A Plateau problem at infinity in $\mathbf{H}^2 \times \mathbf{R}$

Magdalena Rodríguez

Joint work with Leonor Ferrer, Rafe Mazzeo and Francisco Martín

Princeton, 7th November 2018

UNIVERSIDAD DE GRANADA

IEMATH-GR Instituto de Matemáticas

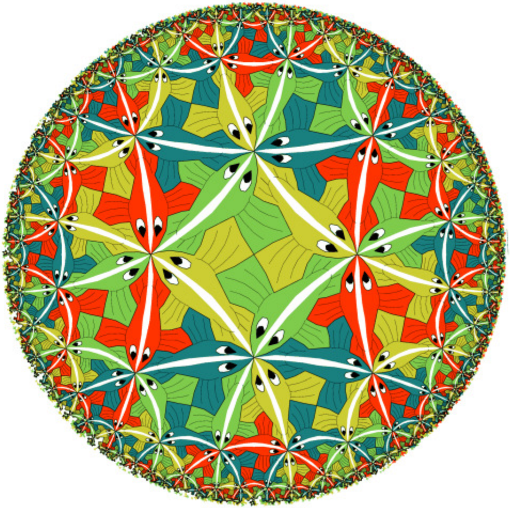
GOBIERNO DE ESPAÑA MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

Unión Europea Fondo Europeo de Desarrollo Regional

"Una manera de hacer Europa"

Proyectos cofinanciados con FEDER (Fondo Europeo de Desarrollo Regional)

$$\mathbb{H}^2 = \{r = |z| < 1\}, \qquad g_{-1} = \frac{4}{(1-|z|^2)^2}|dz|^2$$



Setting

Question 1

Given $\Gamma : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma$?

Nelli and Rosenberg, 2002 $\rightsquigarrow$ $\exists M =$ vertical graph (+ uniqueness)

Kloeckner and Mazzeo, 2017 $\rightsquigarrow$ regularity up to the boundary

Setting

Question 1

Given $\Gamma : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma$?

Nelli and Rosenberg, 2002 $\rightsquigarrow \exists M =$ vertical graph ($+$ uniqueness)

Kloeckner and Mazzeo, 2017 $\rightsquigarrow$ regularity up to the boundary

Setting

Question 1

Given $\Gamma : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma$?

Nelli and Rosenberg, 2002 $\rightsquigarrow$ $\exists M =$ vertical graph ($+$ uniqueness)

Kloeckner and Mazzeo, 2017 $\rightsquigarrow$ regularity up to the boundary

Setting

Question 2

Given $\Gamma^{\pm} : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma^- \cup \Gamma^+$?

Simplest topology $\rightsquigarrow$ $M =$ annuli

Prop. emb. $\rightsquigarrow$ $\Gamma^- < \Gamma^+$

Setting

Question 2

Given $\Gamma^{\pm} : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma^- \cup \Gamma^+$?

Simplest topology $\rightsquigarrow$ $M =$ annuli

Prop. emb. $\rightsquigarrow$ $\Gamma^- < \Gamma^+$

Setting

Question 2

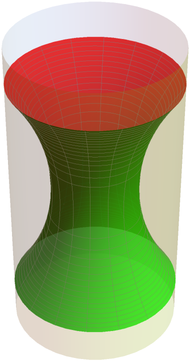
Given $\Gamma^{\pm} : \partial_\infty \mathbb{H}^2 \equiv \mathbb{S}^1 \to \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ cont.,
$\exists M \subset \mathbb{H}^2 \times \mathbb{R}$ prop. emb. min. sup. s.t. $\partial_\infty M = \Gamma^- \cup \Gamma^+$?

Simplest topology $\rightsquigarrow$ $M =$ annuli

Prop. emb. $\rightsquigarrow$ $\Gamma^- < \Gamma^+$

Known examples

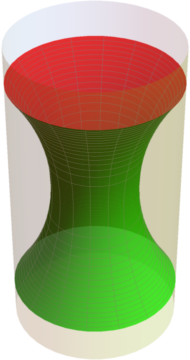
Vertical catenoids (Nelli-Rosenberg, 2002), $\Gamma^+ - \Gamma^- < \pi$



Nelli- Sa Earp- Toubiana (2015) $\rightsquigarrow$ uniqueness

Known examples

Vertical catenoids (Nelli-Rosenberg, 2002), $\Gamma^+ - \Gamma^- < \pi$



Nelli- Sa Earp- Toubiana (2015) $\rightsquigarrow$ uniqueness

An important property

Proposition

Let A be a prop. emb. min annulus s.t. $\partial_\infty A = \Gamma^- \cup \Gamma^+$.
Then neighborhoods of the ends of A are vertical graphs.

