Floer theory in spaces of stable pairs over Riemann surfaces

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Moduli spaces associated with Riemann surfaces

Let Z be a closed Riemann surface. Among the gauge-theoretic moduli spaces $M(Z)$ associated with it, those that are intrinsically compact Kähler manifolds include

- $N_d^\flat$, the projectively flat connections in a $U(2)$-bundle of odd degree d.
  Sending a connection A to the holomorphic structure defined by $\overline{\partial}_A$ defines a biholomorphic map to the moduli space N^{ss} of rank 2 semistable vector bundles.
- $V_{L,\tau}$, the space of *vortices* in a hermitian line bundle $L \to Z$ of degree d:

$$(A, \phi) : \quad \overline{\partial}_A \phi = 0, \quad iF_A + |\phi|^2 \eta = \tau \eta.$$

(η is a fixed area form on Z with $\int_Z \eta = 1$, and $\tau > 0$).
$\overline{\partial}_A$ defines a holomorphic structure in L making ϕ a holomorphic section, so we get a map

$$V_{L,\tau} \to \mathrm{Sym}^d Z, \quad [A, \phi] \mapsto \phi^{-1}(0).$$

This map is biholomorphic for $\tau > 2\pi d$. The resulting Kähler form on $\mathrm{Sym}^d(Z)$ lies in a class varying affine-linearly with τ.

Rank 2 vortices

- A *holomorphic pair* is a holomorphic vector bundle $V \to Z$, together with a non-trivial holomorphic section ϕ. Numerical parameters (r, d)=(rank, degree). $\mathrm{Sym}^d(Z)$ is a fine moduli space of $(1, d)$ holomorphic pairs.
- $V_{E,\tau}$, the space of *vortices* in a hermitian $\mathbb{C}^2$-bundle $E \to Z$:

$$(A, \phi): \quad \overline{\partial}_A \phi = 0, \quad iF_A + \frac{1}{2}(\phi \otimes \phi^*)\eta = \frac{1}{2}\tau\eta \,\mathrm{Id}.$$

  η is a fixed area form on Z, normalized to have total area 1, and $\tau > 0$.
- Bradlow (1990), Bradlow–Daskalopoulos (1993): $V_{E,\tau}$ is a compact Kähler manifold, and the map

$$V_{L,\tau} \to \{(2, d) \text{ holomorphic pairs})\}, \quad [A, \phi] \mapsto [\overline{\partial}_A, \phi]$$

  is biholomorphic onto the coarse moduli space of $(2, d)$ *σ-semistable pairs*, $\sigma = \frac{d}{2} - \tau$.

Stable pairs

- Fix $\sigma > 0$. A (rank, degree) $= (2, d)$ holomorphic pair (E, ϕ) is called *σ-semistable* if, for all line bundles $F \subset E$,
  1. $\deg F \leq \frac{d}{2} + \sigma$; and moreover
  2. $\deg F \leq \frac{d}{2} - \sigma$ if $\phi \in H^0(F)$.

  It's σ-stable if we can sharpen $\leq$ to $<$.
- There are coarse moduli spaces $M_{d,\sigma}$, fine for most σ. We fix a fiber Λ of the determinant submersion
  $\det\colon M_{d,\sigma} \to \mathrm{Pic}^d(Z)$, $[E, \phi] \mapsto \Lambda^2 E$, to define $M_{\Lambda,\sigma}$.
- Thaddeus (1992) gives a precise and beautiful description of the moduli spaces $M_{\Lambda,\sigma}$ which I'll review shortly.
- The compact Kähler manifolds $M_{\Lambda,\sigma}$ are the subject of this lecture.

