Topology of the set of singularities of a solution of the Hamilton-Jacobi Equation

Albert Fathi

IAS Princeton March 15, 2016

In this lecture, a singularity for a locally Lipschitz real valued function U is a point where U is not differentiable.

In this lecture, a singularity for a locally Lipschitz real valued function U is a point where U is not differentiable.

We will denote by $\mathrm{Sing}(U)$ the set of singularities of U. Its, complement, i.e. the set of points where U is differentiable, is denoted by $\mathrm{Diff}(U)$.

In this lecture, a singularity for a locally Lipschitz real valued function U is a point where U is not differentiable.

We will denote by $\mathrm{Sing}(U)$ the set of singularities of U. Its, complement, i.e. the set of points where U is differentiable, is denoted by $\mathrm{Diff}(U)$.

We will give some properties of the set $\mathrm{Sing}(U)$ when U is a viscosity solution of the Hamilton-Jacobi equation, under the "usual" (i.e. Tonelli) regularity of the Hamiltonian.

We recall that global smooth solutions of the Hamilton-Jacobi equation rarely exist.

We recall that global smooth solutions of the Hamilton-Jacobi equation rarely exist.
Therefore a theory of generalized solutions is necessary.

We recall that global smooth solutions of the Hamilton-Jacobi equation rarely exist.
Therefore a theory of generalized solutions is necessary.
We will speak about the viscosity solutions, since we will deal with Hamiltonians on a cotangent space that are convex in the momentum.

We recall that global smooth solutions of the Hamilton-Jacobi equation rarely exist.
Therefore a theory of generalized solutions is necessary.
We will speak about the viscosity solutions, since we will deal with Hamiltonians on a cotangent space that are convex in the momentum.
However to make our lecture accessible to a wide audience, after stating the results in full generality, we will concentrate our methods on distances to closed subsets in Euclidean space.

Our results are valid for any compact manifold, but to fix ideas, we will restrict to the case of the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$.

Our results are valid for any compact manifold, but to fix ideas, we will restrict to the case of the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$.
A function $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is a **Tonelli Hamiltonian** if it satisfies the following conditions:

1) H is C^2.
2) **(C^2 Strict Convexity)** At every (x, p), the second partial derivative $\partial^2_{pp} H(x, p)$ is definite > 0. In particular $H(x, p)$ is strictly convex in p.
3) **(Superlinearity)** $H(x, p)/\|p\| \to +\infty$, as $\|p\| \to +\infty$.

Our results are valid for any compact manifold, but to fix ideas, we will restrict to the case of the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$.
A function $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is a **Tonelli Hamiltonian** if it satisfies the following conditions:

1) H is C^2.
2) **(C^2 Strict Convexity)** At every (x, p), the second partial derivative $\partial^2_{pp} H(x, p)$ is definite > 0. In particular $H(x, p)$ is strictly convex in p.
3) **(Superlinearity)** $H(x, p)/\|p\| \to +\infty$, as $\|p\| \to +\infty$.

A prototype example of such a Tonelli Hamiltonian is

$$H(x, p) = \frac{1}{2}\|p\|^2 + V(x).$$

Our results are valid for any compact manifold, but to fix ideas, we will restrict to the case of the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$.

A function $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is a **Tonelli Hamiltonian** if it satisfies the following conditions:

1) H is C^2.
2) **(C^2 Strict Convexity)** At every (x, p), the second partial derivative $\partial^2_{pp} H(x, p)$ is definite > 0. In particular $H(x, p)$ is strictly convex in p.
3) **(Superlinearity)** $H(x, p)/\|p\| \to +\infty$, as $\|p\| \to +\infty$.

A prototype example of such a Tonelli Hamiltonian is

$$H(x, p) = \frac{1}{2}\|p\|^2 + V(x).$$

The important feature of Tonelli Hamiltonian is that they allow action to be defined by a Lagrangian convex in the speed. This in turn allows to apply the calculus of variations to find minimizers of action (rather than just critical points).

Viscosity solutions for Tonelli Hamiltonians

Viscosity solutions for Tonelli Hamiltonians

Explaining the general definition of a viscosity solution would take us too far away. Instead, we take advantage of the fact that H is Tonelli to give the following equivalent definition.

Viscosity solutions for Tonelli Hamiltonians

Explaining the general definition of a viscosity solution would take us too far away. Instead, we take advantage of the fact that H is Tonelli to give the following equivalent definition.

Definition

The **continuous** function $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ is a viscosity solution of the (evolution) Hamilton-Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0, \tag{0.1}$$

if it is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, and satisfies equation (0.1) almost everywhere.

Viscosity solutions for Tonelli Hamiltonians

Explaining the general definition of a viscosity solution would take us too far away. Instead, we take advantage of the fact that H is Tonelli to give the following equivalent definition.

Definition

The **continuous** function $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ is a viscosity solution of the (evolution) Hamilton-Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0, \tag{0.1}$$

if it is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, and satisfies equation (0.1) almost everywhere.

Note that a concave function is differentiable almost everywhere (it is locally Lipschitz). Therefore U is differentiable almost everywhere, and the last condition in the definition makes sense.

Existence of viscosity solutions

Existence of viscosity solutions

Although, as we already said C^1 solutions for the Hamilton-Jacobi equation may fail to exist, there are always viscosity solutions.

Existence of viscosity solutions

Although, as we already said C^1 solutions for the Hamilton-Jacobi equation may fail to exist, there are always viscosity solutions. This is the reason Pierre-Louis Lions and Mike Crandall introduced them in the 1980's.

Existence of viscosity solutions

Although, as we already said C^1 solutions for the Hamilton-Jacobi equation may fail to exist, there are always viscosity solutions. This is the reason Pierre-Louis Lions and Mike Crandall introduced them in the 1980's. Of course, they relied on previous work of other people, including Hopf and Kruzhkov.

Existence of viscosity solutions

Although, as we already said C^1 solutions for the Hamilton-Jacobi equation may fail to exist, there are always viscosity solutions. This is the reason Pierre-Louis Lions and Mike Crandall introduced them in the 1980's. Of course, they relied on previous work of other people, including Hopf and Kruzhkov.

Theorem

Given any continuous function $u_0 : \mathbb{T}^N \to \mathbb{R}$, there exists a (unique) viscosity solution $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ of the evolution equation $\partial_t U + H(x, \partial_x U) = 0$, such that $u_0(x) = U(x, 0)$, for every $x \in \mathbb{T}^N$.

Existence of viscosity solutions

Although, as we already said C^1 solutions for the Hamilton-Jacobi equation may fail to exist, there are always viscosity solutions. This is the reason Pierre-Louis Lions and Mike Crandall introduced them in the 1980's. Of course, they relied on previous work of other people, including Hopf and Kruzhkov.

Theorem

Given any continuous function $u_0 : \mathbb{T}^N \to \mathbb{R}$, there exists a (unique) viscosity solution $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ of the evolution equation $\partial_t U + H(x, \partial_x U) = 0$, such that $u_0(x) = U(x, 0)$, for every $x \in \mathbb{T}^N$.

The important ingredient that is used in our work is that these solutions have backward "characteristics" at every point, and that these characteristics depend continuously on the end point on the set where the solution is differentiable.

Main result

Our main result is the following

Theorem

Let $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ *be a viscosity solution of the evolution equation*

$$\partial_t U + H(x, \partial_x U) = 0.$$

Main result

Our main result is the following

Theorem

Let $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ *be a viscosity solution of the evolution equation*

$$\partial_t U + H(x, \partial_x U) = 0.$$

The set $\mathrm{Sing}_+(U) = \mathrm{Sing}(U) \cap \mathbb{T}^N \times]0, +\infty[$ *is locally connected.*

Main result

Our main result is the following

Theorem

Let $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ *be a viscosity solution of the evolution equation*

$$\partial_t U + H(x, \partial_x U) = 0.$$

The set $\mathrm{Sing}_+(U) = \mathrm{Sing}(U) \cap \mathbb{T}^N \times]0, +\infty[$ *is locally connected.*
If $\mathrm{Sing}_+(U) \neq \emptyset$*, then every connected component* C *of* $\mathrm{Sing}_+(U)$ *is unbounded in* $\mathbb{T}^N \times [0, +\infty[$*, i.e for every* $t > 0$*, the intersection* $C \cap \mathbb{T}^N \times [t, +\infty[$ *is not empty.*

Main result

Our main result is the following

Theorem

Let $U : \mathbb{T}^N \times [0, +\infty[\to \mathbb{R}$ *be a viscosity solution of the evolution equation*

$$\partial_t U + H(x, \partial_x U) = 0.$$

The set $\mathrm{Sing}_+(U) = \mathrm{Sing}(U) \cap \mathbb{T}^N \times]0, +\infty[$ *is locally connected. If* $\mathrm{Sing}_+(U) \neq \emptyset$*, then every connected component* C *of* $\mathrm{Sing}_+(U)$ *is unbounded in* $\mathbb{T}^N \times [0, +\infty[$*, i.e for every* $t > 0$*, the intersection* $C \cap \mathbb{T}^N \times [t, +\infty[$ *is not empty.*

We will now comment on the two aspects of the result: first the local connectedness, then the unboundedness of the connected components.

Local connectedness

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.

To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.
To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.
To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\text{Sing}_+(U)$ to look locally as the set of singularities of a concave function.
To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'. This set of jumps is of course countable but in the "generic" case it should have non-isolated points.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.

To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'. This set of jumps is of course countable but in the "generic" case it should have non-isolated points. But a countable locally connected set in a metric space has only isolated points.