Ideas by Collin, Hauswirth and Rosenberg
"Slab Theorem" using "Dragging Lemma"

An important property

Proposition

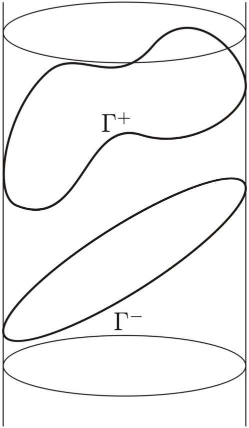
Let A be a prop. emb. min annulus s.t. $\partial_\infty A = \Gamma^- \cup \Gamma^+$.
Then neighborhoods of the ends of A are vertical graphs.

Ideas by Collin, Hauswirth and Rosenberg
"Slab Theorem" using "Dragging Lemma"

Non-existence results

Theorem

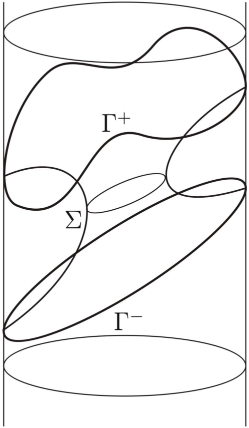
$\Gamma^+ - \Gamma^- > \pi \;\Rightarrow\; \nexists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$



Non-existence results

Theorem

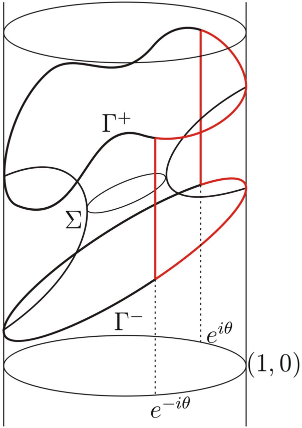
$\Gamma^+ - \Gamma^- > \pi \;\Rightarrow\; \nexists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$



Non-existence results

Theorem

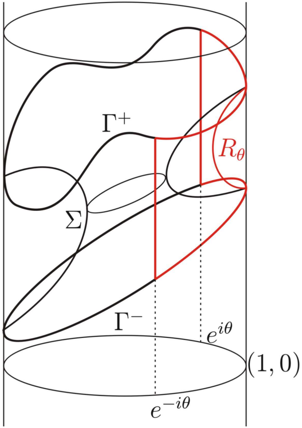
$\Gamma^+ - \Gamma^- > \pi \;\Rightarrow\; \nexists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$



Non-existence results

Theorem

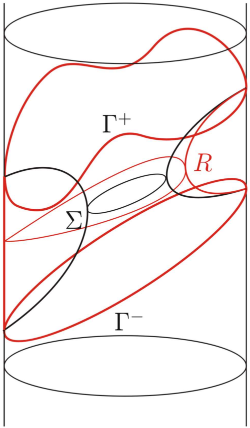
$\Gamma^+ - \Gamma^- > \pi \ \Rightarrow \ \nexists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$



Non-existence results

Theorem

$\Gamma^+ - \Gamma^- > \pi \;\Rightarrow\; \nexists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$



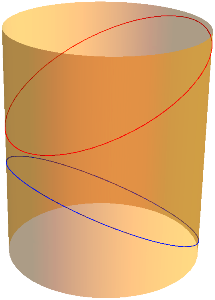
Non-existence results

Proposition

Sup. $\exists A$ *prop. emb. min. annulus s.t.* $\partial_\infty A = \Gamma^- \cup \Gamma^+$ *and* $\exists \theta_1 < \theta_2 \in [0, 2\pi)$ *s.t.:*

- Γ^+ *decreasing in* (θ_1, θ_2) *and increasing in* $(\theta_2, \theta_1 + 2\pi)$.
- Γ^- *increasing in* (θ_1, θ_2) *and decreasing in* $(\theta_2, \theta_1 + 2\pi)$.

