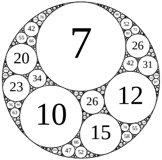

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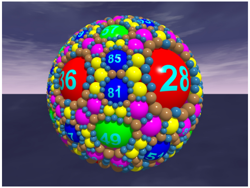

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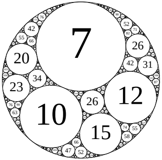

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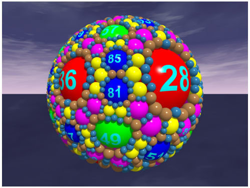

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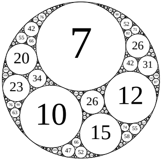

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Answer:

- There are infinitely many such packings! And moreover,
- There are only finitely many such packings!

Basic Setup

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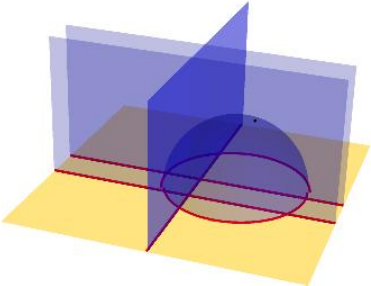

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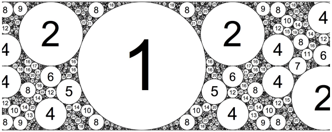

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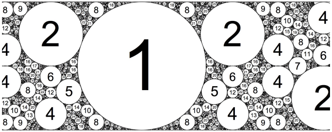

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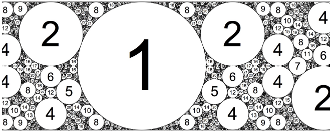

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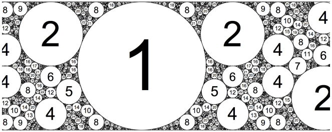

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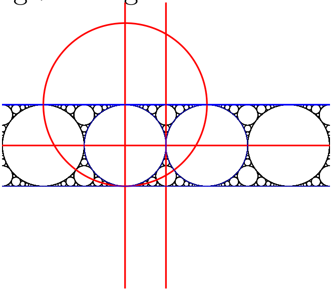

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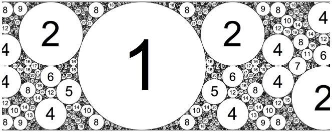

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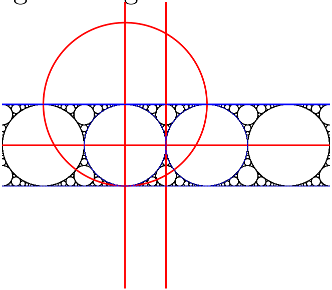

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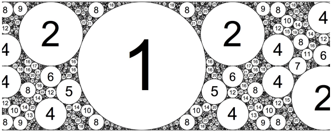

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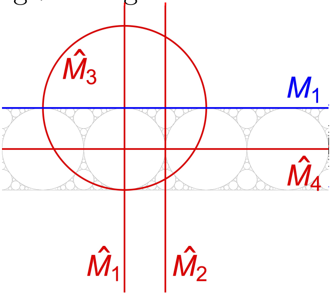

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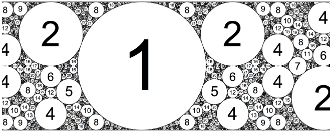

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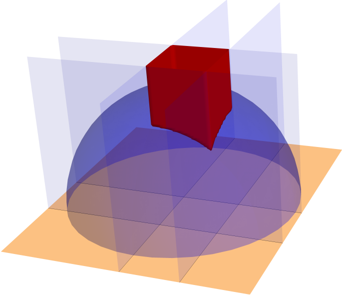

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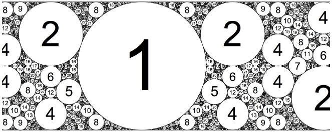

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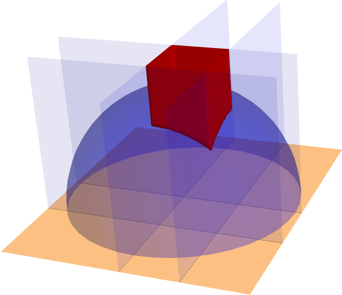

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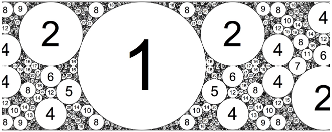

