Stochastic quantization equations

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Stochastic quantization

Stochastic quantization: consider a Euclidean quantum field theory measure as a stationary distribution of a stochastic process.

$$\exp(-S(\phi))\mathcal{D}\phi/Z \quad \Rightarrow \quad \partial_t \phi = -\delta S(\phi)/\delta\phi + \xi$$

where ξ is space-time white noise $\mathbf{E}[\xi(x,t)\xi(\bar{x},\bar{t})] = \delta^{(d)}(x-\bar{x})\delta(t-\bar{t})$.

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- Ornstein-Uhlenbeck (d=0): $S(X) = \frac{1}{2}X^2$

$$dX_t = -X_t dt + dB_t$$

- Φ^4 model: $S(\phi) = \int \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{4}\phi(x)^4 d^d x$

$$\partial_t\phi = \Delta\phi - \phi^3 + \xi$$

- Sine-Gordon (d=2): $S(\phi) = \int \frac{1}{2\beta}(\nabla\phi(x))^2 + \cos(\phi(x))d^2x$

$$\partial_t u = \frac{1}{2}\Delta u + \sin(\beta u) + \xi$$

Stochastic quantization: questions & difficulties

- Field theory: construct the measure $\exp(-S(\phi))\mathcal{D}\phi/Z$
- Stochastic PDE: study well-posedness.

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- The solution to the linear equation (when $d \geq 2$) is a.s. a distribution – so ϕ^3 and $\sin(\beta u)$ are meaningless!
- Let ξ_ε be smooth noise and $\xi_\varepsilon \to \xi$

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \sin(\beta u_\varepsilon) + \xi_\varepsilon$$

Then u_ε does not converge to any nontrivial limit as $\varepsilon \to 0$.

Two solution theories

$$\partial_t \phi = \Delta\phi - \phi^3 + \xi$$
$$\partial_t u = \frac{1}{2}\Delta u + \sin(\beta u) + \xi$$

- Da Prato - Debussche method:
  - Φ^4 in 2D (Da Prato & Debussche '03),
  - sine-Gordon with $\beta^2 < 4\pi$ (Hairer & S. '14).
- Hairer's regularity structure theory:
  - Φ^4 in 3D (Hairer '13),
  - sine-Gordon with for $4\pi \leq \beta^2 < \frac{16\pi}{3}$ (Hairer & S. '14) (expect to apply to all $\beta^2 < 8\pi$; in progress).

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- Alternative theories: Dirichlet forms for Φ^4 in 2D (Albeverio & Rockner '91); Paracontrolled distribution method by Gubinelli et al, for Φ^4 in 3D (Catellier & Chouk '13); Renormalization group method for Φ^4 in 3D (Kupiainen '14).

Outline

- Sine-Gordon equation
  - $\beta^2 < 4\pi$: Apply Da Prato - Debussche method
  - $4\pi \leq \beta^2 < 16\pi/3$: Apply regularity structure
- Φ^4 equation
- Equation from gauge theory

Da Prato - Debussche method

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \sin(\beta u_\varepsilon) + \xi_\varepsilon$$

Write $u_\varepsilon = \Phi_\varepsilon + v_\varepsilon$ where

$$\partial_t \Phi_\varepsilon = \frac{1}{2}\Delta \Phi_\varepsilon + \xi_\varepsilon$$

Then v_ε satisfies

$$\partial_t v_\varepsilon = \frac{1}{2}\Delta v_\varepsilon + \sin(\beta\Phi_\varepsilon)\cos(\beta v_\varepsilon) + \cos(\beta\Phi_\varepsilon)\sin(\beta v_\varepsilon)$$

New random input: $\exp(i\beta\Phi_\varepsilon) = \cos(\beta\Phi_\varepsilon) + i\,\sin(\beta\Phi_\varepsilon).$