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.
To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'. This set of jumps is of course countable but in the "generic" case it should have non-isolated points. But a countable locally connected set in a metric space has only isolated points.
A consequence of our theorem is therefore:

Local connectedness

Since a viscosity solution U is a *semi-concave* function (i.e. locally the sum of a concave and a smooth function) on $\mathbb{T}^N \times]0, +\infty[$, one should expect the set $\mathrm{Sing}_+(U)$ to look locally as the set of singularities of a concave function.
To fix the ideas, let us consider the singularities of a "generic" concave function $f : \mathbb{R} \to \mathbb{R}$. Hence the derivative f' should be a "generic" non-increasing function. The set of singularities for f is the set of jumps of f'. This set of jumps is of course countable but in the "generic" case it should have non-isolated points. But a countable locally connected set in a metric space has only isolated points.
A consequence of our theorem is therefore:

Viscosity solutions of Hamilton-Jacobi Equations for Tonelli Hamiltonians form a very small subset of the set of semi-concave functions

Propagation of singularities

Propagation of singularities

Of course, both parts of the theorem are related to work on propagation of singularities along paths done by Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, Yifeng Yu, and many others.

Propagation of singularities

Of course, both parts of the theorem are related to work on propagation of singularities along paths done by Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, Yifeng Yu, and many others.

In the works above, under some hypothesis, it is shown that for a given $(x_0, t_0) \in \mathrm{Sing}_+(U)$, there exists a path $\gamma : [t_0, t_0 + \epsilon[\to \mathbb{T}^N$, with $\gamma : (t_0) = x_0$, and $(\gamma(t), t) \in \mathrm{Sing}_+(U)$, for all $t \in [t_0, t_0 + \epsilon[$.

Propagation of singularities

Of course, both parts of the theorem are related to work on propagation of singularities along paths done by Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, Yifeng Yu, and many others.

In the works above, under some hypothesis, it is shown that for a given $(x_0, t_0) \in \mathrm{Sing}_+(U)$, there exists a path $\gamma : [t_0, t_0 + \epsilon[\to \mathbb{T}^N$, with $\gamma : (t_0) = x_0$, and $(\gamma(t), t) \in \mathrm{Sing}_+(U)$, for all $t \in [t_0, t_0 + \epsilon[$.

These results are essentially local (i.e. $\gamma(t)$ may have no limit as $t \to t_0 + \epsilon$ or could converge to a point of differentiability)

Propagation of singularities

Of course, both parts of the theorem are related to work on propagation of singularities along paths done by Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, Yifeng Yu, and many others.

In the works above, under some hypothesis, it is shown that for a given $(x_0, t_0) \in \mathrm{Sing}_+(U)$, there exists a path $\gamma : [t_0, t_0 + \epsilon[\to \mathbb{T}^N$, with $\gamma : (t_0) = x_0$, and $(\gamma(t), t) \in \mathrm{Sing}_+(U)$, for all $t \in [t_0, t_0 + \epsilon[$.

These results are essentially local (i.e. $\gamma(t)$ may have no limit as $t \to t_0 + \epsilon$ or could converge to a point of differentiability) except, to my knowledge, in a couple of recent exceptions.

Propagation of singularities

Of course, both parts of the theorem are related to work on propagation of singularities along paths done by Paolo Albano, Piermarco Cannarsa, Wei Cheng, Marco Mazzola, Carlo Sinestrari, Yifeng Yu, and many others.

In the works above, under some hypothesis, it is shown that for a given $(x_0, t_0) \in \mathrm{Sing}_+(U)$, there exists a path $\gamma : [t_0, t_0 + \epsilon[\to \mathbb{T}^N$, with $\gamma : (t_0) = x_0$, and $(\gamma(t), t) \in \mathrm{Sing}_+(U)$, for all $t \in [t_0, t_0 + \epsilon[$.

These results are essentially local (i.e. $\gamma(t)$ may have no limit as $t \to t_0 + \epsilon$ or could converge to a point of differentiability) except, to my knowledge, in a couple of recent exceptions.

If these results were global then they would prove a better result than the second part of theorem stated above: namely, that the path connected components of $\mathrm{Sing}_+(U)$ are unbounded.

The first exception I have in mind is:

The first exception I have in mind is:

P. Albano, *Propagation of singularities for solutions of Hamilton-Jacobi equations*, J. Math. Anal. Appl., **411** (2014), 684–687.

The first exception I have in mind is:

P. Albano, *Propagation of singularities for solutions of Hamilton-Jacobi equations*, J. Math. Anal. Appl., **411** (2014), 684–687.

In this work, the propagation along paths problem is addressed for the closure $\overline{\mathrm{Sing}(U)}$ rather than for the set of singularities itself.

The first exception I have in mind is:

P. Albano, *Propagation of singularities for solutions of Hamilton-Jacobi equations*, J. Math. Anal. Appl., **411** (2014), 684–687.

In this work, the propagation along paths problem is addressed for the closure $\overline{\mathrm{Sing}(U)}$ rather than for the set of singularities itself. The second exception I have in mind is:

The first exception I have in mind is:

P. Albano, *Propagation of singularities for solutions of Hamilton-Jacobi equations*, J. Math. Anal. Appl., **411** (2014), 684–687.

In this work, the propagation along paths problem is addressed for the closure $\overline{\mathrm{Sing}(U)}$ rather than for the set of singularities itself.
The second exception I have in mind is:

P. Cannarsa, M. Mazzola, & C. Sinestrari, *Global Propagation of singularities for time dependent Hamilton-Jacobi equations* DCDS, **35** (2015), 4225–4239.

The first exception I have in mind is:

P. Albano, *Propagation of singularities for solutions of Hamilton-Jacobi equations*, J. Math. Anal. Appl., **411** (2014), 684–687.

In this work, the propagation along paths problem is addressed for the closure $\overline{\mathrm{Sing}(U)}$ rather than for the set of singularities itself. The second exception I have in mind is:

P. Cannarsa, M. Mazzola, & C. Sinestrari, *Global Propagation of singularities for time dependent Hamilton-Jacobi equations* DCDS, **35** (2015), 4225–4239.

The authors show that there is global propagation of singularities along paths for Hamiltonians of the form:

$$H(p) = \frac{1}{2}\langle Ap, p\rangle, \text{ where } A \text{ is a positive definite } N \times N \text{ real matrix.}$$

In this last paper, the authors give a description of the state of the art in 2015.

In this last paper, the authors give a description of the state of the art in 2015.They also note that, in general, the problem has a negative answer if H is allowed Lipschitz dependence in (t, x) (Example 5.6.7, in the book by Cannarsa and Sinestrari). They conclude with:

> Nevertheless, establishing whether genuine singularities propagate indefinitely or not remains a largely open problem.

In this last paper, the authors give a description of the state of the art in 2015.They also note that, in general, the problem has a negative answer if H is allowed Lipschitz dependence in (t, x) (Example 5.6.7, in the book by Cannarsa and Sinestrari). They conclude with:

> Nevertheless, establishing whether genuine singularities propagate indefinitely or not remains a largely open problem.

Our work was motivated by a lecture of Piermarco Cannarsa (April 2014, Avignon), where he mentioned his work with Yifeng Yu: *Dynamics of the propagation of singularities for semiconcave functions, J. Eur. Math. Soc. 11 (2009), 999–1024,*

In this last paper, the authors give a description of the state of the art in 2015.They also note that, in general, the problem has a negative answer if H is allowed Lipschitz dependence in (t, x) (Example 5.6.7, in the book by Cannarsa and Sinestrari). They conclude with:

> Nevertheless, establishing whether genuine singularities propagate indefinitely or not remains a largely open problem.

Our work was motivated by a lecture of Piermarco Cannarsa (April 2014, Avignon), where he mentioned his work with Yifeng Yu:
Dynamics of the propagation of singularities for semiconcave functions, J. Eur. Math. Soc. 11 (2009), 999–1024,
in which they proved under some hypothesis, the non-isolation of singularities for the stationary equation

$$H(x, D_x u) = c.$$

What really intrigued me was that they were using the Jordan-Brouwer separation theorem in the proof.

What really intrigued me was that they were using the Jordan-Brouwer separation theorem in the proof.

This aroused again my strong (unsupported) belief that singularities in weak KAM theory are deeply related to the topology and the Aubry set.

What really intrigued me was that they were using the Jordan-Brouwer separation theorem in the proof.

This aroused again my strong (unsupported) belief that singularities in weak KAM theory are deeply related to the topology and the Aubry set.

Without Piermarco's very inspiring lecture none of this work would have been done.

We start afresh!

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.
We will, denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^N$.

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.
We will, denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^N$.
If $C \subset \mathbb{R}^N$ is a closed subset, we denote by $\delta_C : \mathbb{R}^N \to \mathbb{R}$ the distance function to C.

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.
We will, denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^N$.
If $C \subset \mathbb{R}^N$ is a closed subset, we denote by $\delta_C : \mathbb{R}^N \to \mathbb{R}$ the distance function to C.

$$\delta_C(x) = \inf_{c \in C} \|x - c\|.$$

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.
We will, denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^N$.
If $C \subset \mathbb{R}^N$ is a closed subset, we denote by $\delta_C : \mathbb{R}^N \to \mathbb{R}$ the distance function to C.

$$\delta_C(x) = \inf_{c \in C} \|x - c\|.$$

The function δ_C is 1-Lipschitz.

Distance function

The ideas here can be applied verbatim to distance functions to closed sets in complete Riemannian manifolds.
To keep simple, we will restrict to $\mathbb{R}^N$ with its canonical Euclidean distance
It will give a good idea of the results, and the methods of proof.
We will, denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^N$.
If $C \subset \mathbb{R}^N$ is a closed subset, we denote by $\delta_C : \mathbb{R}^N \to \mathbb{R}$ the distance function to C.

$$\delta_C(x) = \inf_{c \in C} \|x - c\|.$$

The function δ_C is 1-Lipschitz.
On $\mathbb{R}^N \setminus C$, the function δ_C is a prototype of a viscosity solution of a stationary Hamilton-Jacobi equation.

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable.

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable. In fact, we will concentrate on

$$\mathrm{Sing}_+(\delta_C) = \mathrm{Sing}(\delta_C) \setminus C,$$

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable. In fact, we will concentrate on

$$\mathrm{Sing}_+(\delta_C) = \mathrm{Sing}(\delta_C) \setminus C,$$

since it is easy to see that δ_C is differentiable at every point of $\mathring{C}$ and not differentiable at every point of ∂C.