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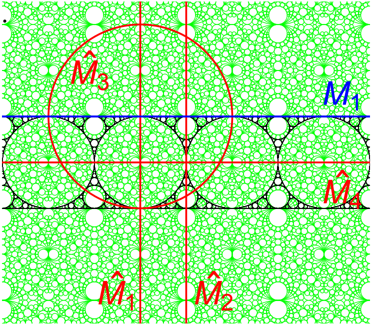

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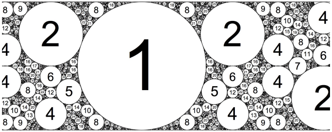

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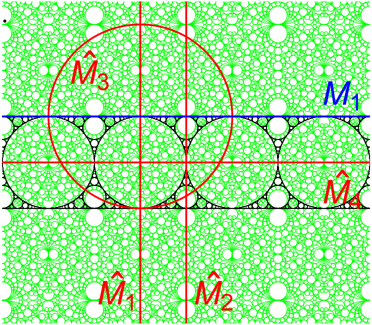

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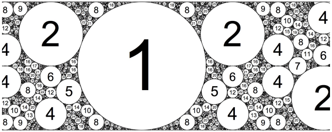

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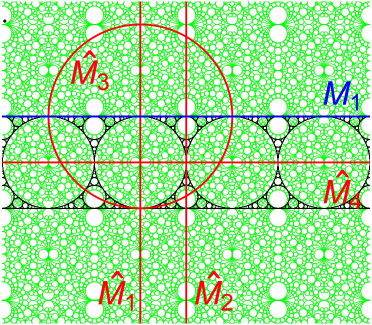

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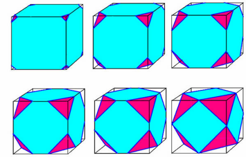

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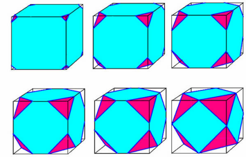

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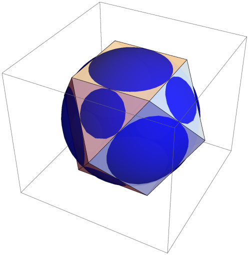

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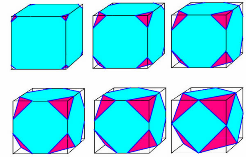

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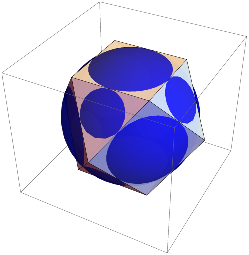

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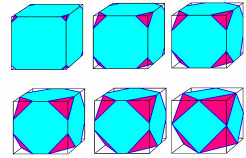

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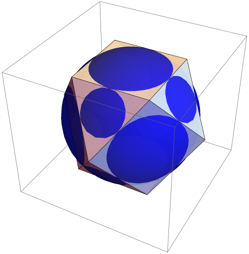

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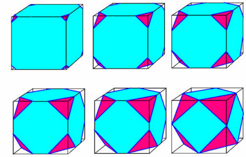

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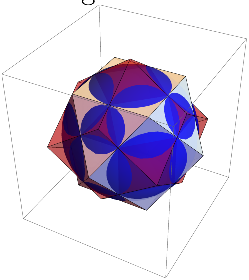

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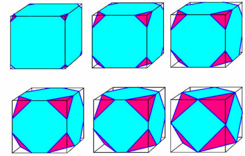

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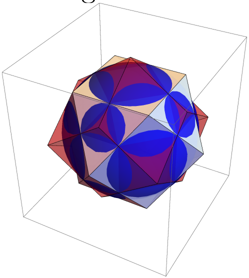

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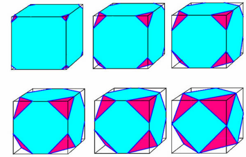

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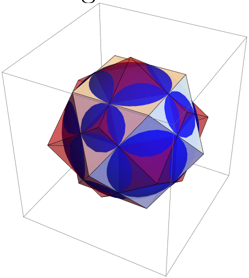

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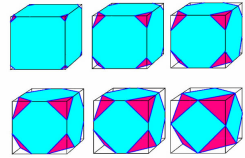

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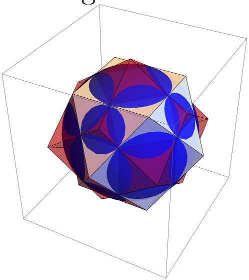