Then $\Gamma^\pm$ *horizontal and* A *is a vertical catenoid.*



Space of annuli and projection map

$$\mathcal{B} = \{(\Gamma^-, \Gamma^+) \subset (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \ \mid 0 < \Gamma^+ - \Gamma^- < \pi\}$$

$$\mathcal{A} = \{A \subset \mathbb{H}^2 \times \mathbb{R} \text{ prop. emb. min. annulus} \mid \partial_\infty A \in \mathcal{B}\}$$

$$\begin{array}{rccc} \Pi : & \mathcal{A} & \to & \mathcal{B} \\ & A & \mapsto & \partial_\infty A \end{array} \qquad (\Pi \text{ is not onto})$$

Theorem

$\mathcal{A}$ is a Banach submanifold in the space of prop. emb. surf. in $\mathbb{H}^2 \times \mathbb{R}$ and Π is Fredholm of index 0.

Space of annuli and projection map

$$\mathcal{B} = \{(\Gamma^-, \Gamma^+) \subset (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \ \mid 0 < \Gamma^+ - \Gamma^- < \pi\}$$

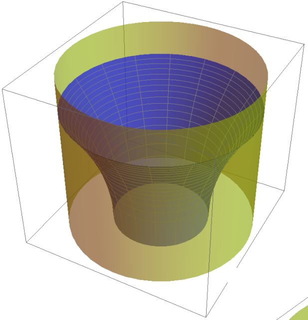
$$\mathcal{A} = \{A \subset \mathbb{H}^2 \times \mathbb{R} \text{ prop. emb. min. annulus} \mid \partial_\infty A \in \mathcal{B}\}$$

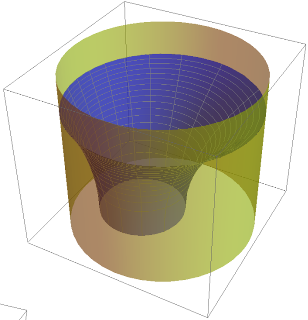
$$\begin{array}{rccc} \Pi : & \mathcal{A} & \to & \mathcal{B} \\ & A & \mapsto & \partial_\infty A \end{array} \qquad (\Pi \text{ is not onto})$$

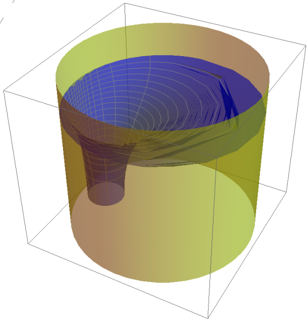
Theorem

$\mathcal{A}$ is a Banach submanifold in the space of prop. emb. surf. in $\mathbb{H}^2 \times \mathbb{R}$ and Π is Fredholm of index 0.

Π is not proper







Compactness

$\{A_n\} \subset \mathcal{A}$ s.t. $\Pi(A_n) = (\Gamma_n^-, \Gamma_n^+) \to (\Gamma^-, \Gamma^+) \in \mathcal{B}$, $A_n \to$?

We know $A_n \setminus (B(q_n, R_n) \times \mathbb{R})$ is the union of vertical graphs $u_n^{\pm}$.
Possible cases (up to a subsequence):

1. q_n, R_n independent of n
2. R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$
3. $R_n \to +\infty$

Compactness

$\{A_n\} \subset \mathcal{A}$ s.t. $\Pi(A_n) = (\Gamma_n^-, \Gamma_n^+) \to (\Gamma^-, \Gamma^+) \in \mathcal{B}$, $A_n \to$?

We know $A_n \setminus (B(q_n, R_n) \times \mathbb{R})$ is the union of vertical graphs $u_n^\pm$.

Possible cases (up to a subsequence):

1. q_n, R_n independent of n
2. R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$
3. $R_n \to +\infty$

$\{A_n\} \subset \mathcal{A}$ s.t. $\Pi(A_n) = (\Gamma_n^-, \Gamma_n^+) \to (\Gamma^-, \Gamma^+) \in \mathcal{B}$, $A_n \to$?

We know $A_n \setminus (B(q_n, R_n) \times \mathbb{R})$ is the union of vertical graphs $u_n^{\pm}$.
Possible cases (up to a subsequence):

1. q_n, R_n independent of n
2. R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$
3. $R_n \to +\infty$

Compactness

$\{A_n\} \subset \mathcal{A}$ s.t. $\Pi(A_n) = (\Gamma_n^-, \Gamma_n^+) \to (\Gamma^-, \Gamma^+) \in \mathcal{B}$, $A_n \to$?