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable. In fact, we will concentrate on

$$\mathrm{Sing}_+(\delta_C) = \mathrm{Sing}(\delta_C) \setminus C,$$

since it is easy to see that δ_C is differentiable at every point of $\mathring{C}$ and not differentiable at every point of ∂C.
In that case, our theorem becomes

Theorem

The set $\mathrm{Sing}_+(\delta_C)$ is locally connected. Moreover, for every bounded connected component U of $\mathbb{R}^N \setminus C$, the intersection $U \cap \mathrm{Sing}_+(\delta_C)$ is non-empty and connected.

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable. In fact, we will concentrate on

$$\mathrm{Sing}_+(\delta_C) = \mathrm{Sing}(\delta_C) \setminus C,$$

since it is easy to see that δ_C is differentiable at every point of $\mathring{C}$ and not differentiable at every point of ∂C.
In that case, our theorem becomes

Theorem

The set $\mathrm{Sing}_+(\delta_C)$ is locally connected. Moreover, for every bounded connected component U of $\mathbb{R}^N \setminus C$, the intersection $U \cap \mathrm{Sing}_+(\delta_C)$ is non-empty and connected.

The non-emptyness above follows from the fact that δ_C achieves a maximum on such a bounded component, and it cannot be differentiable at such a maximum, as we will see later.

We study the set of points $\mathrm{Sing}(\delta_C)$ where δ_C is not differentiable. In fact, we will concentrate on

$$\mathrm{Sing}_+(\delta_C) = \mathrm{Sing}(\delta_C) \setminus C,$$

since it is easy to see that δ_C is differentiable at every point of $\mathring{C}$ and not differentiable at every point of ∂C.
In that case, our theorem becomes

Theorem

The set $\mathrm{Sing}_+(\delta_C)$ is locally connected. Moreover, for every bounded connected component U of $\mathbb{R}^N \setminus C$, the intersection $U \cap \mathrm{Sing}_+(\delta_C)$ is non-empty and connected.

The non-emptyness above follows from the fact that δ_C achieves a maximum on such a bounded component, and it cannot be differentiable at such a maximum, as we will see later.
We will also discuss the non-bounded connected components of $\mathbb{R}^N \setminus C$ later on.

We recall some well-known facts about this function δ_C.

We recall some well-known facts about this function δ_C.
For every $x \in \mathbb{R}^N$, the set

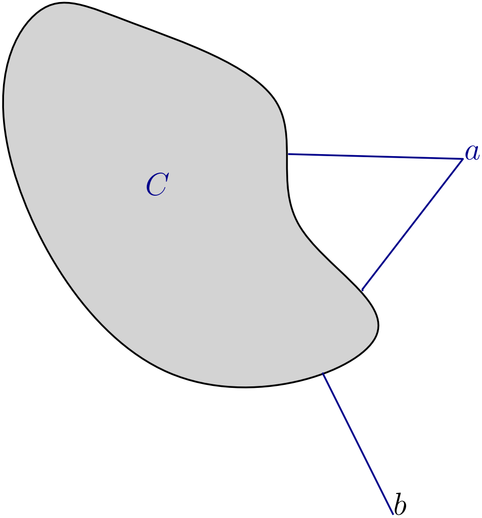
$$P_C(x) = \{c \in C \mid \|x - c\| = \delta_C(x)\}$$

of points of C, where the distance $\delta_C(x)$ is attained, is a non-empty compact subset of C.

We recall some well-known facts about this function δ_C.
For every $x \in \mathbb{R}^N$, the set

$$P_C(x) = \{c \in C \mid \|x - c\| = \delta_C(x)\}$$

of points of C, where the distance $\delta_C(x)$ is attained, is a non-empty compact subset of C.
This set $P_C(x) = \{c \in C \mid \|x - c\| = \delta_C(x)\}$ is called the set of projections of x on C.



The projection $P_C(a)$ consists of 2 points, and $P_C(b)$ is a singleton.

We denote by $\mathrm{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\mathrm{Reg}_+(C) = \mathrm{Reg}(C) \setminus C$.

We denote by $\mathrm{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\mathrm{Reg}_+(C) = \mathrm{Reg}(C) \setminus C$.
The set $\mathrm{Reg}(C)$ is the disjoint union of C and $\mathrm{Reg}_+(C)$.

We denote by $\mathrm{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\mathrm{Reg}_+(C) = \mathrm{Reg}(C) \setminus C$.
The set $\mathrm{Reg}(C)$ is the disjoint union of C and $\mathrm{Reg}_+(C)$.
For a point $x \in \mathrm{Reg}(C)$, we will denote by x_C the unique point in $P_C(x)$.

We denote by $\text{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\text{Reg}_+(C) = \text{Reg}(C) \setminus C$.
The set $\text{Reg}(C)$ is the disjoint union of C and $\text{Reg}_+(C)$.
For a point $x \in \text{Reg}(C)$, we will denote by x_C the unique point in $P_C(x)$.
It is a classical simple exercise, using compactness arguments, to show that projection map $\text{Reg}_C \to C, x \mapsto x_C$ is continuous.

We denote by $\mathrm{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\mathrm{Reg}_+(C) = \mathrm{Reg}(C) \setminus C$.
The set $\mathrm{Reg}(C)$ is the disjoint union of C and $\mathrm{Reg}_+(C)$.
For a point $x \in \mathrm{Reg}(C)$, we will denote by x_C the unique point in $P_C(x)$.
It is a classical simple exercise, using compactness arguments, to show that projection map $\mathrm{Reg}_C \to C, x \mapsto x_C$ is continuous.

Lemma

If $x \in \mathbb{R}^N$, and $c \in P_C(x)$, the open segment

$$]c, x[= \{(1-t)x + tc \mid t \in]0,1[\}$$

is contained in $\mathrm{Reg}(C)$. *In fact, for every $s \in]0,1[$, we have*

$$P_C((1-s)x + sc) = \{c\}, \text{ and } \delta_C((1-s)x + sc) = (1-s)\delta_C(x).$$

We denote by $\mathrm{Reg}(C)$ the set of points in $\mathbb{R}^N$ where $P_C(x)$ is single valued, and by $\mathrm{Reg}_+(C) = \mathrm{Reg}(C) \setminus C$.
The set $\mathrm{Reg}(C)$ is the disjoint union of C and $\mathrm{Reg}_+(C)$.
For a point $x \in \mathrm{Reg}(C)$, we will denote by x_C the unique point in $P_C(x)$.
It is a classical simple exercise, using compactness arguments, to show that projection map $\mathrm{Reg}_C \to C, x \mapsto x_C$ is continuous.

Lemma

If $x \in \mathbb{R}^N$, and $c \in P_C(x)$, the open segment

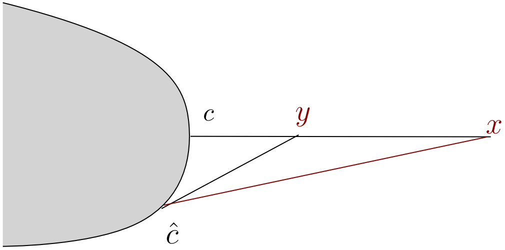
$$]c, x[= \{(1-t)x + tc \mid t \in]0,1[\}$$

is contained in $\mathrm{Reg}(C)$*. In fact, for every $s \in]0,1[$, we have*

$$P_C((1-s)x + sc) = \{c\}, \text{ and } \delta_C((1-s)x + sc) = (1-s)\delta_C(x).$$

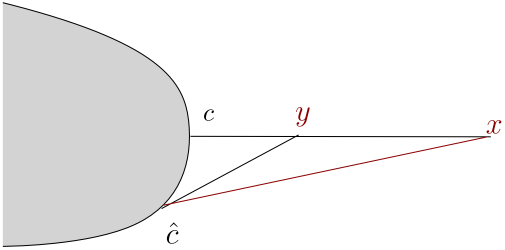
Therefore, the set $\mathrm{Reg}_+ C$ is dense in $\mathbb{R}^N \setminus C$.

Proof of the Lemma



$$y = (1 - s)x + sc$$
$$\hat{c} \in P_C(y)$$
$$y - c = (1 - s)(y - x)$$

Proof of the Lemma

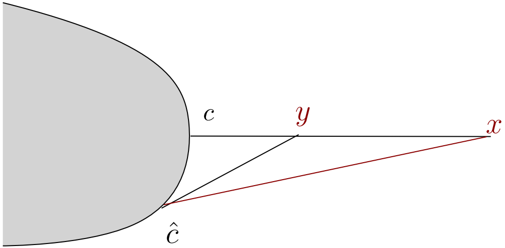


$$y = (1-s)x + sc$$
$$\hat{c} \in P_C(y)$$
$$y - c = (1-s)(y-x)$$

We have the inequalities

$$\begin{aligned}\|x-y\| + \|y-\hat{c}\| \geq \|x-\hat{c}\| &\geq \|x-c\| \\ &= \|x-y\| + \|y-c\| \\ &\geq \|x-y\| + \|y-\hat{c}\|\end{aligned}$$

Proof of the Lemma



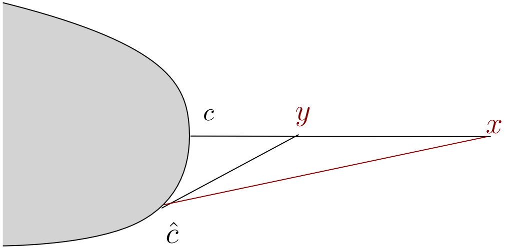
$$y = (1-s)x + sc$$
$$\hat{c} \in P_C(y)$$
$$y - c = (1-s)(y-x)$$

We have the inequalities

$$\begin{aligned} \|x-y\| + \|y-\hat{c}\| \geq \|x-\hat{c}\| &\geq \|x-c\| \\ &= \|x-y\| + \|y-c\| \\ &\geq \|x-y\| + \|y-\hat{c}\| \end{aligned}$$

Therefore all inequalities are equality.