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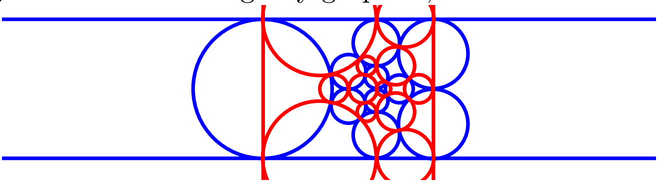

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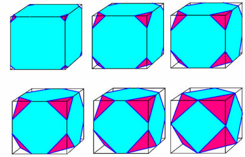

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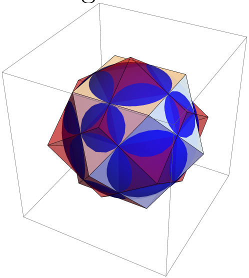

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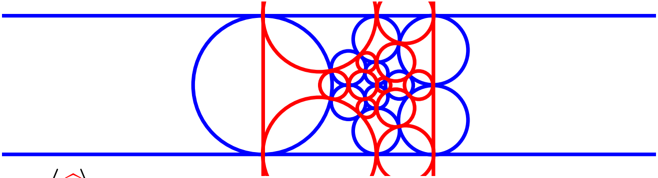

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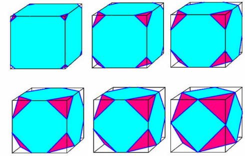

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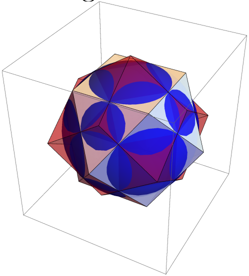

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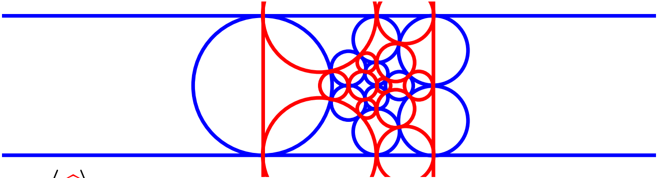

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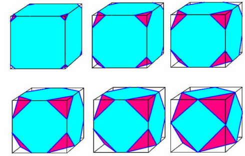

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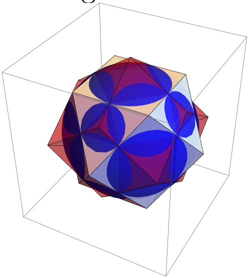

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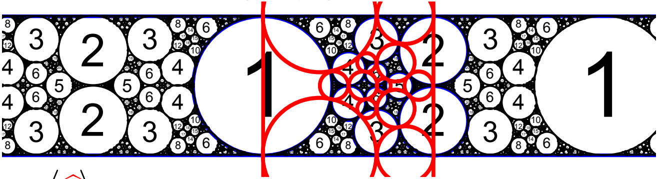

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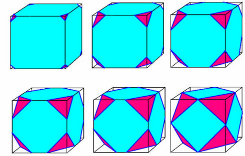

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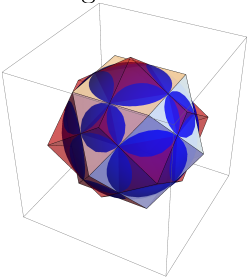

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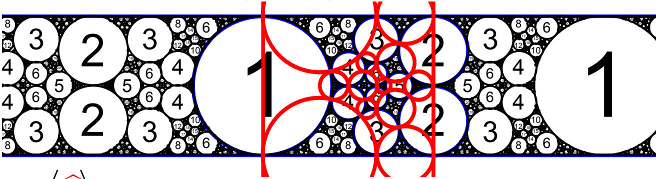

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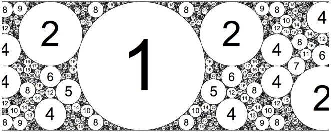

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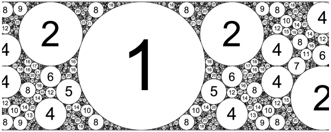

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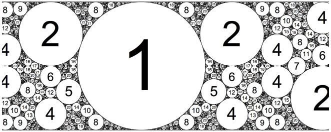

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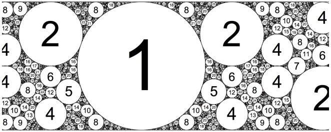