We know $A_n \setminus (B(q_n, R_n) \times \mathbb{R})$ is the union of vertical graphs $u_n^\pm$.
Possible cases (up to a subsequence):

1. q_n, R_n independent of n
2. R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$
3. $R_n \to +\infty$

Compactness

$\{A_n\} \subset \mathcal{A}$ s.t. $\Pi(A_n) = (\Gamma_n^-, \Gamma_n^+) \to (\Gamma^-, \Gamma^+) \in \mathcal{B}$, $A_n \to$?

We know $A_n \setminus (B(q_n, R_n) \times \mathbb{R})$ is the union of vertical graphs $u_n^{\pm}$.
Possible cases (up to a subsequence):

1. q_n, R_n independent of n
2. R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$
3. $R_n \to +\infty$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:

Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q,R)$

$\Rightarrow M_n = A_n \cap (B(q,R) \times \mathbb{R})$ are compact annuli

with ∂M_n locally uniformly bounded

and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q,R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$

and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$

By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:

Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q,R)$

$\Rightarrow M_n = A_n \cap (B(q,R) \times \mathbb{R})$ are compact annuli
with ∂M_n locally uniformly bounded
and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q,R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$
and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$
By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:
Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q, R)$
$\Rightarrow M_n = A_n \cap (B(q, R) \times \mathbb{R})$ are compact annuli
with ∂M_n locally uniformly bounded
and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q, R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$
and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$
By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:

Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q,R)$

$\Rightarrow M_n = A_n \cap (B(q,R) \times \mathbb{R})$ are compact annuli

with ∂M_n locally uniformly bounded

and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q,R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$

and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$

By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:

Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^{\pm} \to u^{\pm}$ on $\mathbb{H}^2 \setminus B(q, R)$
$\Rightarrow M_n = A_n \cap (B(q, R) \times \mathbb{R})$ are compact annuli
with ∂M_n locally uniformly bounded
and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q, R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$
and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$
By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:

Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q,R)$

$\Rightarrow M_n = A_n \cap (B(q,R) \times \mathbb{R})$ are compact annuli

with ∂M_n locally uniformly bounded

and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q,R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$

and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$

By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem

In case 1, $A_n \to A \in \mathcal{A}$ (subsequence) and $\Pi(A) = (\Gamma^-, \Gamma^+)$.

Proof:
Elliptic estim. + Arzelà-Ascoli Th. $\rightsquigarrow$ $u_n^\pm \to u^\pm$ on $\mathbb{H}^2 \setminus B(q, R)$
$\Rightarrow M_n = A_n \cap (B(q, R) \times \mathbb{R})$ are compact annuli
with ∂M_n locally uniformly bounded
and the area blowup set Z of A_n lies in the solid cylinder.

Take $C \subset (\mathbb{H}^2 \setminus \overline{B(q, R)}) \times \mathbb{R}$ catenoid symmetric w.r.t. $\mathbb{H}^2 \times \{t_0\}$
and the continuous family of catenoids going from C to $\mathbb{H}^2 \times \{t_0\}$
By White's Th., $Z \cap (\mathbb{H}^2 \times \{t_0\}) = \emptyset$

This holds $\forall t_0 \in \mathbb{R} \Rightarrow Z = \emptyset$

Compactness

CASE 1: $q_n = q \in \mathbb{H}^2$, $R_n = R > 0$, independent of n

Theorem (White)

$\{M_n\} \subset \mathbb{H}^2 \times \mathbb{R}$ prop. embeb. min. surf.
Sup. the area and genus of M_n are uniformly bounded indep. of n.
Then (a subseq.) M_n converge to a prop. embeb. min. surf. M.
Moreover, for each c.c. Σ of M either:

1. *the convergence to Σ is smooth with multiplicity one, or*
2. *the convergence to Σ is smooth (with some multiplicity > 1) away from a discrete set.*

Compactness

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$.
T_n = hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:
Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.
By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs
$\Rightarrow$ $\Sigma_n \to C$ = annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$
$\Rightarrow$ C is a catenoid

Compactness

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. T_n = hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:

Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.

By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs

$\Rightarrow \Sigma_n \to C$ = annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$

$\Rightarrow C$ is a catenoid

Compactness

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. $T_n =$ hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:

Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.

