Hodge theory for combinatorial geometries

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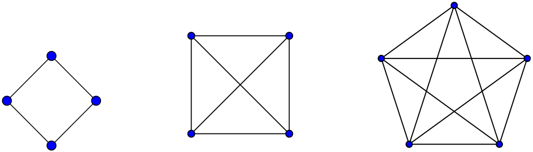
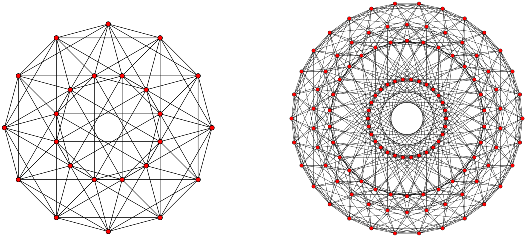
A *graph* is a 1-dimensional space, with vertices and edges.



Graphs are the simplest geometric structures.



Hassler Whitney (1932): The *chromatic polynomial* of a graph G is the function

$$\chi_G(q) = (\text{the number of proper colorings of } G \text{ with } q \text{ colors}).$$

Example

$$\begin{array}{ccc} \bullet & - & \bullet \\ | & & | \\ \bullet & - & \bullet \end{array}$$

$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \quad \chi_G(1) = 0,\ \chi_G(2) = 2,\ \ldots$$

What can be said about the chromatic polynomial in general?

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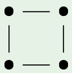


$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \quad \chi_G(1) = 0, \ \chi_G(2) = 2, \ \ldots$$

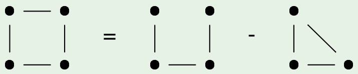
Read's conjecture (1968)

The absolute values of the coefficients of the chromatic polynomial $\chi_G(q)$ form a log-concave sequence for any graph G, that is,

$$a_i^2 \geq a_{i-1}\, a_{i+1} \quad \textit{for all } i.$$

Example

How do we compute the chromatic polynomial? We write



and use

$$\begin{aligned} \chi_{G'}(q) &= q(q-1)^3 \\ \chi_{G''}(q) &= q(q-1)(q-2). \end{aligned}$$

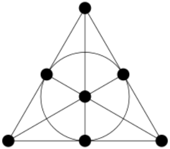
Therefore

$$\chi_G(q) = \chi_{G'}(q) - \chi_{G''}(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

This algorithmic description of $\chi_G(q)$ makes the prediction of the conjecture interesting.

For any finite set of vectors A in a vector space over a field, define

$$f_i(A) = (\text{number of independent subsets of } A \text{ with size } i).$$



Example

If A is the set of all nonzero vectors in $\mathbb{F}_2^3$, then

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

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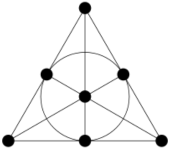

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How do we compute $f_i(A)$? We use

$$f_i(A) = f_i(A \setminus v) + f_{i-1}(A \,/\, v).$$

Welsh's conjecture (1969)

The sequence f_i form a log-concave sequence for any finite set of vectors A in any vector space over any field, that is,

$$f_i^2 \geq f_{i-1}\, f_{i+1} \text{ for all } i.$$

Hassler Whitney (1935)

Whitney provided axioms for *independence*, and

defined any finite structure adhering to these axioms to be *matroids*.

A graph gives a matroid, where a subset of its edges is

"independent" if it does not contain any cycle.

A configuration of vectors gives a matroid, where a subset of vectors is

"independent" if it is linearly independent.

One can define the *characteristic polynomial* of a matroid by the recursion

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q).$$

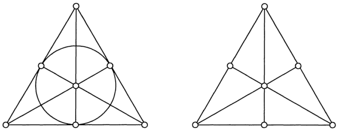
Rota's conjecture (1970)

The coefficients of the characteristic polynomial $\chi_M(q)$ form a log-concave sequence for any matroid M, that is,

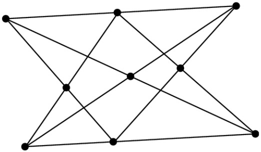
$$\mu_i^2 \geq \mu_{i-1}\mu_{i+1} \quad \textit{for all } i.$$

This implies the conjecture on G and the conjecture on A.

Fano matroid is realizable over a field k iff $\mathrm{char}(k) = 2$.



Non-Fano matroid is realizable over a field k iff $\mathrm{char}(k) \neq 2$.



Non-Pappus matroid is not realizable over any field.

How many matroids are realizable over a field?

0% of matroids are realizable.

In other words, almost all matroids are (conjecturally) not realizable over any field.

Testing the realizability of a matroid over a given field is not easy.

When $k = \mathbb{Q}$, this is equivalent to Hilbert's tenth problem over $\mathbb{Q}$.

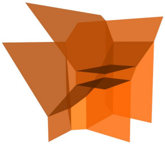
In a recent joint work with *Karim Adiprasito* and *Eric Katz*, we proved

the log-concavity conjectures in their full generality.

Here are the three fundamental objects that appear in the proof:

(1) A matroid M can be viewed as a piecewise linear object Δ_M,

the *tropical linear space* of M (*Ardila-Klivans*).



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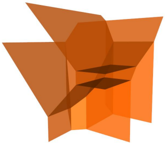

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(3) The vector space $A^1(\Delta)_{\mathbb{R}}$ contains a convex cone $\mathscr{K}_\Delta$, the *ample cone* of Δ.

Let M be a matroid of rank $r+1$.

Main Theorem

Let ℓ be an element of the ample cone of $\mathscr{K}_M$ and let $k \le r/2$.

(1) *Hard Lefschetz: The multiplication by ℓ defines an isormophism*

$$A^k(\Delta_M)_{\mathbb{R}} \longrightarrow A^{r-k}(\Delta_M)_{\mathbb{R}}, \qquad h \longmapsto \ell^{r-2k} \cdot h.$$

(2) *Hodge standard: The multiplication by ℓ defines a definite form of sign $(-1)^k$*

$$PA^k(\Delta_M)_{\mathbb{R}} \times PA^k(\Delta_M)_{\mathbb{R}} \longrightarrow A^r(\Delta_M)_{\mathbb{R}} \simeq \mathbb{R}, \qquad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

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Why does this imply the log-concavity conjectures? Essentially because

$$b^2 \geq ac \text{ if and only if } \begin{vmatrix} b & a \\ c & b \end{vmatrix} \geq 0.$$

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The intermediate objects are tropical varieties with good cohomology rings,

but not in general associated to a matroid (unlike in the case of polytopes).