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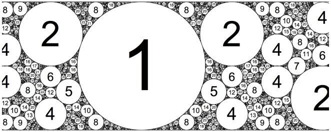

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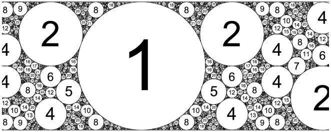

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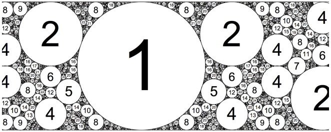

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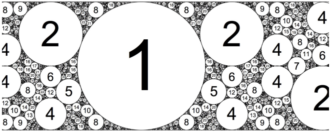

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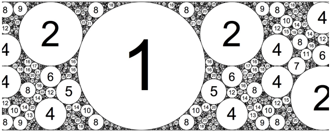

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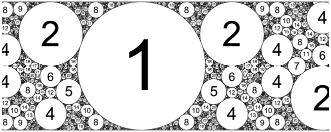

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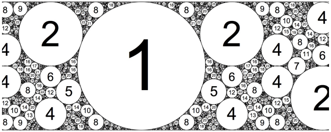

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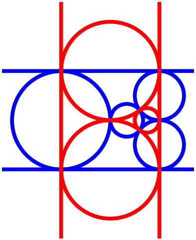

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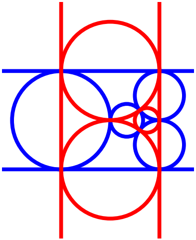

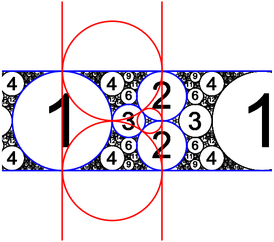

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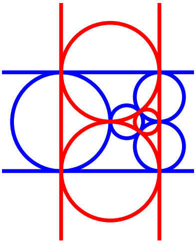

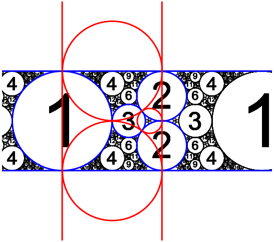

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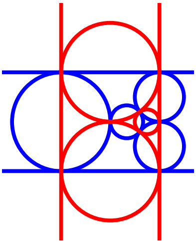

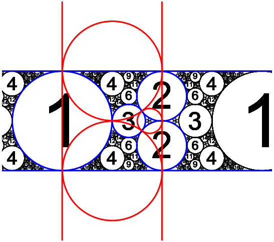

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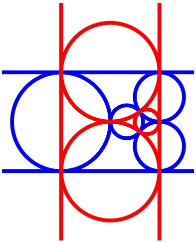

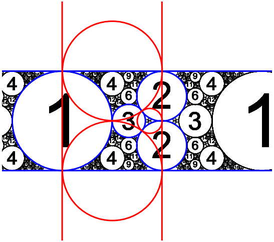

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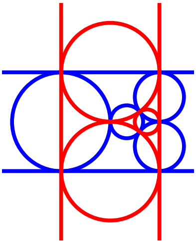

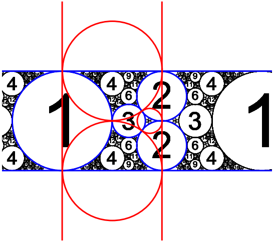

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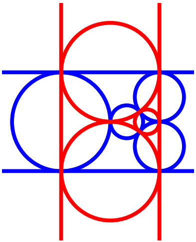

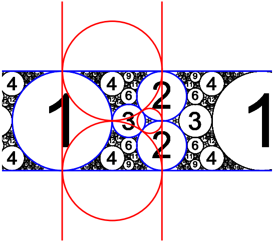

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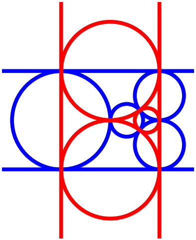

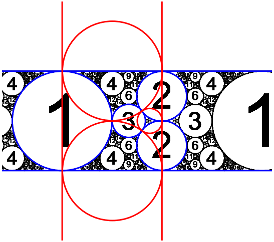

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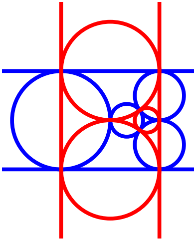