By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs

$\Rightarrow \Sigma_n \to C =$ annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$

$\Rightarrow C$ is a catenoid

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. T_n = hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:
Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.
By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs
$\Rightarrow$ $\Sigma_n \to C$ = annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$
$\Rightarrow$ C is a catenoid

Compactness

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. $T_n =$ hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:
Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.
By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs
$\Rightarrow \Sigma_n \to C =$ annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$
$\Rightarrow C$ is a catenoid

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. T_n = hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:
Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.
By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs
$\Rightarrow \Sigma_n \to C$ = annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$
$\Rightarrow C$ is a catenoid

Compactness

CASE 2: R_n independent of n, $q_n \to q_\infty \in \partial_\infty \mathbb{H}^2$

Theorem

In case 2, $A_n \to D^- \cup D^+$, min. graphs bounded by $\Gamma^\pm$, together with a vertical segment joining $(q_\infty, t^-) \in \Gamma^-$ to $(q_\infty, t^+) \in \Gamma^+$. $T_n =$ hyp. transl. mapping q_n to $0 \Rightarrow \Sigma_n = T_n(A_n) \to C$ a catenoid. In particular, $t^+ - t^- < \pi$.

Proof:
Since $\mathbb{H}^2 \setminus B(q_n, R) \to \mathbb{H}^2$, $u_n^\pm \to u^\pm$ entire minimal graphs.
By uniqueness, $A_n \to D^- \cup D^+$.

Σ_n annulus and $\Sigma_n \setminus (B(0,R) \times \mathbb{R})$ union of 2 graphs
$\Rightarrow \ \Sigma_n \to C =$ annulus bounded by $\mathbb{H}^2 \times \{t^\pm\}$
$\Rightarrow \ C$ is a catenoid

CASE 3: $R_n \to +\infty$

Theorem

In case 3, $\exists \ell \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ vertical segment in the limit set of A_n s.t. $\exists p_n \in A_n$ with hor. normal vector converging to a point in ℓ and $T_n =$ hyp. transl. mapping p_n to $(0, t_n) \Rightarrow \Sigma_n = T_n(A_n) \to \Sigma =$ generalized catenoid. In particular, $length(\ell) = \pi$.

CASE 3: $R_n \to +\infty$

Theorem

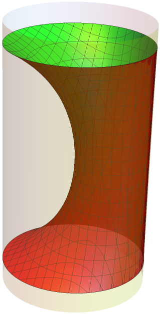
In case 3, $\exists \ell \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *vertical segment in the limit set of* A_n *s.t.* $\exists p_n \in A_n$ *with hor. normal vector converging to a point in* ℓ *and* $T_n =$ *hyp. transl. mapping* p_n *to* $(0, t_n) \Rightarrow \Sigma_n = T_n(A_n) \to \Sigma =$ *generalized catenoid. In particular,* $length(\ell) = \pi$.

Compactness

CASE 3: $R_n \to +\infty$

Theorem

In case 3, $\exists \ell \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *vertical segment in the limit set of* A_n*s.t.* $\exists p_n \in A_n$ *with hor. normal vector converging to a point in* ℓ *and* $T_n =$ *hyp. transl. mapping* p_n *to* $(0, t_n) \Rightarrow \Sigma_n = T_n(A_n) \to \Sigma =$ *generalized catenoid. In particular,* *length*$(\ell) = \pi$.



Existence

$G \subset$Isom$(\mathbb{H}^2 \times \mathbb{R})$ finite group leaving invariant some catenoid C.

Sup. no element of $\mathcal{J}^0(C)$ is left invariant by G.

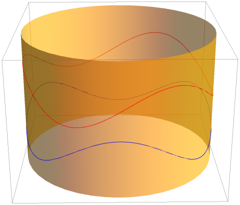
$\mathcal{J}^0(C)$ is spanned by $\varphi_1 = \left(\frac{1}{r} - r\right) cos\theta$ and $\varphi_2 = \left(\frac{1}{r} - r\right) sin\theta$, generated by hyperbolic translations

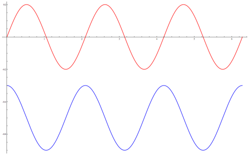
$G \subset$Isom$(\mathbb{H}^2 \times \mathbb{R})$ finite group leaving invariant some catenoid C.

Sup. no element of $\mathcal{J}^0(C)$ is left invariant by G.