Gauge theory vs. symplectic geometry

- The equations for flat connections and rank 1 vortices are dimensional reductions of equations in 4 dimensions with gauge symmetry: instanton, Seiberg–Witten with a closed, non-exact 2-form perturbation.
- The rank 2 vortex equations are (almost) the dimensional reductions of 4-dimensional non-abelian SW equations studied by Feehan–Leness and others.
- Instanton, SW invariants of 3- and 4-manifolds containing Z are intimately related to symplectic topology of $N^\flat(Z)$ and $\mathrm{Sym}^d Z$ respectively, in particular to Lagrangian submanifolds and holomorphic curves.
- When d is even, the moduli space $N^\flat$ of projectively flat connections is singular, and problematic for Floer theory. Instanton Floer theory is also hard to set up beyond the case of homology 3-spheres, because of problems with singularities.
- Aspiration: use a space of stable pairs $M_{\Lambda,\sigma}$ (with d even) as a substitute for $N^\flat$, and construct 3-manifold invariants via Floer theory in $M_{\Lambda,\sigma}$.

Structure of $M_{\Lambda,\sigma}$ (Thaddeus)

(E, ϕ) σ-semistable: for all line bundles $F \subset E$, $\deg F \leq \frac{d}{2} + \sigma$, and moreover $\deg F \leq \frac{d}{2} - \sigma$ if $\phi \in H^0(F)$.
Take $d \geq 0$ even.

- ϕ is always a section of *some* line bundle $F_\phi \subset E$ (of maximal degree). Since $\deg F_\phi \geq 0$, we have $\sigma \leq d/2$.
- We get a sequence of non-empty moduli spaces $M_i = M_{\Lambda,(d/2)-i-\epsilon}$, for $i = 0, 1, \ldots \frac{d}{2} - 1$ and $\epsilon \in (0, 1)$.
- In M_0, we must have $\deg F_\phi \leq 0$, so (F_ϕ, ϕ) is a deg 0 rank 1 holomorphic pair (must be $(\mathcal{O}_Z, 1)$), while E is an extension of $\mathcal{O}_Z$ by Λ. Must be non-split, but that's the only constraint. We get
$$M_0 = \mathbb{P}H^1(\Lambda^{-1}) = \mathbb{P}H^0(K_Z\Lambda)^*.$$
- In M_1, F_ϕ could have degree 1; the deg 1 holo. pairs form Z. In fact, M_1 is the blow-up of M_0 along Z embedded via $|K_Z\Lambda|$.
- M_{i+1} is a *flip* of M_i for $i > 0$.
- All are smooth projective of dimension $d + g - 2$; simply connected; Picard rank 2 for $i > 0$.

The last in the line

- We're most interested in the *last* in the sequence of flips, $M_{top} = M_{d/2-1}$. That is, $\sigma \in (0,1)$; (E,ϕ) *is σ-semistable* if E is a semistable bundle and F_ϕ does not destabilize E.
- There's an Abel–Jacobi map

$$M_{top} \to N^{ss}_{\Lambda}, \quad [E,\phi] \mapsto [E]$$

  whose fibers are the projective spaces $\mathbb{P}H^0(E)$.
- For $d > 2g-2$, Abel–Jacobi is surjective and we think of it as a sort of 'resolution', in that N^{ss}_d is singular (of dim $3g-3$) while M_{top} is non-singular (of dimension $g+d-2$).
- We'll focus on M_{top} because it's closest to the world of stable bundles and flat connections.

Which degree?

- Recall that Heegaard Floer theory is based on $\mathrm{Sym}^d Z$ with $d = g(Z)$. The reason for $d = g$ is that a handlebody U bounding Z defines interesting Lagrangian submanifolds of $\mathrm{Sym}^g Z$ specifically.
- These Lagrangians (which are tori) can be constructed
  1. *explicitly:* the product of g disjoint circles that bound in U;
  2. *implicitly:* as limits of solutions to the SW equations on the cylindrical completion of in $U \setminus B^3$, with a Taubes-type perturbation; or as iterated vanishing cycles of degenerations.
- The analogous degree for rank 2 stable pairs (and the rank 2 SW equations over handlebodies) turns out to be $d = 2g + 2$.
- From now, on M_Z denotes M_{top} for a fixed determinant Λ of degree $2g + 2$.
- It is smooth projective of dimension $3g$.
- *Fortuitous observation:* M_Z is Fano!
  Specific to $(d, \sigma) = (2g + 2, \text{small})$.