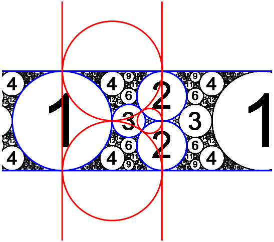

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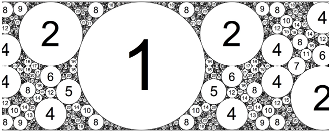

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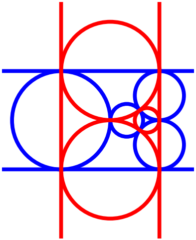

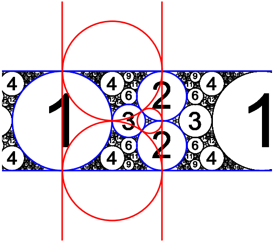

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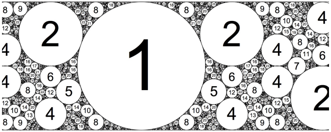

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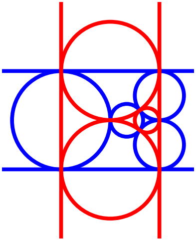

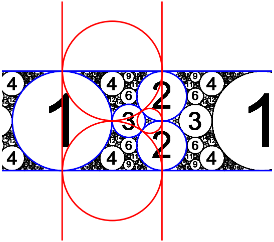

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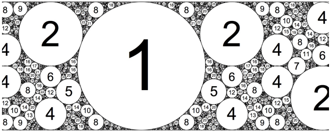

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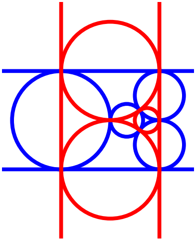

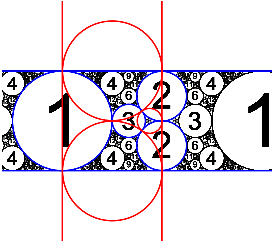

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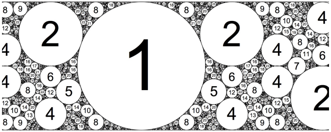

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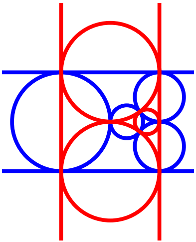

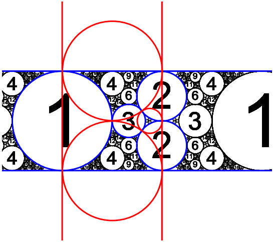

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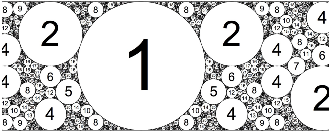

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But *can* construct, e.g., $\mathbb{Z}[\varphi]$-superintegral packings on right-angled dodecahedron (arithmetic and co-compact)

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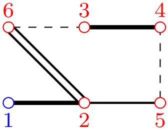

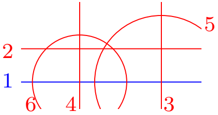

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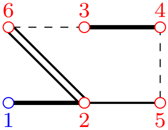

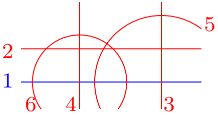

Structure Thm: (K-N) If walls $\widetilde{\mathcal{C}}$ of reflection lattice decompose as $\widetilde{\mathcal{C}} = \mathcal{C} \bigsqcup \widehat{\mathcal{C}}$

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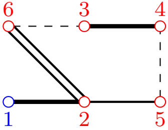

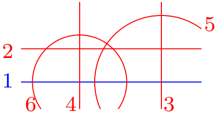

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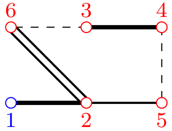

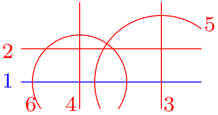

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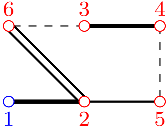

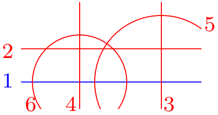

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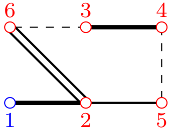

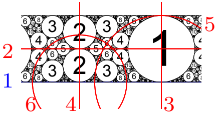

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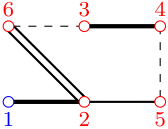

