

# Stability and Instability of Black Holes

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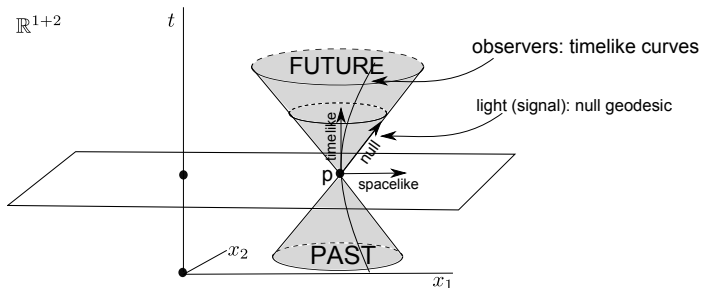
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# Introduction

- General relativity is a successful theory of gravitation.
- Objects of study: (4-dimensional) Lorentzian manifolds  $(\mathcal{M}, g)$  which satisfy the Einstein-vacuum equations:

$$\text{Ric}(g) = 0.$$

Here  $g$  is a Lorentzian metric, namely a differential assignment of a non-degenerate, symmetric, bilinear form of signature  $(-, +, +, +)$  in  $T_{\mathbf{p}}\mathcal{M}$  at each  $\mathbf{p} \in \mathcal{M}$ . New feature of  $g$ : Causality. If  $X \in T_{\mathbf{p}}\mathcal{M}$  is called timelike if  $g(X, X) < 0$ , null if  $g(X, X) = 0$  and spacelike if  $g(X, X) > 0$ .



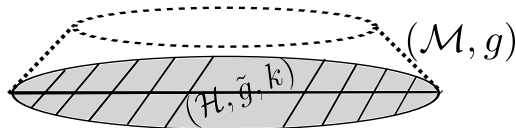
# Initial Value Problem

The equations  $Ric(g) = 0$  are hyperbolic. This can be seen by using the wave gauge:  $\square_g x^a = 0$ . Then,

$$\square_g g_{\mu\nu} = Q_{\mu\nu}(g, \nabla g),$$

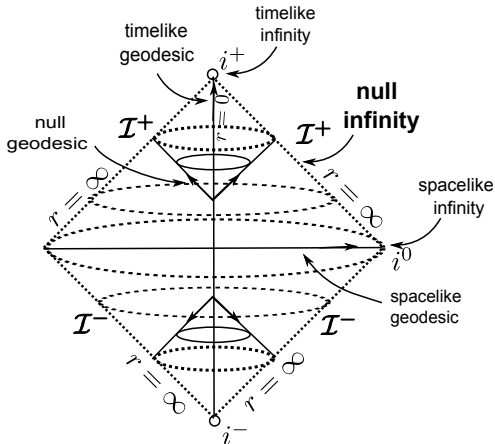
and hence the appropriate problem of study is the IVP:

- Initial data consist of a triple  $(\mathcal{H}, \tilde{g}, k)$ , where  $(\mathcal{H}, \tilde{g})$  is a 3-dimensional Riemannian manifold and  $k$  a symmetric (0,2)-tensor ( $(\tilde{g}, k)$  are tensors to be the first and second fundamental forms, respectively).



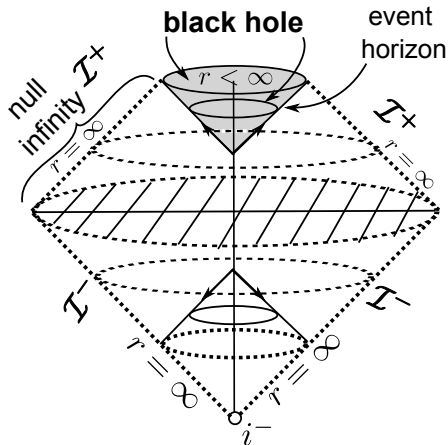
# Explicit solutions I: Trivial Solution

Trivial initial data  $(\mathbb{R}^3, \tilde{g}_{\text{flat}}, k = 0)$  give rise to Minkowski spacetime. Its topology is trivial  $\mathbb{R}^4$  and the metric is the flat Lorentzian metric, called Minkowski metric. **PROBLEM:** Understand global causal structure, in particular, null geodesics reaching arbitrarily large distances. Conformal diagram:



# Explicit Solutions II: Black Holes

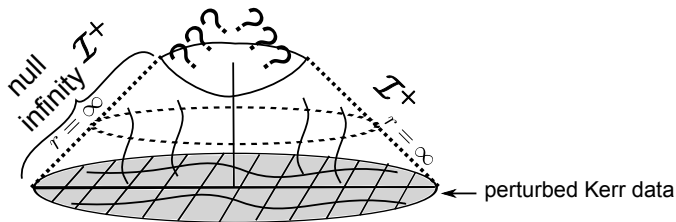
- What other type of conformal diagram may arise?
- Black holes! Fundamental prediction of GR.



- Main example: Kerr family of rotating black holes.
- Explicit solutions are physically relevant only if they are *stable*.