Proof of the Lemma



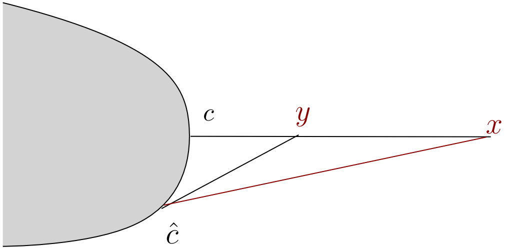
$$y = (1-s)x + sc$$
$$\hat{c} \in P_C(y)$$
$$y - c = (1-s)(y-x)$$

We have the inequalities

$$\begin{aligned} \|x-y\| + \|y-\hat{c}\| \geq \|x-\hat{c}\| &\geq \|x-c\| \\ &= \|x-y\| + \|y-c\| \\ &\geq \|x-y\| + \|y-\hat{c}\| \end{aligned}$$

Therefore all inequalities are equality. This implies that $\|y-c\| = \|y-\hat{c}\|$, and that x, y and $\hat{c}$ are aligned.

Proof of the Lemma



$$y = (1-s)x + sc$$
$$\hat{c} \in P_C(y)$$
$$y - c = (1-s)(y-x)$$

We have the inequalities

$$\begin{aligned} \|x-y\| + \|y-\hat{c}\| \geq \|x-\hat{c}\| &\geq \|x-c\| \\ &= \|x-y\| + \|y-c\| \\ &\geq \|x-y\| + \|y-\hat{c}\| \end{aligned}$$

Therefore all inequalities are equality. This implies that $\|y-c\| = \|y-\hat{c}\|$, and that x, y and $\hat{c}$ are aligned. Hence $c = \hat{c}$, and $\delta_C(y) = \|y-c\| = (1-s)\|x-c\| = (1-s)\delta_C(x)$.

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Theorem

The function δ_C is differentiable at $x \notin C$ if and only if $x \in \mathrm{Reg}_+(C)$.

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Theorem

The function δ_C is differentiable at $x \notin C$ if and only if $x \in \mathrm{Reg}_+(C)$.At a point $x \notin C$ where δ_C is differentiable, the gradient $\nabla_x \delta_C$ of δ_C at x is given by

$$\nabla_x \delta_C = \frac{x - x_C}{\|x - x_C\|}.$$

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Theorem

The function δ_C is differentiable at $x \notin C$ if and only if $x \in \mathrm{Reg}_+(C)$.At a point $x \notin C$ where δ_C is differentiable, the gradient $\nabla_x \delta_C$ of δ_C at x is given by

$$\nabla_x \delta_C = \frac{x - x_C}{\|x - x_C\|}.$$

Therefore the derivative of δ_C is continuous on the set of points where it is defined.

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Theorem

The function δ_C is differentiable at $x \notin C$ if and only if $x \in \mathrm{Reg}_+(C)$.At a point $x \notin C$ where δ_C is differentiable, the gradient $\nabla_x \delta_C$ of δ_C at x is given by

$$\nabla_x \delta_C = \frac{x - x_C}{\|x - x_C\|}.$$

Therefore the derivative of δ_C is continuous on the set of points where it is defined.

Note that this implies that the gradient $\nabla_x \delta_C$ is always of norm 1, wherever it exists.

Differentiability of δ_C

As is well-known there is a strong relationship between projections and differentiability of δ_C.

Theorem

The function δ_C is differentiable at $x \notin C$ if and only if $x \in \mathrm{Reg}_+(C)$.At a point $x \notin C$ where δ_C is differentiable, the gradient $\nabla_x \delta_C$ of δ_C at x is given by

$$\nabla_x \delta_C = \frac{x - x_C}{\|x - x_C\|}.$$

Therefore the derivative of δ_C is continuous on the set of points where it is defined.

Note that this implies that the gradient $\nabla_x \delta_C$ is always of norm 1, wherever it exists. Hence δ_C cannot be differentiable at its maximum on a bounded connected component of $\mathbb{R}^N \setminus C$.

Combining the lemma above and the theorem we get

Combining the lemma above and the theorem we get

Corollary

The function δ_C is differentiable on the set

$$\mathcal{C}_C = \{(1-t)x + tc \mid x \notin C, c \in P_C(x), t \in]0,1[\}.$$

We define the map $\Phi : \mathrm{Reg}_+(C) \times [0,1] \to \mathbb{R}^N$, with

$$\varphi(x,t) = tx + (1-t)x_C.$$

Combining the lemma above and the theorem we get

Corollary

The function δ_C is differentiable on the set

$$\mathcal{C}_C = \{(1-t)x + tc \mid x \notin C, c \in P_C(x), t \in]0,1[\}.$$

We define the map $\Phi : \mathrm{Reg}_+(C) \times [0,1] \to \mathbb{R}^N$, with

$$\varphi(x,t) = tx + (1-t)x_C.$$

This map is continuous since this is the case for $x \mapsto x_C$.

Here are the properties of Φ:

Here are the properties of Φ:

(1) For every $s \in [0,1]$, and every $x \in \mathrm{Reg}_+(C)$, we have

$$\delta_C(\Phi(x,s)) = (1-s)\delta_C(x).$$

Here are the properties of Φ:

(1) For every $s \in [0,1]$, and every $x \in \mathrm{Reg}_+(C)$, we have

$$\delta_C(\Phi(x,s)) = (1-s)\delta_C(x).$$

(2) We have $\Phi(x,0) = x$, for every $x \in \mathrm{Reg}_+(C)$.

Here are the properties of Φ:

(1) For every $s \in [0,1]$, and every $x \in \mathrm{Reg}_+(C)$, we have

$$\delta_C(\Phi(x,s)) = (1-s)\delta_C(x).$$

(2) We have $\Phi(x,0) = x$, for every $x \in \mathrm{Reg}_+(C)$.

(3) For every $s < 1$, we have $\Phi((x,t),s) \in \mathrm{Reg}_+(C) \subset \mathbb{R}^N \setminus C$. Therefore the image $\Phi(\mathrm{Reg}_+(C) \times [0,1[)$ avoids $\mathrm{Sing}_+(U)$, which is the complement of $\mathrm{Reg}_+(C)$ in $\mathbb{R}^N \setminus C$

Here are the properties of Φ:

(1) For every $s \in [0,1]$, and every $x \in \mathrm{Reg}_+(C)$, we have

$$\delta_C(\Phi(x,s)) = (1-s)\delta_C(x).$$

(2) We have $\Phi(x,0) = x$, for every $x \in \mathrm{Reg}_+(C)$.

(3) For every $s < 1$, we have $\Phi((x,t),s) \in \mathrm{Reg}_+(C) \subset \mathbb{R}^N \setminus C$. Therefore the image $\Phi(\mathrm{Reg}_+(C) \times [0,1[)$ avoids $\mathrm{Sing}_+(U)$, which is the complement of $\mathrm{Reg}_+(C)$ in $\mathbb{R}^N \setminus C$

The first item follows from the Lemma above. Item (2), follows from the definition of Φ. Item (3), for $s \in]0,1[$ follows from the fact that $]c,x[= \{sc + (1-s)x \mid s \in]0,1[\} \subset \mathrm{Reg}_+(C)$ for every $x \notin C$, and every $c \in P_C(x)$

The structure of $\mathrm{Sing}_+(\delta_C)$ will follow from the existence of that map Φ and its properties given above.

Connectedness criteria

We need tools to prove connectedness of a subset.

Connectedness criteria

We need tools to prove connectedness of a subset. We start with the following one, which we leave as an exercise.

Lemma

Let S be a subset of a **metric** *space X. Then S is connected if and only if it satisfies the following condition:*

> *For every subset $F \subset X \setminus S$,* **closed** *in X, the set S is contained in a unique connected component of $X \setminus F$.*

Connectedness criteria

We need tools to prove connectedness of a subset. We start with the following one, which we leave as an exercise.

Lemma

Let S be a subset of a **metric** *space X. Then S is connected if and only if it satisfies the following condition:*

> *For every subset $F \subset X \setminus S$,* **closed** *in X, the set S is contained in a unique connected component of $X \setminus F$.*

To be able to prove the connectedness properties, we need a way to single out a unique component of $X \setminus F$.

This is done by the following (apparently new) theorem.

This is done by the following (apparently new) theorem.

Theorem (Proper Homotopy Track)

Suppose $F \subset M$ *is a* **closed** *subset of the connected finite-dimensional manifold* M*. If* $\Phi : F \times [0,1[\to M$ *is a proper homotopy with* $\Phi(x,0) = x$*, for every* $x \in F$*, then the track* $\Phi(F \times [0,1[)$ *of the homotopy* Φ *covers all the connected components of* $M \setminus F$ *except at most one.*

This is done by the following (apparently new) theorem.

Theorem (Proper Homotopy Track)

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Recall that a continuous map $f : X \to Y$ is said to be proper if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

This is done by the following (apparently new) theorem.

Theorem (Proper Homotopy Track)

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Recall that a continuous map $f : X \to Y$ is said to be proper if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

What the theorem says is that there is a high price to pay to send a closed subset of a manifold to "infinity".

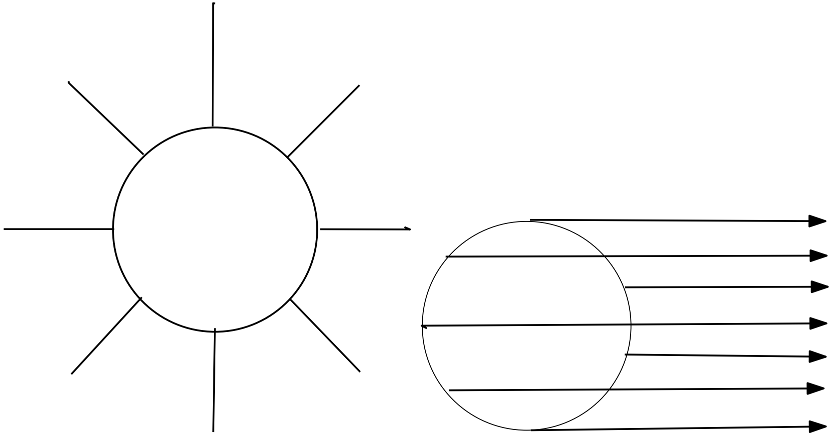
This is done by the following (apparently new) theorem.