A non-abelian Heegaard Floer theory??

$M_Z = M_{top}$ for $d = 2g + 2$. Smooth projective, Fano of dim $3g$.

- That M_Z is Fano implies that any pair of simply connected embedded Lagrangians have well-defined Floer cohomology.
- *Conjecture: In degree $2g + 2$, a handlebody U bounding Z defines an embedded Lagrangian submanifold $L_U \subset M_Z$, diffeomorphic to $(S^3)^{\times g}$.*
- If true, these could be used to form a Heegaard-Floer type theory based on Floer cohomology for the pair of Lagrangians coming from a Heegaard splitting.
- When $g = 1$ (so $d = 4$), M_Z is the blow-up of $\mathbb{C}P^3$ along Z, embedded via a degree 4 linear system. The conjecture is true here (the Lagrangians are vanishing cycles for a Lefschetz pencil with M_Z as fiber). We haven't yet managed to prove it for $g > 1$.

Fibered 3-manifolds

- Gauge theory also has a symplectic interpretation on *fibered* 3-manifolds $Y^3 \to S^1$.
- Let Z be the fiber, ϕ the monodromy. For any $d \geq 0$, the *symplectic fixed point Floer homology group*, for the symplectic action of ϕ on $\mathrm{Sym}^d Z$, is isomorphic to a summand in the monopole Floer homology of Y (with suitable perturbations). The summand corresponds to a subset of the Spin^c-structures.
- This suggests that the fixed point Floer homology for the action of ϕ on rank 2 stable pairs is also worth exploring. All degrees d are of interest in this setting, but since we are interested in the Fukaya category of M_Z we shall also focus on the (related) fixed point Floer homology for M_Z.

Set up for fixed point Floer homology

Equivalent data:

- (M, ω, ϕ) cpt. manifold, symplectic form, symplectic automorphism

  $$\updownarrow$$

  $(T \to S^1, \Omega)$ proper fiber bundle, closed fiberwise-symp. 2-form.

  $$(M, \omega, \phi) \longrightarrow \text{mapping torus } (p_\phi \colon T_\phi \to S^1, \omega_\phi)$$

  $$\text{fiber, monodromy} \longleftarrow (p : T \to S^1, \Omega)$$

  Here $T_\phi = (M \times \mathbb{R})/(x,t) \sim (\phi(x), t+1)$ and $p_\phi^* \omega_\phi = \omega$.
  Monodromy is for the symplectic connection $H^\Omega = (\ker Dp)^\Omega$.
- Fixed points $\leftrightarrow$ horizontal sections
- Adding closed 2-form η, zero on fibers, $(T \to S^1, \Omega + \eta)$ gives symp. isotopy $(M, \omega, \{\phi_t\}_{t \in [0,1]})$.
  Flux $\phi_t \in H^1(M; \mathbb{R})$ lies in $\mathrm{im}(1 - \phi_0^*)$ iff η exact on T.

Fixed-point Floer homology

- To each monotone symplectic automorphism $\phi \in \mathrm{Aut}(M,\omega)$,

$$[\omega_\phi] = \lambda c_1(T^{vert} T_\phi) \in H^2(T_\phi;\mathbb{R}), \quad \lambda > 0,$$

  we can attach its fixed-point Floer homology $HF(M;\phi)$.
- Finitely generated, $(\mathbb{Z}/2)$-graded abelian group;
  Euler characteristic = Lefschetz number Λ_ϕ.
  Module over quantum cohomology $QH^*(M) = (H^*(M;\mathbb{Z}),\star)$.
- Invariant under isotopies $\{\phi_t\}$ with flux in $\mathrm{im}(1-\phi_0^*)|_{H^1(M;\mathbb{R})}$.
- If ϕ has non-degenerate fixed points,