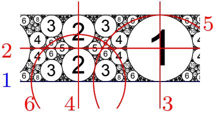

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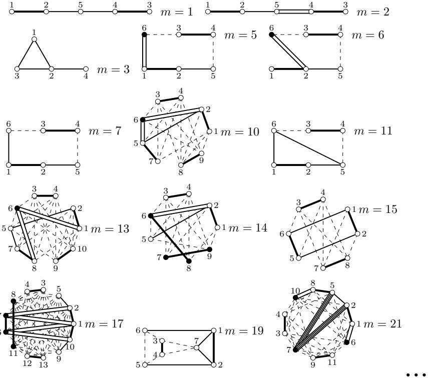

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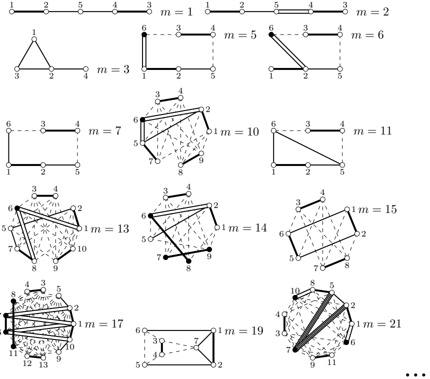

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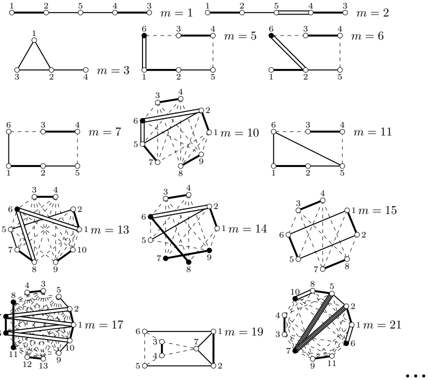

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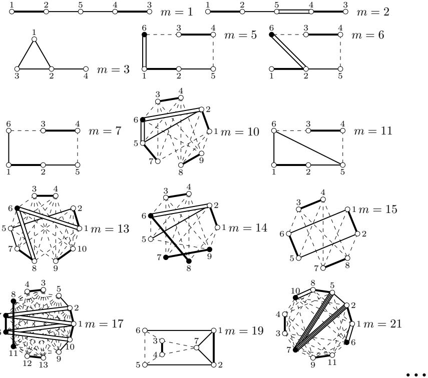

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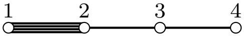

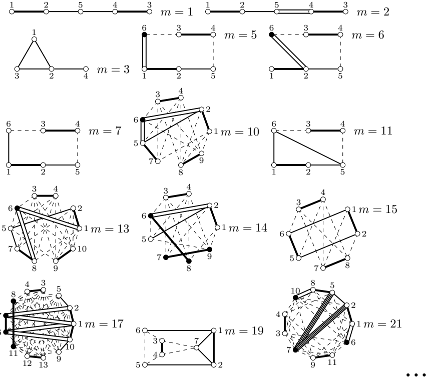

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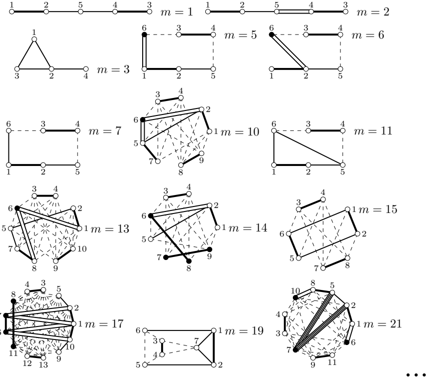

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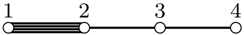
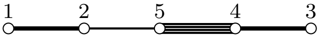


Subgroup:



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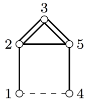

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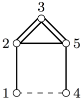

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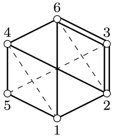

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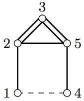

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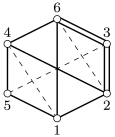

Can likely prove that no subgroup of these has packing.