Theorem (Proper Homotopy Track)

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Recall that a continuous map $f : X \to Y$ is said to be proper if for every compact subset $K \subset Y$, the inverse image $f^{-1}(K)$ is compact.

What the theorem says is that there is a high price to pay to send a closed subset of a manifold to "infinity".



In the figure, we illustrate the two classical ways to send $F = \mathbb{S}^{N-1}$, the unit sphere for the Euclidean norm on $M = \mathbb{R}^N$. In the homothety case, the unbounded component is covered, in the translation case the bounded component is covered.

It is possible to show that, for $F = \mathbb{S}^{N-1} \subset M = \mathbb{R}^N$, the theorem above is equivalent to Brouwer's fixed point theorem.

It is possible to show that, for $F = \mathbb{S}^{N-1} \subset M = \mathbb{R}^N$, the theorem above is equivalent to Brouwer's fixed point theorem.

Therefore the proof must use some non-trivial tool from topology.

It is possible to show that, for $F = \mathbb{S}^{N-1} \subset M = \mathbb{R}^N$, the theorem above is equivalent to Brouwer's fixed point theorem.

Therefore the proof must use some non-trivial tool from topology.

We postpone further discussion and proof of the theorem above to the end of the lecture.

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M.

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M. If we can find a continuous homotopy $\Phi : (M \setminus S) \times [0,1[\to M$ such that:

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M. If we can find a continuous homotopy $\Phi : (M \setminus S) \times [0, 1[\to M$ such that:

(i) *for every $x \in M \setminus S$, we have $\Phi(x, 0) = x$,*

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M. If we can find a continuous homotopy $\Phi : (M \setminus S) \times [0,1[\to M$ such that:

(i) *for every $x \in M \setminus S$, we have $\Phi(x,0) = x$,*

(ii) *the image $\Phi((M \setminus S) \times [0,1[)$ avoids S,*

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M. If we can find a continuous homotopy $\Phi : (M \setminus S) \times [0,1[\to M$ such that:

(i) *for every $x \in M \setminus S$, we have $\Phi(x,0) = x$,*

(ii) *the image $\Phi((M \setminus S) \times [0,1[)$ avoids S,*

(iii) *for every subset $F \subset M \setminus S$ which is* **closed** *in M, the restriction $\Phi|F : F \times [0,1[\to M$ is proper,*

The homotopy track theorem leeds to a criteria for connectedness that is well adapted to our situation.

Proposition (Homotopical Criterion for Connectedness)

Let $S \subset M$ be a subset of the connected finite-dimensional manifold M. If we can find a continuous homotopy $\Phi : (M \setminus S) \times [0, 1[\to M$ such that:

(i) *for every $x \in M \setminus S$, we have $\Phi(x, 0) = x$,*

(ii) *the image $\Phi((M \setminus S) \times [0, 1[)$ avoids S,*

(iii) *for every subset $F \subset M \setminus S$ which is* **closed** *in M, the restriction $\Phi|F : F \times [0, 1[\to M$ is proper,*

then S is connected.

Proof.

We have to show that for each subset $F \subset M \setminus S$, **closed in** X, the set S is contained in a unique connected component of $X \setminus F$.

Proof.

We have to show that for each subset $F \subset M \setminus S$, **closed in** X, the set S is contained in a unique connected component of $X \setminus F$. By the Proper Homotopy Track Theorem, there is at most one connected component C of $M \setminus F$ which is not entirely contained in $\Phi(F \times [0,1[)$. Since S avoids both F, and the image of Φ, necessarily S is contained in C. □

We would like to apply this proposition in our situation.

Proof.

We have to show that for each subset $F \subset M \setminus S$, **closed in** X, the set S is contained in a unique connected component of $X \setminus F$. By the Proper Homotopy Track Theorem, there is at most one connected component C of $M \setminus F$ which is not entirely contained in $\Phi(F \times [0,1[)$. Since S avoids both F, and the image of Φ, necessarily S is contained in C. □

We would like to apply this proposition in our situation. In this case, we have $M = \mathbb{R}^N \setminus C, S = \mathrm{Sing}_+(\delta_C), M \setminus S = \mathrm{Reg}_+(C)$, and the homotopy

$$\begin{aligned}\Phi : \mathrm{Reg}_+(C) \times [0,1[&\to \mathrm{Reg}_+(C) \subset \mathbb{R}^N \setminus C \\ (x,s) &\mapsto s x_C + (1-s)x.\end{aligned}$$

Proof.

We have to show that for each subset $F \subset M \setminus S$, **closed in** X, the set S is contained in a unique connected component of $X \setminus F$. By the Proper Homotopy Track Theorem, there is at most one connected component C of $M \setminus F$ which is not entirely contained in $\Phi(F \times [0,1[)$. Since S avoids both F, and the image of Φ, necessarily S is contained in C. □

We would like to apply this proposition in our situation. In this case, we have $M = \mathbb{R}^N \setminus C, S = \mathrm{Sing}_+(\delta_C), M \setminus S = \mathrm{Reg}_+(C)$, and the homotopy

$$\begin{aligned} \Phi : \mathrm{Reg}_+(C) \times [0,1[&\to \mathrm{Reg}_+(C) \subset \mathbb{R}^N \setminus C \\ (x,s) &\mapsto s x_C + (1-s)x. \end{aligned}$$

Condition (i) of the proposition $\Phi(x,0) = x$ is satisfied. Condition (ii) is also satisfied since already the image of Φ avoids $\mathrm{Sing}_+(\delta_C)$, the complement of $\mathrm{Reg}_+(C)$ in $\mathbb{R}^N \setminus C$.

Proof.

We have to show that for each subset $F \subset M \setminus S$, **closed in** X, the set S is contained in a unique connected component of $X \setminus F$. By the Proper Homotopy Track Theorem, there is at most one connected component C of $M \setminus F$ which is not entirely contained in $\Phi(F \times [0,1[)$. Since S avoids both F, and the image of Φ, necessarily S is contained in C. □

We would like to apply this proposition in our situation. In this case, we have $M = \mathbb{R}^N \setminus C, S = \mathrm{Sing}_+(\delta_C), M \setminus S = \mathrm{Reg}_+(C)$, and the homotopy

$$\begin{aligned} \Phi : \mathrm{Reg}_+(C) \times [0,1[&\to \mathrm{Reg}_+(C) \subset \mathbb{R}^N \setminus C \\ (x,s) &\mapsto s x_C + (1-s)x. \end{aligned}$$

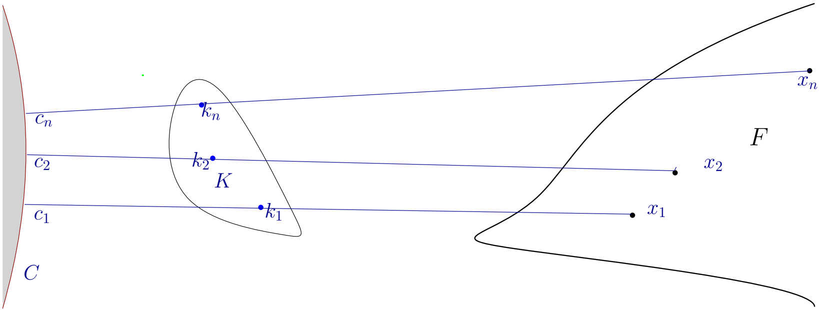
Condition (i) of the proposition $\Phi(x,0) = x$ is satisfied. Condition (ii) is also satisfied since already the image of Φ avoids $\mathrm{Sing}_+(\delta_C)$, the complement of $\mathrm{Reg}_+(C)$ in $\mathbb{R}^N \setminus C$.
The crux of the matter is the validity of the properness condition.

Therefore let us consider a subset $F \subset \mathrm{Reg}_+(C)$ which is closed in $\mathbb{R}^N \setminus C$, and a compact set $K \subset \mathbb{R}^N \setminus C$.

Therefore let us consider a subset $F \subset \mathrm{Reg}_+(C)$ which is closed in $\mathbb{R}^N \setminus C$, and a compact set $K \subset \mathbb{R}^N \setminus C$.
We want to show that from any sequence $(x_n, s_n) \in F \times [0, 1[$, with $\Phi(x_n, s_n) = k_n \in K$, we can extract from (x_n, s_n), a subsequence converging to a point in $F \times [0, 1[$.

Therefore let us consider a subset $F \subset \mathrm{Reg}_+(C)$ which is closed in $\mathbb{R}^N \setminus C$, and a compact set $K \subset \mathbb{R}^N \setminus C$.
We want to show that from any sequence $(x_n, s_n) \in F \times [0, 1[$, with $\Phi(x_n, s_n) = k_n \in K$, we can extract from (x_n, s_n), a subsequence converging to a point in $F \times [0, 1[$.
We call c_n the projection of x_n on C.

Therefore let us consider a subset $F \subset \mathrm{Reg}_+(C)$ which is closed in $\mathbb{R}^N \setminus C$, and a compact set $K \subset \mathbb{R}^N \setminus C$.
We want to show that from any sequence $(x_n, s_n) \in F \times [0, 1[$, with $\Phi(x_n, s_n) = k_n \in K$, we can extract from (x_n, s_n), a subsequence converging to a point in $F \times [0, 1[$.
We call c_n the projection of x_n on C.



We have

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

where the inequality follows from $s_n \leq 1$.