$\mathcal{J}^0(C)$ is spanned by $\varphi_1 = \left(\frac{1}{r} - r\right) \cos\theta$ and $\varphi_2 = \left(\frac{1}{r} - r\right) \sin\theta$, generated by hyperbolic translations

Example:
$G = \langle R_k \rangle$, $R_k =$ rotation by $2\pi/k$ around the axis of C, $k \geq 2$





Existence

$$\mathcal{B}_G = \{\Gamma = (\Gamma^-, \Gamma^+) \in \mathcal{B},\ G\text{-invariant}\}$$
$$\mathcal{A}_G = \{A \in \mathcal{A},\ G\text{-invariant}\}, \qquad \Pi_G = \Pi|_{\mathcal{A}_G}$$

Theorem

Π_G is onto, i.e. $\forall \Gamma \in \mathcal{B}_G$, $\exists A \in \mathcal{A}_G$ s.t. $\Pi(A) = \Gamma$.

Proof:

$\mathcal{B}_G$ and $\mathcal{A}_G$ are Banach manifolds and Π_G is Fredholm of index 0.
Moreover, Π_G is proper: $\{A_n\} \subset \mathcal{A}_G$ s.t. $\Pi(A_n) \to \Gamma \in \mathcal{B}_G$.

Boundary cond. avoids case 3; symmetry avoids case 2.
$\{A_n\}$ in case 1 $\Rightarrow A_n \to A = G$-invariant, $\Pi(A) = \Gamma$.

Π_G has a well-defined $\mathbb{Z}$-valued degree

$$\deg(\Pi_G) = \sum_{A \in \Pi_G^{-1}(\Gamma)} (-1)^{\text{index}(A)}$$

where Γ is any reg. val. of Π_G and index$(A) = \#$ neg. eigenval. of Jacobi operator acting on G-inv. functions

Existence

$\mathcal{B}_G = \{\Gamma = (\Gamma^-, \Gamma^+) \in \mathcal{B},\ G\text{-invariant}\}$
$\mathcal{A}_G = \{A \in \mathcal{A},\ G\text{-invariant}\}, \quad \Pi_G = \Pi|_{\mathcal{A}_G}$

Theorem

Π_G is onto, i.e. $\forall \Gamma \in \mathcal{B}_G$, $\exists A \in \mathcal{A}_G$ s.t. $\Pi(A) = \Gamma$.

Proof:

$\mathcal{B}_G$ and $\mathcal{A}_G$ are Banach manifolds and Π_G is Fredholm of index 0.

Moreover, Π_G is proper: $\{A_n\} \subset \mathcal{A}_G$ s.t. $\Pi(A_n) \to \Gamma \in \mathcal{B}_G$.

Boundary cond. avoids case 3; symmetry avoids case 2.
$\{A_n\}$ in case 1 $\Rightarrow A_n \to A = G$-invariant, $\Pi(A) = \Gamma$.

Π_G has a well-defined $\mathbb{Z}$-valued degree

$$\deg(\Pi_G) = \sum_{A \in \Pi_G^{-1}(\Gamma)} (-1)^{\text{index}(A)}$$

where Γ is any reg. val. of Π_G and index$(A) = \#$ neg. eigenval. of Jacobi operator acting on G-inv. functions

Existence

$\mathcal{B}_G = \{\Gamma = (\Gamma^-, \Gamma^+) \in \mathcal{B},\ G\text{-invariant}\}$
$\mathcal{A}_G = \{A \in \mathcal{A},\ G\text{-invariant}\}, \quad \Pi_G = \Pi|_{\mathcal{A}_G}$

Theorem

Π_G is onto, i.e. $\forall \Gamma \in \mathcal{B}_G$, $\exists A \in \mathcal{A}_G$ s.t. $\Pi(A) = \Gamma$.

Proof:
$\mathcal{B}_G$ and $\mathcal{A}_G$ are Banach manifolds and Π_G is Fredholm of index 0.
Moreover, Π_G is proper: $\{A_n\} \subset \mathcal{A}_G$ s.t. $\Pi(A_n) \to \Gamma \in \mathcal{B}_G$.

Boundary cond. avoids case 3; symmetry avoids case 2.
$\{A_n\}$ in case 1 $\Rightarrow A_n \to A = G$-invariant, $\Pi(A) = \Gamma$.

Π_G has a well-defined $\mathbb{Z}$-valued degree

$$\deg(\Pi_G) = \sum_{A \in \Pi_G^{-1}(\Gamma)} (-1)^{\text{index}(A)}$$

where Γ is any reg. val. of Π_G and $\text{index}(A) = \#$ neg. eigenval. of Jacobi operator acting on G-inv. functions

Existence

$\mathcal{B}_G = \{\Gamma = (\Gamma^-, \Gamma^+) \in \mathcal{B},\ G\text{-invariant}\}$
$\mathcal{A}_G = \{A \in \mathcal{A},\ G\text{-invariant}\}, \quad \Pi_G = \Pi|_{\mathcal{A}_G}$

Theorem

Π_G is onto, i.e. $\forall \Gamma \in \mathcal{B}_G$, $\exists A \in \mathcal{A}_G$ s.t. $\Pi(A) = \Gamma$.

