A Riemann-Roch-Grothendieck theorem in Bott-Chern cohomology

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Princeton, April 21st 2014

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- In general $H_{\mathrm{BC}}^{\cdot}(X,\mathbf{C})$ strictly finer than $H_{\mathrm{DR}}^{\cdot}(X,\mathbf{C})$.
- $H_{\mathrm{BC}}^{(=)}(S,\mathbf{R}) = \bigoplus_{0\le p\le n} H_{\mathrm{BC}}^{(p,p)}(S,\mathbf{R})$.
- Holomorphic vector bundles have characteristic classes in $H_{\mathrm{BC}}^{(=)}(S,\mathbf{R})$ (Bott and Chern).

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$$\mathrm{ch}_{\mathrm{BC}}\left(R^{\cdot}p_*F\right) = p_*\left[\mathrm{Td}_{\mathrm{BC}}\left(TX\right)\mathrm{ch}_{\mathrm{BC}}\left(F\right)\right] \text{ in } H^{(=)}_{\mathrm{BC}}\left(S, \mathbf{R}\right).$$

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Also $c_{1,\mathrm{BC}}\left(\det R^{\cdot}p_*F\right) = p_*\left[\mathrm{Td}_{\mathrm{BC}}\left(TX\right)\mathrm{ch}_{\mathrm{BC}}\left(F\right)\right]^{(1,1)}$.

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- Our goal is to prove this result in full generality.

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- $\ldots$ so as to obtain a nondegenerate Hermitian form.

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- If $\dim_{\mathbf{R}} M = 2$, $\eta\left(\alpha^{(0)}, \beta^{(2)}\right) = -i \int_M \alpha \wedge \overline{\beta} \ldots$

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- η is a Hermitian form of signature (∞, ∞).

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- If M complex, $\overline{\partial}^* = \partial, \partial^* = \overline{\partial}$.

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- Curvature R^{E} is the Hodge Laplacian $\frac{1}{2}\left[\nabla^{E}, \nabla^{E*}\right]$.

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- $[d, d^*] = 0, \left[\overline{\partial}, \overline{\partial}^*\right] = -i\overline{\partial}\partial\omega.$
- Holomorphic Laplacian vanishes if and only if $\overline{\partial}\partial\omega = 0.$

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- ...which have the best features of both.

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- Second case: S is a point.
- Third case: Kähler fibration.

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- The theorem to be proved is the known fact $1 = 1$.

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- An aside: how to prove RRH analytically while preserving $\overline{\partial}^X$?

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- ...with $\alpha_0 = p_*\left[\operatorname{Td}\left(TX, g^{TX}\right)\operatorname{ch}\left(F, g^F\right)\right], \alpha_\infty = \operatorname{ch}\left(Rp_*F, g^{Rp_*F}\right)$.

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- For the c_1, curvature theorem for Quillen metrics.

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- Forget about the Kähler property...
- ...and formally imitate the proof of RRG in the Kähler case.

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- The term $\overline{\partial}^S\partial^S\omega^S$ appears 'because' it is a Laplacian in the exotic Hodge theory of S.

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- Exotic Laplacian $\overline{\partial}^X \partial^X \omega^X$ obstruction to local index theorem.

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$$-\left(\overline{\partial}^{X}\partial^{X}i\omega^{X}\right)^{c}-\frac{1}{16}\left\|\left(\overline{\partial}^{X}-\partial^{X}\right)\omega^{X}\right\|^{2}_{\Lambda^{\cdot}\left(T^{*}_{\mathbf{R}}X\right)}.$$

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The term $\left(\overline{\partial}^X\partial^X i\omega^X\right)^c$ is of length 4 in the Clifford algebra. Local index theory accepts only terms of length ≤ 2.

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- Imitate the construction of the α_t in the Kähler case...
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- The forms α_t converge as $t \to 0$.
- This proves the theorem in this special case.

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- This proves RRG!

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- The general proof gives us $1 = 1$ even when $M = S$!

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