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

where the inequality follows from $s_n \leq 1$.
Since K is compact, disjoint from C, and $\delta_C = 0$ only on C, we can find $\alpha, \beta > 0$ such that $\alpha \leq \delta_C \leq \beta$ on K.

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

where the inequality follows from $s_n \leq 1$.
Since K is compact, disjoint from C, and $\delta_C = 0$ only on C, we can find $\alpha, \beta > 0$ such that $\alpha \leq \delta_C \leq \beta$ on K. This yields

$$\beta \geq (1 - s_n)\|x_n - c_n\| = \|k_n - c_n\| \geq \alpha.$$

Since K is compact $\kappa = \sup_{z \in K}\|z\| < +\infty$.

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

where the inequality follows from $s_n \leq 1$.
Since K is compact, disjoint from C, and $\delta_C = 0$ only on C, we can find $\alpha, \beta > 0$ such that $\alpha \leq \delta_C \leq \beta$ on K. This yields

$$\beta \geq (1 - s_n)\|x_n - c_n\| = \|k_n - c_n\| \geq \alpha.$$

Since K is compact $\kappa = \sup_{z \in K}\|z\| < +\infty$. This implies that $\sup_n \|c_n\| \leq \beta + \kappa < +\infty$.
Therefore, extracting if necessary, we can assume that

$$k_n \to k \in K, c_n \to c \in C, \text{ and } s_n \to s \in [0, 1].$$

We have

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \delta_C(k_n) = \|k_n - c_n\|,$$

where the inequality follows from $s_n \leq 1$.
Since K is compact, disjoint from C, and $\delta_C = 0$ only on C, we can find $\alpha, \beta > 0$ such that $\alpha \leq \delta_C \leq \beta$ on K. This yields

$$\beta \geq (1 - s_n)\|x_n - c_n\| = \|k_n - c_n\| \geq \alpha.$$

Since K is compact $\kappa = \sup_{z \in K}\|z\| < +\infty$. This implies that $\sup_n\|c_n\| \leq \beta + \kappa < +\infty$.
Therefore, extracting if necessary, we can assume that

$$k_n \to k \in K, c_n \to c \in C, \text{ and } s_n \to s \in [0, 1].$$

If we assume $\xi = \sup_n\|x_n\| < +\infty$, then extracting further, we can assume that $x_n \to x$.

We obtained above

$$\delta_C(x_n) = \|x_n - c_n\| \geq (1 - s_n)\|x_n - c_n\| = \|k_n - c_n\| \geq \alpha > 0.$$

This yields in the limit

$$\delta_C(x) \geq (1 - s)\delta_C(x) \geq \alpha > 0.$$

Therefore $s > 1$, and $x \notin C$. Since $x = \lim x_n$, and all the x_n are in F, which is closed in $\mathbb{R}^N \setminus C$, we conclude that $x \in F$. In particular, we get

Lemma

If $F \subset \mathrm{Reg}_+(C)$ is a closed and bounded subset of $\mathbb{R}^N \setminus C$, then the restriction $\Phi : F \times [0, 1[\to \mathbb{R}^N \setminus C$ is proper.

This lemma, together with the Homotopical Criterion for Connectedness, proves:

Theorem

If U is bounded connected component of $\mathbb{R}^N \setminus C$, then $U \cap \operatorname{Sing}(\delta_C)$ is connected.

This lemma, together with the Homotopical Criterion for Connectedness, proves:

Theorem

If U is bounded connected component of $\mathbb{R}^N \setminus C$, then $U \cap \mathrm{Sing}(\delta_C)$ is connected.

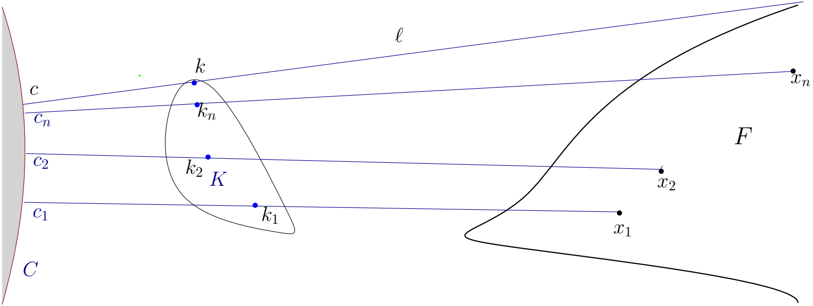
We now consider the case when $F \subset \mathrm{Reg}_+(C)$ is not bounded. The lack of properness comes from the fact that the sequence x_n goes to ∞, i.e. $\|x_n\| \to +\infty$.

This lemma, together with the Homotopical Criterion for Connectedness, proves:

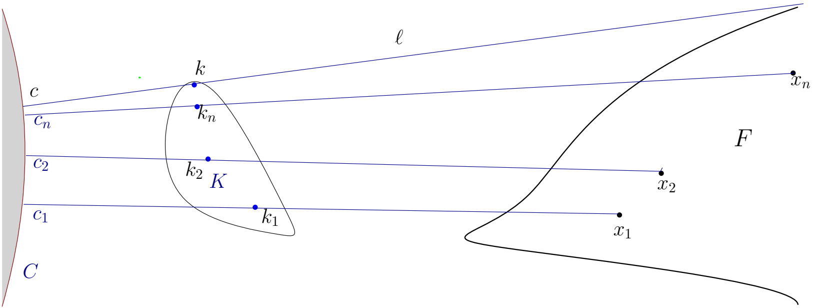
Theorem

If U is bounded connected component of $\mathbb{R}^N \setminus C$, then $U \cap \mathrm{Sing}(\delta_C)$ is connected.

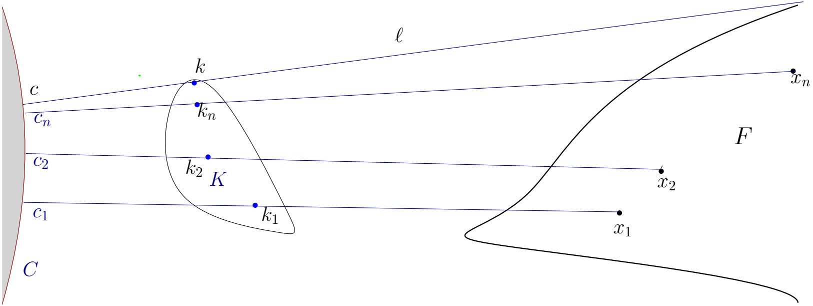
We now consider the case when $F \subset \mathrm{Reg}_+(C)$ is not bounded.
The lack of properness comes from the fact that the sequence x_n goes to ∞, i.e. $\|x_n\| \to +\infty$.
In this case, extracting if necessary, the segment $[c_n, x_n]$ "tends" to a half line ℓ starting at $c \in C$.



ℓ
k
c
x_n
c_n
k_n
F
c_2
k_2
K
x_2
c_1
k_1
x_1
C



Since for every $z \in [c_n, x_n]$, we have $\delta_C(z) = \|z - c_n\|$, in the limit, we obtain



Since for every $z \in [c_n, x_n]$, we have $\delta_C(z) = \|z - c_n\|$, in the limit, we obtain

$$\delta_C(z) = \|z - c\|, \text{ for every } z \in \ell.$$

Therefore, the lack of properness comes from the Aubry set of C, which we we now introduce.

The Aubry set of a closed set

The Aubry set of a closed set

Definition

The Aubry set $\mathcal{I}(C)$ is the set of points $x \in \mathbb{R}^N$, for which there exists a half line ℓ starting at a point $c \in C$, and such that

$$\delta_C(z) = \|z - c\|, \text{ for every } z \in \ell.$$

The Aubry set of a closed set

Definition

The Aubry set $\mathcal{I}(C)$ is the set of points $x \in \mathbb{R}^N$, for which there exists a half line ℓ starting at a point $c \in C$, and such that

$$\delta_C(z) = \|z - c\|, \text{ for every } z \in \ell.$$

It is not difficult to show that $\mathcal{I}(C)$ is a closed subset of $\mathbb{R}^N$ which is contained in $\mathrm{Reg}(C)$, and avoids every bounded connected component of $\mathbb{R}^N \setminus C$.

The Aubry set of a closed set

Definition

The Aubry set $\mathcal{I}(C)$ is the set of points $x \in \mathbb{R}^N$, for which there exists a half line ℓ starting at a point $c \in C$, and such that

$$\delta_C(z) = \|z - c\|, \text{ for every } z \in \ell.$$

It is not difficult to show that $\mathcal{I}(C)$ is a closed subset of $\mathbb{R}^N$ which is contained in $\mathrm{Reg}(C)$, and avoids every bounded connected component of $\mathbb{R}^N \setminus C$.

The analysis above yields the theorem:

The analysis above yields the theorem:

Theorem

For every connected component U of $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$, the intersection $U \cap \mathrm{Sing}(\delta_C)$ is connected. Moreover, if U is not bounded then this intersection $U \cap \mathrm{Sing}(\delta_C)$ is also unbounded.

The analysis above yields the theorem:

Theorem

For every connected component U of $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$, the intersection $U \cap \mathrm{Sing}(\delta_C)$ is connected. Moreover, if U is not bounded then this intersection $U \cap \mathrm{Sing}(\delta_C)$ is also unbounded.

The statement of the theorem is also reminiscent of the work:

The analysis above yields the theorem:

Theorem

For every connected component U of $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$, the intersection $U \cap \mathrm{Sing}(\delta_C)$ is connected. Moreover, if U is not bounded then this intersection $U \cap \mathrm{Sing}(\delta_C)$ is also unbounded.

The statement of the theorem is also reminiscent of the work:
P. Cannarsa & R. Peirone, *Unbounded components of the singular set of the distance function in $\mathbb{R}^n$*, TAMS 353 (2001) 4567–4581.

The analysis above yields the theorem:

Theorem

For every connected component U of $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$, the intersection $U \cap \mathrm{Sing}(\delta_C)$ is connected. Moreover, if U is not bounded then this intersection $U \cap \mathrm{Sing}(\delta_C)$ is also unbounded.