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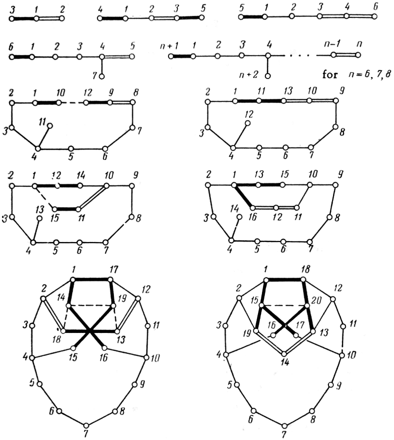

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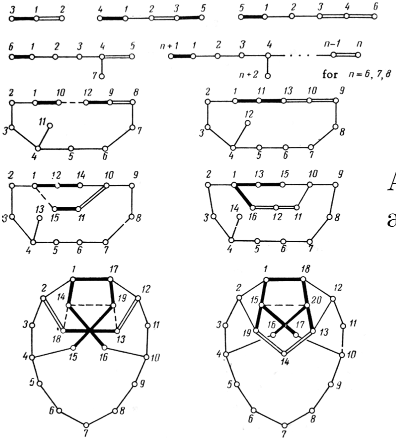

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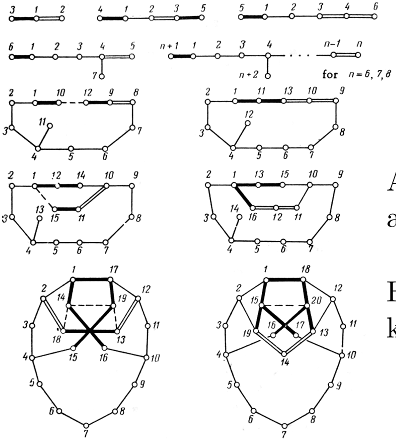

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Higher dimensional examples known, $n = 20$ due to Borcherds

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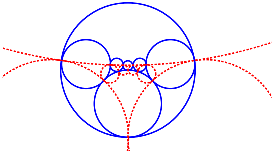

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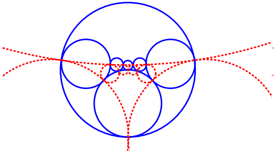

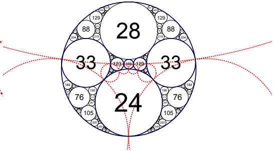

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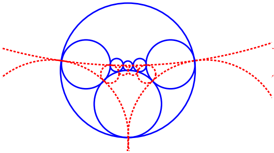
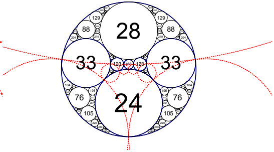
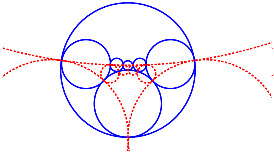
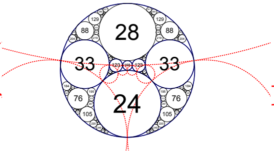




Integral

Non-superintegral: Take Π = Hex pyramid





Integral

But turns out

NOT superintegral!

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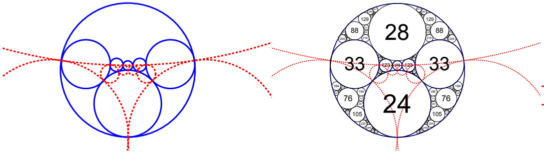

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Is the supergroup arithmetic?

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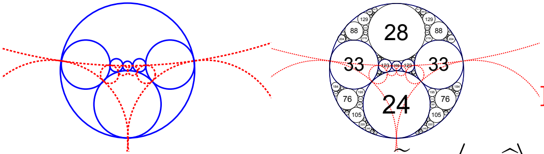

Integral

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Is the supergroup arithmetic? $\widetilde{\Gamma} = \left\langle \mathcal{C}, \widehat{\mathcal{C}} \right\rangle$

is reflection group with Gram

$\mathcal{G} = \{C_i \star C_j\}$,

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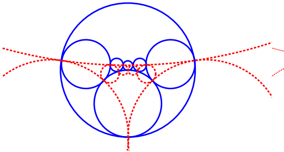

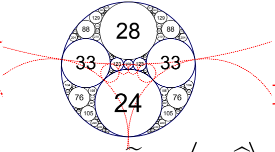

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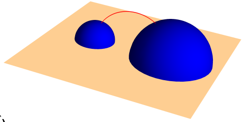

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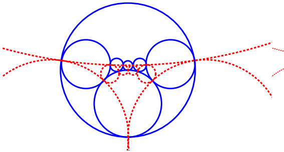

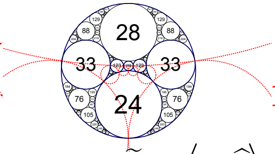

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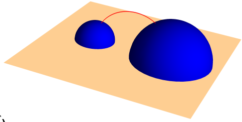