$$HF(M;\phi) = H_*\left(CF_*(\phi),\partial_J\right), \quad CF_*(\phi) = \mathbb{Z}^{\mathrm{fix}\,\phi},$$

  graded by Lefschetz signs.
- Matrix entries $\langle \partial_J x_-, x_+\rangle$ count *J-holomorphic sections u* of $T_\phi \times \mathbb{R} \to S^1 \times \mathbb{R}$ with $\lim_{t\to\pm\infty} u(\cdot,t) = x_\pm$ (where J is a suitable translation-invariant almost complex structure).

Monodromy acting on stable pair spaces

- Let Z be a closed, connected, oriented surface and $\Lambda \to Z$ a complex line bundle. There's a central extension of the mapping class group $\Gamma = \pi_0 \mathrm{Diff}^+(Z)$,

$$1 \to H^1(Z;\mathbb{Z}) \to \widetilde{\Gamma} \to \Gamma \to 1.$$

$\widetilde{\Gamma} := \{(\phi, \tilde{\phi}) \text{ up to isotopy}\}$: $\phi \in \mathrm{Diff}^+(Z)$ and $\tilde{\phi} \colon \Lambda \xrightarrow{\cong} \phi^*\Lambda$.

- Fix a complex structure in Z and a holomorphic structure in Λ. Let $M = M_{\Lambda,\sigma}$ be the space of σ-stable pairs over Z with determinant Λ. Let ω_M be a Kähler form.
- There's a homomorphism

$$\mu \colon \widetilde{\Gamma} \to (\mathrm{Aut}\,/\mathrm{Ham})(M, \omega) :$$

constructed as follows:

  - Build from $\tilde{\phi}$ a line bundle $\Lambda_{\tilde{\phi}} \to T_\phi$. Choose fiberwise complex structure in T_ϕ, holomorphic structure in $\Lambda_{\tilde{\phi}}$.
  - Associated bundle M-bundle $M_\phi \to S^1$ has $H^2(M_\phi) = H^2(M)$.
  - Choose any closed, fiberwise Kähler 2-form Ω in M_ϕ extending ω_M. Take monodromy.

Stable pair Floer homology

- When $\deg \Lambda = 2g_Z + 2$, and ω_M an anticanonical Kähler form, $\Phi := \mu(\tilde{\phi})$ is a monotone symplectic automorphism. Define

  $$HSP(\tilde{\phi}) := HF(M, \Phi),$$

  a $\mathbb{Z}/2$-graded abelian group, module over $QH^*(M)$.
- It breaks into generalized eigenspaces for $c_1(M) \star \cdot$:

  $$HSP(\tilde{\phi}) \otimes \mathbb{C} = \bigoplus_{\lambda} HSP(\tilde{\phi}; \mathbb{C})_{\lambda},$$

  Non-zero summands can only be for λ zero or an eigenvalue of $c_1(M) \star \cdot$ acting on $QH^*(M; \mathbb{C})$.

The genus 1 case: quantum cohomology

When Z is an elliptic curve, $M_Z = \mathrm{Bl}_Z(\mathbb{C}P^3)$. Here $Z = Q_0 \cap Q_1$ (complete intersection of quadric surfaces).

Proposition

The generalized eigenspace decomposition for $c_1(M_Z) \star \cdot$ acting on $QH^*(M_Z)$ is as follows:

$$\begin{aligned} QH^*(M_Z) \otimes \mathbb{C} = QH_{-1} & & \textit{(dim 4)} \\ \oplus QH_0 \oplus QH_8 \oplus QH_{-4-4i} \oplus QH_{-4+4i} & & \textit{(lines)}. \end{aligned}$$

There is a $\mathbb{C}$-algebra isomorphism $QH^*_{-1} \cong H^*(Z;\mathbb{C})$. Thus, as algebras,

$$QH^*(M_Z) \otimes \mathbb{C} = H^*(Z;\mathbb{C}) \oplus \mathbb{C}^4.$$

Proof is by direct calculation.
The four simple eigenvalues agree with critical values of the Hori–Vafa–Givental mirror superpotential.