The statement of the theorem is also reminiscent of the work:
P. Cannarsa & R. Peirone, *Unbounded components of the singular set of the distance function in* $\mathbb{R}^n$, TAMS 353 (2001) 4567–4581.
To obtain the local connectedness, one has to go further and localise the argument given above to appropriate open subsets of $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$.

Proof of Proper Homotopy Track Theorem

We would like now to explain the proof of Proper Homotopy Track Theorem.

Proof of Proper Homotopy Track Theorem

We would like now to explain the proof of Proper Homotopy Track Theorem.

Theorem

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Proof of Proper Homotopy Track Theorem

We would like now to explain the proof of Proper Homotopy Track Theorem.

Theorem

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Although the general case requires some more serious arguments, for the sake of simplicity, we will give a proof in the case where $F = \partial D$, the boundary of the smooth compact domain D, and the image $\Phi(\partial D \times [0,1[)$ of Φ does not cover the whole of $M \setminus D$.

Proof of Proper Homotopy Track Theorem

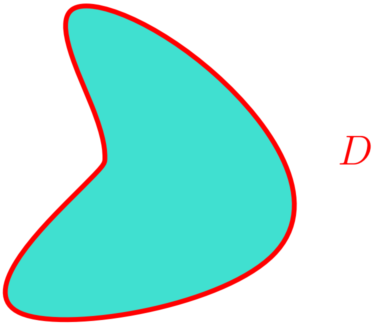
We would like now to explain the proof of Proper Homotopy Track Theorem.

Theorem

Suppose $F \subset M$ is a **closed** *subset of the connected finite-dimensional manifold M. If $\Phi : F \times [0,1[\to M$ is a proper homotopy with $\Phi(x,0) = x$, for every $x \in F$, then the track $\Phi(F \times [0,1[)$ of the homotopy Φ covers all the connected components of $M \setminus F$ except at most one.*

Although the general case requires some more serious arguments, for the sake of simplicity, we will give a proof in the case where $F = \partial D$, the boundary of the smooth compact domain D, and the image $\Phi(\partial D \times [0,1[)$ of Φ does not cover the whole of $M \setminus D$.

The situation looks like the figure



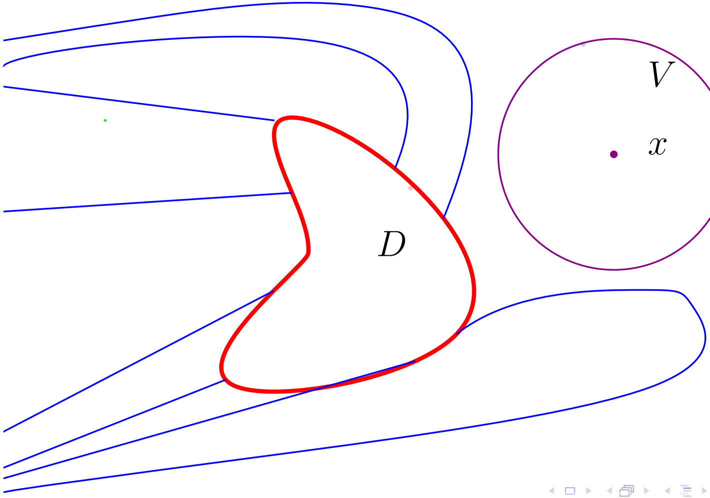
The red part is $F = \partial D$.
We will show that $D \subset \Phi(\partial D \times [0,1[)$.

Suppose $x \notin D$ is not contained in $\Phi(\partial D \times [0, 1[)$.

Suppose $x \notin D$ is not contained in $\Phi(\partial D \times [0,1[)$. Since the map Φ is proper its image $\Phi(\partial D \times [0,1[)$ is closed.

Suppose $x \notin D$ is not contained in $\Phi(\partial D \times [0, 1[)$. Since the map Φ is proper its image $\Phi(\partial D \times [0, 1[)$ is closed. Therefore we can find a small open neighborhood $V \subset M \setminus D$ of x with $V \cap \Phi(\partial D \times [0, 1[) = \emptyset$.

Suppose $x \notin D$ is not contained in $\Phi(\partial D \times [0,1[)$. Since the map Φ is proper its image $\Phi(\partial D \times [0,1[)$ is closed. Therefore we can find a small open neighborhood $V \subset M \setminus D$ of x with $V \cap \Phi(\partial D \times [0,1[) = \emptyset$.



Since ∂D is smooth, we can define the smooth manifold N by

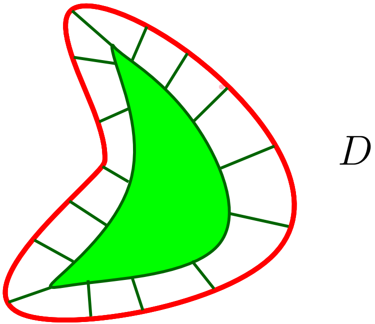
$$N = M \setminus \mathring{D} \cup_{\partial D} \partial D \times [0, 1[,$$

where $z \in \partial D$ is identified with $(z, 0) \in \partial D \times [0, 1[$.

Since ∂D is smooth, we can define the smooth manifold N by

$$N = M \setminus \mathring{D} \cup_{\partial D} \partial D \times [0, 1[,$$

where $z \in \partial D$ is identified with $(z, 0) \in \partial D \times [0, 1[$.
Using that ∂D is a smooth submanifold, we can find a closed neighborhood of ∂D in D diffeomorphic to $\partial D \times [0, 1]$, with ∂D identified with $\partial D \times \{0\}$. The set $\hat{D} = D \setminus \partial D \times [0, 1[$ is a smaller copy of D included in $\mathring{D}$.
The abstract space N is homeomorphic to $M \setminus \hat{D}$, an open subset of M.

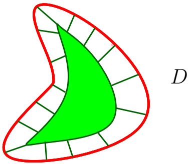


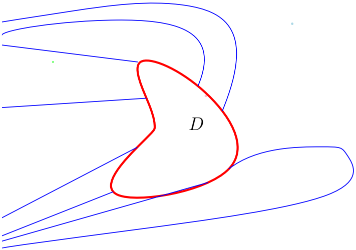
The set $\hat{D}$ is in lgreen. The part of D between ∂D and $\hat{D}$ is diffeomorphic to $\partial D \times [0, 1[$.

The manifold N is diffeomorphic to the complement of the green region.

Since $\Phi|\partial D \times \{0\}$ is the "identity", we can extend Φ by the identity on $M \setminus \mathring{D}$ to a continuous map $\tilde{\Phi} : N \to \mathbb{T}^N \times]0, +\infty[$.

Since $\Phi | \partial D \times \{0\}$ is the "identity", we can extend Φ by the identity on $M \setminus \mathring{D}$ to a continuous map $\tilde{\Phi} : N \to \mathbb{T}^N \times]0, +\infty[$.





$\tilde{\Phi}$ is the identity outside of $\mathring{D}$, and $\tilde{\Phi}$ sends the dark green segment through a point in ∂D to the blue segment through the same point in ∂D.

The map $\tilde{\Phi}$ is proper like Φ.

The map $\tilde{\Phi}$ is proper like Φ.

Since the manifolds N and M have the same dimension, and the target M of $\tilde{\Phi}$ is connected, this proper map has a well defined degree $\mod 2$.

The map $\tilde{\Phi}$ is proper like Φ.

Since the manifolds N and M have the same dimension, and the target M of $\tilde{\Phi}$ is connected, this proper map has a well defined degree $\mod 2$.

This degree is "essentially" the number of points $\mod 2$ of the inverse image by $\tilde{\Phi}$ of a "generic point" in M.

The map $\tilde{\Phi}$ is proper like Φ.

Since the manifolds N and M have the same dimension, and the target M of $\tilde{\Phi}$ is connected, this proper map has a well defined degree $\mod 2$.

This degree is “essentially” the number of points $\mod 2$ of the inverse image by $\tilde{\Phi}$ of a “generic point” in M.

We claim that the degree of $\tilde{\Phi}$ is 1.

The map $\tilde{\Phi}$ is proper like Φ.

Since the manifolds N and M have the same dimension, and the target M of $\tilde{\Phi}$ is connected, this proper map has a well defined degree $\mod 2$.

This degree is "essentially" the number of points $\mod 2$ of the inverse image by $\tilde{\Phi}$ of a "generic point" in M.

We claim that the degree of $\tilde{\Phi}$ is 1.
To prove our claim, we observe that the neighborhood V of x is disjoint from $\Phi(\partial D \times [0,1[)$, therefore contained in $\tilde{\Phi}^{-1}(V) \subset M \setminus D$. Moreover, since $V \subset M \setminus D$, on which $\tilde{\Phi}$ is the identity, we see that every point in the open set V gets covered exactly once by $\tilde{\Phi}$, hence the degree is one.

The map $\tilde{\Phi}$ is proper like Φ.

Since the manifolds N and M have the same dimension, and the target M of $\tilde{\Phi}$ is connected, this proper map has a well defined degree $\mod 2$.

This degree is "essentially" the number of points $\mod 2$ of the inverse image by $\tilde{\Phi}$ of a "generic point" in M.

We claim that the degree of $\tilde{\Phi}$ is 1.
To prove our claim, we observe that the neighborhood V of x is disjoint from $\Phi(\partial D \times [0,1[)$, therefore contained in $\tilde{\Phi}^{-1}(V) \subset M \setminus D$. Moreover, since $V \subset M \setminus D$, on which $\tilde{\Phi}$ is the identity, we see that every point in the open set V gets covered exactly once by $\tilde{\Phi}$, hence the degree is one.
Since the degree of $\tilde{\Phi}$ is 1, degree theory implies that $\tilde{\Phi}$ is surjective. In particular, the domain D is contained in the image of $\tilde{\Phi}$. But $\tilde{\Phi}$ is the identity on $N \setminus D \times [0,1[= M \setminus D$, therefore $D \subset \Phi(\partial D \times [0,1[)$, as we wanted to prove.