# The Stability Problem

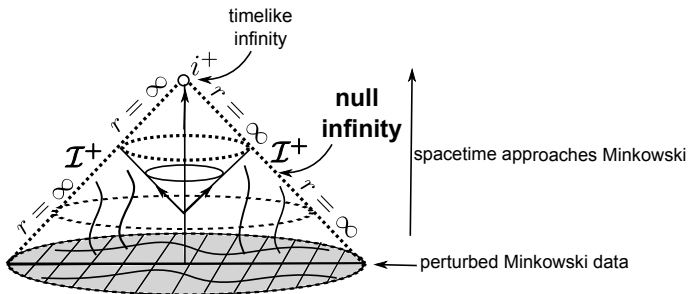
If one perturbs the Kerr initial data  $(\mathcal{H}, \tilde{g}, k)$ , then does one obtain a space-time which contains a black hole region and approaches a Kerr spacetime?



The stability problem for Kerr is one of the main outstanding open problems in general relativity.

# Christodoulou-Klainerman Stability of Minkowski

Most notable global result is the work of Christodoulou-Klainerman on perturbations of Minkowski.



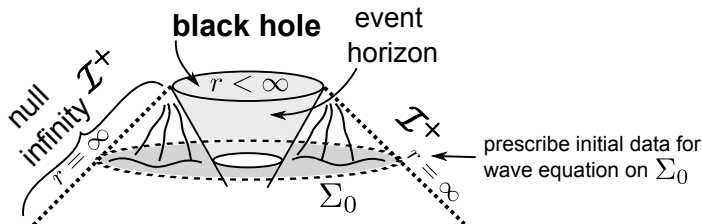
They showed that perturbations propagate like waves and thus decay in view of the decay properties of the wave equation on Minkowski (actual work 500p). Arising spacetime is geodesically complete.

# The Wave Equation on Black Holes

- The C-K work made apparent the need to understand the quantitative decay properties of the scalar wave equation

$$\square_g \psi = 0.$$

- This is much simpler than studying the fully non-linear problem, since, for instance, the background spacetime is fixed.



- We want to understand the evolution of  $\psi$  up to and including the event horizon.



# The Subextremal Kerr Family of Black Holes

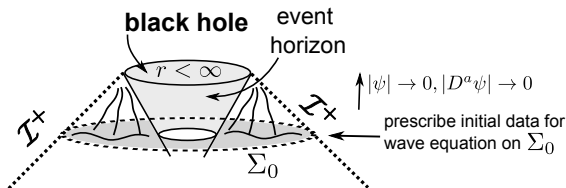
- Angular momentum  $a$  of Kerr black holes has an upper limit  $a_{\max}$  in terms of terms of their mass.
- Subextremal:  $a < a_{\max}$ . Extremal:  $a = a_{\max}$ .

Dafermos-Rodnianski: If  $a < a_{\max}$  then if  $\square_g \psi = 0$  then

$$|\psi| \leq C \cdot D \cdot \frac{1}{\tau} \quad \text{and}$$

$$|D^a \psi| \leq C \cdot D' \cdot \frac{1}{\tau}$$

as  $\tau \rightarrow +\infty$ , where  $\tau$  is an appropriate 'time' parameter, up to and including the event horizon.



# The Extremal Case

My work focuses on the extremal case:

S.A.: If  $a = a_{\max}$  and  $\square_g \psi = 0$ , then

- If  $\psi$  is axisymmetric then

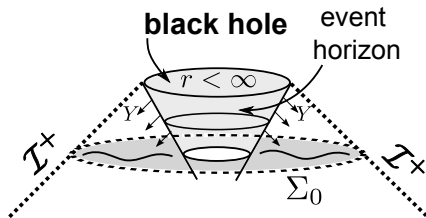
$$|\psi| \leq C \cdot D \cdot \frac{1}{\tau^{\text{crit}}} \quad \text{and}$$

as  $\tau \rightarrow +\infty$ , where  $\tau$  is an appropriate 'time' parameter.

- Generically,  $Y\psi$  does not decay along the event horizon, and in fact

$$|YY\psi| \rightarrow +\infty$$

as  $\tau \rightarrow +\infty$  along the event horizon. Here,  $Y$  is a vector field transversal to the event horizon. Similar blow-up results hold for  $L^2$  norms.



# Source of Instability: Conservation Laws

- For  $|a| = M$ , the quantity

$$\int_{S_v} \left[ Y\psi + \frac{1}{2M} \cdot \psi \right] d\mu_{S^2}$$

is conserved along the event horizon.

- These laws have been generalized for Maxwell equations and Linearized Gravity, again for  $|a| = M$  (Harvey et al).
- Similar conservation laws hold on null infinity. These give rise to the well-known Newman–Penrose constants.
- Complete characterization of all null hypersurfaces admitting such conservation laws is given in terms of a new foliation-invariant elliptic operator (S.A. 2013).

# Open Problems

- Non-linear stability of subextremal Kerr.
- Non-linear instability of extremal Kerr.

**THANK YOU!**