The genus 1 case: Floer homology

Theorem (A. Lee–P.)

For Z of genus 1 and $\widetilde{\phi} \in \widetilde{\Gamma}$ homologically non-degenerate, there are isomorphisms of $(\mathbb{Z}/2)$-graded abelian groups

$$HSP(\widetilde{\phi}) \cong HSP(\widetilde{\phi})_{-1} \oplus \mathbb{Z}^4_{even}$$
$$HSP(\widetilde{\phi})_{-1} \cong HF(Z;\phi).$$

Homologically non-degenerate means $\phi^* - 1$ invertible on $H^1(Z;\mathbb{Q})$. Equivalently, ϕ is not a power of a Dehn twist.
Under this condition:

$$HSP(\widetilde{\phi}) \cong HSP(\widetilde{\phi})_{-1} \oplus \mathbb{Z}^4_{even}$$
$$HSP(\widetilde{\phi})_{-1} \cong HF(Z;\phi).$$

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Notes on $HF(Z;\phi)$:

- In the homologically non-degenerate case, it's $\mathbb{Z}^F$. Here $F = (\phi^* - 1)^{-1}(L)/L$ where $L = H^1(Z;\mathbb{Z})$.
- It lives in degree d, where $(-1)^d = \text{sign}\det(\phi^* - 1)$.
- Y.-J. Lee–Taubes: it's SW monopole Floer homology for T_ϕ, summed over Spin^c-structures $\mathfrak{s}$ with $c_1(\mathfrak{s})[Z] = [2]$, with negative monotone perturbations.

Proving the first clause: $HSP(\widetilde{\phi}) = HSP(\widetilde{\phi})_{-1} \oplus \mathbb{Z}^4$

- A certain lift $\widetilde{\tau} \in \widetilde{\Gamma}$ of a Dehn twist $\tau \in \Gamma$ acts on M_Z by a Dehn twist around a Lagrangian 3-sphere V.
- The count $m^0(V)$ of Maslov 2 holomorphic discs on V is necessarily an eigenvalue of $c_1(M_Z) \star \cdot$. By an argument of I. Smith, $m^0(V) = -1$.
- It follows that $c_1 + I$ is nilpotent on $HF(V, L)$ for any other monotone Lagrangian L with $m^0(V) = -1$.
- There is an exact triangle

$$\cdots \to HF(V, \mu(\widetilde{\phi})(V)) \to HSP(\widetilde{\phi}) \to HSP(\widetilde{\tau} \circ \widetilde{\phi}) \to \cdots .$$

  Taking the sum of all the generalized eigenspaces for eigenvalues $\lambda \neq -1$, the sequence remains exact but the first term dies.
- Hence for any composite of lifted Dehn twists, this part of HSP is the same as for the identity: $\mathbb{Z}^4_{even}$.

$HSP(\widetilde{\phi})_{-1} \cong HF(Z;\phi)$: how we *don't* prove it

- I. Smith uses Lagrangian correspondences to embed the Fukaya category of a genus $g > 1$ surface into that of the blow-up of $\mathbb{C}P^{2g+1}$ along an intersection of two quadrics.
- There's still a Lagrangian correspondence from Z to M_Z, but it appears that it *does not* induce a functor

  $$\mathcal{F}(Z) \to \mathcal{F}(M_Z)_{-1}$$

  because of holomorphic discs attached to the correspondence. (Perhaps the obstruction can be cancelled by a bulk deformation of $\mathcal{F}(Z)$—cf. ideas in a slightly different context of Fukaya.)