Proof:
$\mathcal{B}_G$ and $\mathcal{A}_G$ are Banach manifolds and Π_G is Fredholm of index 0.
Moreover, Π_G is proper: $\{A_n\} \subset \mathcal{A}_G$ s.t. $\Pi(A_n) \to \Gamma \in \mathcal{B}_G$.

> *Boundary cond. avoids case 3; symmetry avoids case 2.*
> *$\{A_n\}$ in case 1 $\Rightarrow A_n \to A = G$-invariant, $\Pi(A) = \Gamma$.*

Π_G has a well-defined $\mathbb{Z}$-valued degree

$$\deg(\Pi_G) = \sum_{A \in {\Pi_G}^{-1}(\Gamma)} (-1)^{\text{index}(A)}$$

where Γ is any reg. val. of Π_G and index$(A) = \#$ neg. eigenval. of Jacobi operator acting on G-inv. functions

Existence

$\mathcal{J}^0(C)$ is spanned by $\varphi_1 = \left(\frac{1}{r} - r\right) cos\theta$ and $\varphi_2 = \left(\frac{1}{r} - r\right) sin\theta$, generated by hyperbolic translations

$\Rightarrow$ in this context, C is nondegenerate

$\Rightarrow$ Implicit Function Th. gives us local existence.

A neighborhood of C in $\mathcal{A}_G$ projects diffeomorphically to a neighborhood $\mathcal{U}$ in $\mathcal{B}_G$.

Sard-Smale Th. $\Rightarrow \exists \Gamma \in \mathcal{U}$ reg. val. of Π_G s.t. ${\Pi_G}^{-1}(\Gamma) \neq \emptyset$.

$\partial_\infty C$ bounds only $C \Rightarrow \deg(\Pi_G) = \pm 1 \neq 0$.

$\mathcal{J}^0(C)$ is spanned by $\varphi_1 = \left(\frac{1}{r} - r\right) cos\theta$ and $\varphi_2 = \left(\frac{1}{r} - r\right) sin\theta$, generated by hyperbolic translations
$\Rightarrow$ in this context, C is nondegenerate
$\Rightarrow$ Implicit Function Th. gives us local existence.
A neighborhood of C in $\mathcal{A}_G$ projects diffeomorphically to a neighborhood $\mathcal{U}$ in $\mathcal{B}_G$.
Sard-Smale Th. $\Rightarrow \exists \Gamma \in \mathcal{U}$ reg. val. of Π_G s.t. ${\Pi_G}^{-1}(\Gamma) \neq \emptyset$.

$\partial_\infty C$ bounds only $C \Rightarrow$ $\deg(\Pi_G) = \pm 1 \neq 0$.

Existence

$\mathcal{J}^0(C)$ is spanned by $\varphi_1 = \left(\frac{1}{r} - r\right) cos\theta$ and $\varphi_2 = \left(\frac{1}{r} - r\right) sin\theta$, generated by hyperbolic translations
$\Rightarrow$ in this context, C is nondegenerate
$\Rightarrow$ Implicit Function Th. gives us local existence.
A neighborhood of C in $\mathcal{A}_G$ projects diffeomorphically to a neighborhood $\mathcal{U}$ in $\mathcal{B}_G$.
Sard-Smale Th. $\Rightarrow \exists \Gamma \in \mathcal{U}$ reg. val. of Π_G s.t. ${\Pi_G}^{-1}(\Gamma) \neq \emptyset$.

$\partial_\infty C$ bounds only $C \Rightarrow \deg(\Pi_G) = \pm 1 \neq 0$.

Open problem

Open problem

Sharp conditions for the existence of this Plateau problem at infinity?