The local connectedness argument

We now describe the “localization” process that provide the local connectedness.

The local connectedness argument

We now describe the "localization" process that provide the local connectedness.
Since $\text{Sing}_+(\delta_C) \cap \mathcal{I}(C) = \emptyset$, we start with an open bounded set O contained in $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$.

The local connectedness argument

We now describe the "localization" process that provide the local connectedness.
Since $\mathrm{Sing}_+(\delta_C) \cap \mathcal{I}(C) = \emptyset$, we start with an open bounded set O contained in $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$.
We now construct a homotopy $\Phi_O : O \cap \mathrm{Reg}_+(C) \times [0,1[\to O$ from Φ.

The local connectedness argument

We now describe the "localization" process that provide the local connectedness.
Since $\mathrm{Sing}_+(\delta_C) \cap \mathcal{I}(C) = \emptyset$, we start with an open bounded set O contained in $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$.
We now construct a homotopy $\Phi_O : O \cap \mathrm{Reg}_+(C) \times [0,1[\to O$ from Φ.
For every $x \in O \cap \mathrm{Reg}_+(C)$, we define

$$s(x) = \sup\{s \in [0,1] \mid \Phi(x,s') \in O, \text{ for every } s' \in [0,s]\}.$$

The local connectedness argument

We now describe the "localization" process that provide the local connectedness.

Since $\mathrm{Sing}_+(\delta_C) \cap \mathcal{I}(C) = \emptyset$, we start with an open bounded set O contained in $\mathbb{R}^N \setminus (C \cup \mathcal{I}(C))$.

We now construct a homotopy $\Phi_O : O \cap \mathrm{Reg}_+(C) \times [0,1[\to O$ from Φ.

For every $x \in O \cap \mathrm{Reg}_+(C)$, we define

$$s(x) = \sup\{s \in [0,1] \mid \Phi(x, s') \in O, \text{ for every } s' \in [0,s]\}.$$

Since O is open, and Φ is continuous, the function s is > 0 and lower semi-continuous on $O \cap \mathrm{Reg}_+(C)$.

The positivity, and the lower-semi-continuity of s imply that the set

$$W_O = \{(x,s) \mid s < s(x,t)\}$$

is an open subset of $(O \cap \mathrm{Reg}_+(C)) \times [0,1[$ containing $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$, that is diffeomorphic to $(O \cap \mathrm{Reg}_+(C)) \times [0,1[$ by a diffeomorphism which is the identity on $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$.

The positivity, and the lower-semi-continuity of s imply that the set

$$W_O = \{(x, s) \mid s < s(x, t)\}$$

is an open subset of $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ containing $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$, that is diffeomorphic to $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ by a diffeomorphism which is the identity on $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$.
Therefore the homotopy Φ_O we are looking for is the restriction of Φ to W_O.

The positivity, and the lower-semi-continuity of s imply that the set

$$W_O = \{(x, s) \mid s < s(x, t)\}$$

is an open subset of $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ containing $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$, that is diffeomorphic to $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ by a diffeomorphism which is the identity on $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$.

Therefore the homotopy Φ_O we are looking for is the restriction of Φ to W_O.

Again the crux of the matter is to understand when $\Phi_O : W_O \cap (F \times [0, 1[) \to O$ is proper for a set $F \subset O \cap \mathrm{Reg}_+(C)$ which is closed in O.

The positivity, and the lower-semi-continuity of s imply that the set

$$W_O = \{(x, s) \mid s < s(x, t)\}$$

is an open subset of $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ containing $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$, that is diffeomorphic to $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ by a diffeomorphism which is the identity on $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$.
Therefore the homotopy Φ_O we are looking for is the restriction of Φ to W_O.
Again the crux of the matter is to understand when $\Phi_O : W_O \cap (F \times [0, 1[) \to O$ is proper for a set $F \subset O \cap \mathrm{Reg}_+(C)$ which is closed in O.
Like we did before, let us fix a compact subset $K \subset O$, and assume that $x_n \in F, s_n \in [0, s(x_n)[$ are such that $\Phi(x_n, s_n) = k_n \in K$.

The positivity, and the lower-semi-continuity of s imply that the set

$$W_O = \{(x, s) \mid s < s(x, t)\}$$

is an open subset of $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ containing $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$, that is diffeomorphic to $(O \cap \mathrm{Reg}_+(C)) \times [0, 1[$ by a diffeomorphism which is the identity on $(O \cap \mathrm{Reg}_+(C)) \times \{0\}$.
Therefore the homotopy Φ_O we are looking for is the restriction of Φ to W_O.
Again the crux of the matter is to understand when $\Phi_O : W_O \cap (F \times [0, 1[) \to O$ is proper for a set $F \subset O \cap \mathrm{Reg}_+(C)$ which is closed in O.
Like we did before, let us fix a compact subset $K \subset O$, and assume that $x_n \in F, s_n \in [0, s(x_n)[$ are such that $\Phi(x_n, s_n) = k_n \in K$.
The problem is to show that, up to extraction, the sequence converges to a point in $W_O \cap (F \times [0, 1[)$.

Like before we introduce the projection c_n of x_n on C.
Since O is bounded, as before, up to extraction, we can assume that $x_n \to x, c_n \to c \in C, k_n \to k \in K$, and $s_n \to s \in [0, 1]$.

Like before we introduce the projection c_n of x_n on C.
Since O is bounded, as before, up to extraction, we can assume that $x_n \to x, c_n \to c \in C, k_n \to k \in K$, and $s_n \to s \in [0, 1]$.
Since c_n is the projection of x_n on c, we obtain that $c \in P_c(x)$.
Moreover, from $k_n \in [c_n, x_n]$, we also obtain $k \in [c, x]$

Like before we introduce the projection c_n of x_n on C.
Since O is bounded, as before, up to extraction, we can assume that $x_n \to x, c_n \to c \in C, k_n \to k \in K$, and $s_n \to s \in [0,1]$.
Since c_n is the projection of x_n on c, we obtain that $c \in P_c(x)$.
Moreover, from $k_n \in [c_n, x_n]$, we also obtain $k \in [c, x]$
If we could show that $[k, x] \in O$, then necessarily $x \in F$, and by continuity $\Phi(x, s') \in [k, x] \subset O$, for each $s' \in [0, s]$, which implies $s < s(x)$, and therefore $(x, s) \in W_O \cap (F \times [0, 1[)$, as required.

This suggests to introduce the following concept.

This suggests to introduce the following concept.

Definition

An open subset $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ is said to be adapted to $\mathrm{Sing}(\delta_C)$, if it satisfies

This suggests to introduce the following concept.

Definition

An open subset $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ is said to be adapted to $\mathrm{Sing}(\delta_C)$, if it satisfies

(1) $O \cap \mathcal{I}(C) = \emptyset$,

(2) for every $x \notin C$, if $c \in P_C(x)$, and $k \in [c,x] \cap O$, then $[k,x] \subset O$.

This suggests to introduce the following concept.

Definition

An open subset $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ is said to be adapted to $\mathrm{Sing}(\delta_C)$, if it satisfies

(1) $O \cap \mathcal{I}(C) = \emptyset$,

(2) for every $x \notin C$, if $c \in P_C(x)$, and $k \in [c, x] \cap O$, then $[k, x] \subset O$.

It is not difficult to see that a connected component of an open subset $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ adapted to $\mathrm{Sing}(\delta_C)$ is itself adapted to $\mathrm{Sing}(\delta_C)$.

Therefore the analysis above yields the following proposition.

Therefore the analysis above yields the following proposition.

Proposition

Let $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ *be an open subset adapted to* $\mathrm{Sing}(\delta_C)$*. For every connected component* V *of* O*, the intersection* $V \cap \mathrm{Sing}(\delta_C)$ *is connected.*

Therefore the analysis above yields the following proposition.

Proposition

Let $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ *be an open subset adapted to* $\mathrm{Sing}(\delta_C)$. *For every connected component* V *of* O, *the intersection* $V \cap \mathrm{Sing}(\delta_C)$ *is connected.*

The local connectedness of $\mathrm{Sing}_+(\delta_C)$ follows from the following proposition.

Therefore the analysis above yields the following proposition.

Proposition

Let $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ *be an open subset adapted to* $\mathrm{Sing}(\delta_C)$*. For every connected component* V *of* O*, the intersection* $V \cap \mathrm{Sing}(\delta_C)$ *is connected.*

The local connectedness of $\mathrm{Sing}_+(\delta_C)$ follows from the following proposition.

Proposition

Let $O \subset \mathbb{R}^N(C \cup \mathcal{I}(C))$ *be an open subset. We can find an open subset* $\hat{O} \subset O$ *such that* $\hat{O}$ *is adapted to* $\mathrm{Sing}(\delta_C)$*, and* $O \cap \mathrm{Sing}(U) = \hat{O} \cap \mathrm{Sing}(\delta_C)$*.*

The proof of the proposition above is easy. In fact, let us define the set A_O as the union of intervals $[c, x]$ with $x \notin O$, and $c \in P_C(x)$.

The proof of the proposition above is easy. In fact, let us define the set A_O as the union of intervals $[c, x]$ with $x \notin O$, and $c \in P_C(x)$. It is not difficult to show that A_O is closed, and $A_O \cap O \subset \mathrm{Reg}(C)$.

The proof of the proposition above is easy. In fact, let us define the set A_O as the union of intervals $[c, x]$ with $x \notin O$, and $c \in P_C(x)$. It is not difficult to show that A_O is closed, and $A_O \cap O \subset \mathrm{Reg}(C)$. Therefore the set $\hat{O} = O \setminus (A_O \cup \mathcal{I}_+(U))$ is open and contains $O \cap \mathrm{Sing}(U)$, and by construction $\hat{O}$ is adapted to $\mathrm{Sing}(\delta_C)$.