Toy model: Morse theory on blow-ups

- Let $\widetilde{X}$ be the blow-up of a complex manifold X along a complex-codimension 2 submanifold Y.
- Let f be a Morse function on X, generic in that it has no critical points on Y while $f|_Y$ is Morse. Its pullback $\widetilde{f}$ to $\widetilde{X}$ is again Morse.
- On the exceptional divisor $E = \mathbb{P}N_{Y/X}$, $\widetilde{f}$ has exactly one critical point λ_y over each critical point $y \in \mathrm{crit}(f|_Y)$. Namely, λ_y is the unique complex line in $(N_{Y/X})_y$ contained in $\ker(D_y f \colon N_y Y \to \mathbb{R})$.
- We have $\mathrm{ind}_{\tilde{X}}(\lambda_y) = \mathrm{ind}_Y(y) + 2$.
- Hence if f and $f|_Y$ are perfect Morse functions, so is $\widetilde{f}$.

Computing the chain complex

- The mapping torus T_Φ of $\Phi = \mu(\widetilde{\phi})$ is the 'family blow-up', relative to S^1, of $\mathbb{C}P^3 \times S^1$ along T_ϕ.
- Using an explicit model for the symplectic blow-up, we can arrange that the fixed points of the symplectic monodromy are in bijection with those of ϕ, together with 4 coming from a hamiltonian automorphism of $\mathbb{C}P^3$. (This is much like the toy model.)
- Take ϕ to be the action of a homologically non-degenerate element in $SL_2(\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$. When $\det(\phi^* - 1)|_{H^1(Z;\mathbb{Q})} > 0$, all fixed points are even, so the Floer differential is trivial and we're done.
- When $\det(\phi^* - 1)|_{H^1(Z;\mathbb{Q})} < 0$, there are exactly 4 even fixed points. The differential on the Floer complex must be trivial so as to have rank $HSP_{even} \geq 4$.
- I hid a snag with this argument...

Continuity of Floer homology

- Snag: The explicit model is for *low-weight blow-ups* (i..e cohomology classes $[\omega_t] = 4h - t[E]$ for t small), while Floer homology was defined for an *anticanonical* symplectic form ω_1.
- We choose to handle this using *continuity of Floer homology.*
- Fixed-point Floer homology can be 'classically' defined for automorphisms of any compact symplectic 6-manifold: the continuity maps used to prove invariance are not available because of bubbling.
- *Continuity principle*, Y.-J. Lee, Usher: In a family $(M, \omega_t, \Phi_t)_{t\in[0,1]}$ where all $HF(\Phi_t)$ are well-defined over the same field, rank $HF(\Phi_t)$ is constant provided that the symplectic action A_t on the *period group* P varies in a simple way: $A_t = f(t)A_{t_0}$, where $f(t) \geq 0$.
- Use this principle to see that we can deform from low-weight blow-up forms ω_t to an anticanonical form ω_1.
- Avoid bubbling in this borderline case by using Kähler forms and keeping the chosen complex structure unchanged (up to small perturbations) through the deformation.

Where from here?

- Higher genus? M_Z contains an interesting codimension g submanifold: extensions

$$0 \to F \to E \to \Lambda F_\phi \to 0$$

  where (F, ϕ) is a holomorphic pair of the highest allowed degree, g. It's a $\mathbb{P}^g$-bundle over $\mathrm{Sym}^g Z$. *Guess*: $HSP(\phi)$ contains g copies of the fixed-point Floer cohomology action on $\mathrm{Sym}^g Z$, coming from fixed points here.
  This locus hints at a relationship with Heegaard Floer theory.
- We can also obtain results on Lagrangian Floer cohomology for $g = 1$, and again see a relation with SW theory. The results for $g = 1$ are consistent with the notion that M_Z is a space of interest with respect to Floer-theoretic invariants of 3-manifolds.
- The critical next step is the construction of embedded Lagrangian submanifolds from handlebodies. I'm working on this, by means of degenerations of Z.
- If it can be done one gets a well-defined Floer cohomology group from any Heegaard splitting. If it is stabilization-invariant, one gets a 3-manifold invariant.