Existence: GENERAL RESULT

$A \in \mathcal{A}^*$ if A is proper, Alexandrov embedded min. annulus with embedded ends and

$$u_r^-(1,\theta) - v_r^-(1,\theta) < 0, \quad \forall \theta \in \mathbb{S}^1$$

where u^- is the graph parametrizing the bottom end of A and v^- is the entire minimal graph bounded by Γ^-

Existence: GENERAL RESULT

$$\widetilde{\Pi} : \mathcal{A}^* \times \mathbb{R} \times \mathbb{C} \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{C}$$

$$\widetilde{\Pi}(A, a, z) = \left(\Gamma^-, \Gamma^+ + a + \mathrm{Re}(ze^{i\theta}), G_0(A), G(A)\right)$$

where $G_0(A) = f_0(A) - f_0(A)^{-1}$,

$f_0(A) = \mathbf{Flux}(A, \Gamma^-, \partial_t) = \int_{\mathbb{S}^1} u_r^-(1,\theta)d\theta$

and $G(A) = \frac{1}{f_0(A)} \int_{\mathbb{S}^1} e^{i\theta}\left(u_r^-(1,\theta) - v_r^-(1,\theta)\right) d\theta \in \mathbb{H}^2$

is called the center of A.

Existence: GENERAL RESULT

$$\widetilde{\Pi} : \mathcal{A}^* \times \mathbb{R} \times \mathbb{C} \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{C}$$

$$\widetilde{\Pi}(A, a, z) = \left(\Gamma^-, \Gamma^+ + a + \mathrm{Re}(ze^{i\theta}), G_0(A), G(A)\right)$$

where $G_0(A) = f_0(A) - f_0(A)^{-1}$,

$$f_0(A) = \textbf{Flux}(A, \Gamma^-, \partial_t) = \int_{\mathbb{S}^1} u_r^-(1,\theta)\mathrm{d}\theta$$

and $G(A) = \frac{1}{f_0(A)} \int_{\mathbb{S}^1} e^{i\theta}\left(u_r^-(1,\theta) - v_r^-(1,\theta)\right)\mathrm{d}\theta \in \mathbb{H}^2$

is called the **center** of A.

Existence: GENERAL RESULT

$$\widetilde{\Pi} : \mathcal{A}^* \times \mathbb{R} \times \mathbb{C} \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{C}$$

$$\widetilde{\Pi}(A, a, z) = \left(\Gamma^-, \Gamma^+ + a + \mathrm{Re}(ze^{i\theta}), G_0(A), G(A)\right)$$

where $G_0(A) = f_0(A) - f_0(A)^{-1}$,

$$f_0(A) = \mathbf{Flux}(A, \Gamma^-, \partial_t) = \int_{\mathbb{S}^1} u_r^-(1,\theta)\mathrm{d}\theta$$

and $G(A) = \frac{1}{f_0(A)} \int_{\mathbb{S}^1} e^{i\theta} \left(u_r^-(1,\theta) - v_r^-(1,\theta)\right) \mathrm{d}\theta \in \mathbb{H}^2$

is called the **center** of A.

Existence: GENERAL RESULT

$$\widetilde{\mathcal{A}}^* = \widetilde{\Pi}^{-1}\left((\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{H}^2\right)$$

Theorem

The map $\widetilde{\Pi} : \widetilde{\mathcal{A}^*} \times \mathbb{R} \times \mathbb{C} \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{H}^2$ *is:*

1. *Locally invertible near any catenoid.*
2. *A proper Fredholm map of index 0 and degree* ± 1.

In particular, $\forall(\Gamma^-, \Gamma^+) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$, *there exist constants* a_0, a_1, a_2 *s.t.* $(\Gamma^-, \Gamma^+ + a_0 + a_1 \cos\theta + a_2 \sin\theta)$ *bound a proper Alexandrov-embedded min. annulus with embedded ends.*

Remark

We can prescribe one of the boundary curves, the vertical flux and the center of the neck.

Existence: GENERAL RESULT

$$\widetilde{\mathcal{A}}^* = \widetilde{\Pi}^{-1}\left((\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{H}^2\right)$$

Theorem

The map $\widetilde{\Pi} : \widetilde{\mathcal{A}^*} \times \mathbb{R} \times \mathbb{C} \to (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times \mathbb{H}^2$ *is:*

1. *Locally invertible near any catenoid.*
2. *A proper Fredholm map of index 0 and degree* ± 1.

In particular, $\forall(\Gamma^-, \Gamma^+) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$, *there exist constants* a_0, a_1, a_2 *s.t.* $(\Gamma^-, \Gamma^+ + a_0 + a_1 \cos\theta + a_2 \sin\theta)$ *bound a proper Alexandrov-embedded min. annulus with embedded ends.*

Remark

We can prescribe one of the boundary curves, the vertical flux and the center of the neck.