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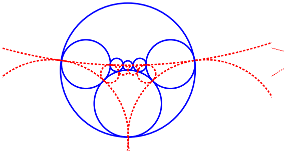

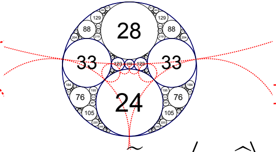

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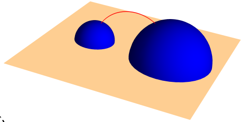

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KAT $\Longrightarrow$

$$\mathcal{G}(\text{hex pyr}) = \begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \\
1 & -1 & 1 & 5 & 7 & 5 & 1 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 \\
1 & 1 & -1 & 1 & 5 & 7 & 5 & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 \\
1 & 5 & 1 & -1 & 1 & 5 & 7 & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 0 \\
1 & 7 & 5 & 1 & -1 & 1 & 5 & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 0 \\
1 & 5 & 7 & 5 & 1 & -1 & 1 & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 0 \\
1 & 1 & 5 & 7 & 5 & 1 & -1 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 \\
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0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 1 & 5 & 7 & 5 & 1 & -1 & 1 \\
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\end{pmatrix}$$

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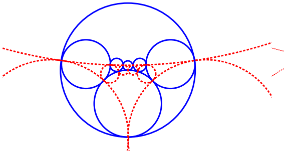

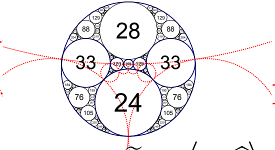

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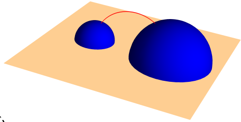

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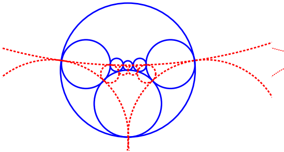
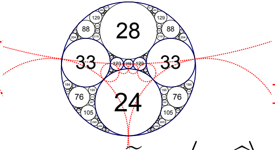
Non-arithmetic!

Non-superintegral: Take Π = Hex pyramid





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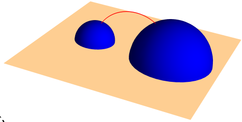

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Non-arithmetic! (à la Deligne-Mostow)

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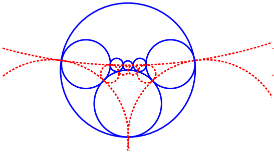

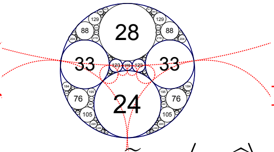

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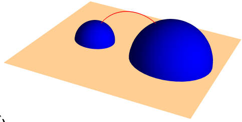

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Non-arithmetic! (à la Deligne-Mostow) $\widetilde{\Gamma} < G(\mathbb{Z}[\frac{1}{3}])$,

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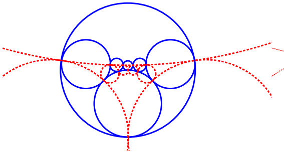

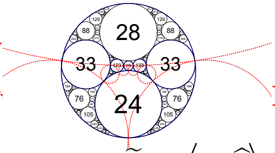

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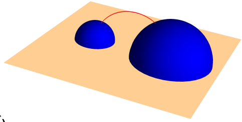

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Non-arithmetic! (à la Deligne-Mostow) $\widetilde{\Gamma} < G(\mathbb{Z}[\frac{1}{3}])$, but $\widetilde{\Gamma}$ is **thin** in $G(\mathbb{R}) \times G(\mathbb{Q}_3)$, since it is lattice in first factor.

Local-Global?

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Recall for classical Apollonian packing

Thm: (Bourgain-K, 2014)
Almost all admissible numbers arise. $\frac{\#\mathcal{B}\cap[1,X]}{\#\text{admissibles}\cap[1,X]} \to 1.$

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Extended by E. Fuchs, K. Stage, X. Zhang (2017) to:

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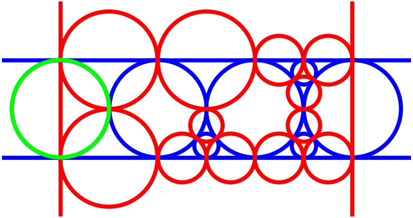

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