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- If $\exp(i\beta\Phi_\varepsilon)$ had nontrivial limit Ψ (actually not!) then

$$\partial_t v = \frac{1}{2}\Delta v + \Psi\, f(v)$$

PDE argument

Let f be a smooth function, and $\Psi \in C^\gamma$ with $\gamma > -1$,

$$\partial_t v = \frac{1}{2}\Delta v + \Psi\, f(v)$$

Let $K = (\partial_t - \frac{1}{2}\Delta)^{-1}$ be the heat kernel. Then:

$$\mathcal{M} : \; v \mapsto K * (\Psi\, f(v))$$

defines a map from C^1 to C^1 itself:

- Young's Thm: $g \in C^\alpha, h \in C^\beta$, $\alpha + \beta > 0 \Rightarrow gh \in C^{\min(\alpha,\beta)}$

$$\Psi\, f(v) \in C^\gamma \qquad (\gamma > -1)$$

  (Example of $\alpha + \beta < 0$: BdB with $B \in C^{\frac{1}{2}-\delta}$ Brownian motion)

- Schauder's estimate: "heat kernel gives two more regularities"

$$\mathcal{M}v \in C^{\gamma+2} \subseteq C^1$$

Da Prato - Debussche method

- Back to our equation

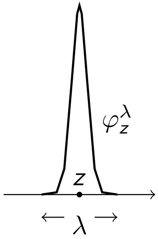
$$\partial_t v_\epsilon = \frac{1}{2}\Delta v_\epsilon + \sin(\beta\Phi_\varepsilon)\cos(\beta v_\varepsilon) + \cos(\beta\Phi_\varepsilon)\sin(\beta v_\varepsilon)$$

Q: Does $\exp(i\beta\Phi_\varepsilon)$ converge to a limit in C^γ with $\gamma > -1$?

- Kolmogorov: To show random processes $F_\varepsilon \to F \in C^\gamma$,

$$\begin{cases} \mathbb{E}\left|\langle F_\varepsilon, \varphi_z^\lambda\rangle\right|^p \lesssim \lambda^{\gamma p} \\ \lambda^{-\gamma p}\,\mathbb{E}\left|\langle F_\varepsilon - F, \varphi_z^\lambda\rangle\right|^p \to 0 \end{cases}$$

for $\forall p \geq 1$.

- 

Da Prato - Debussche method: second moment

Question $e^{i\beta\Phi_\varepsilon} \to ?$ in C^γ with $\gamma > -1$

- Want: $\mathbb{E}\left[\left|\int \varphi^\lambda(z) e^{i\beta\Phi_\varepsilon(z)} dz\right|^2\right] \lesssim \lambda^{2\gamma}$
- Using characteristic function of Gaussian

$$\mathbb{E}\left[e^{i\beta\Phi_\varepsilon(z)} e^{-i\beta\Phi_\varepsilon(z')}\right]$$
$$= e^{-\frac{\beta^2}{2}\mathbb{E}\left[(\Phi_\varepsilon(z) - \Phi_\varepsilon(z'))^2\right]}$$

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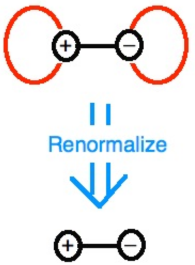
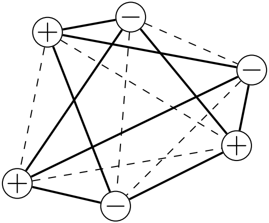


$$\begin{cases} \mathbb{E}\big[\Phi_\varepsilon(z)\Phi_\varepsilon(z')\big] \sim -\frac{1}{2\pi}\log(|z-z'|+\varepsilon) \\ \mathbb{E}\big[\Phi_\varepsilon(z)^2\big] \sim -\frac{1}{2\pi}\log\varepsilon \to \infty \end{cases}$$

Renormalise: $e^{i\beta\Phi_\varepsilon} \quad \rightsquigarrow \quad \Psi_\varepsilon \overset{def}{=} \varepsilon^{-\beta^2/(4\pi)}\, e^{i\beta\Phi_\varepsilon}$

$$\mathbb{E}\Big[\Psi_\varepsilon(z)\bar{\Psi}_\varepsilon(z')\Big] \sim (|z-z'|+\varepsilon)^{-\frac{\beta^2}{2\pi}}$$ indicates $\Psi_\varepsilon \to \Psi \in C^{-\frac{\beta^2}{2\pi}}$

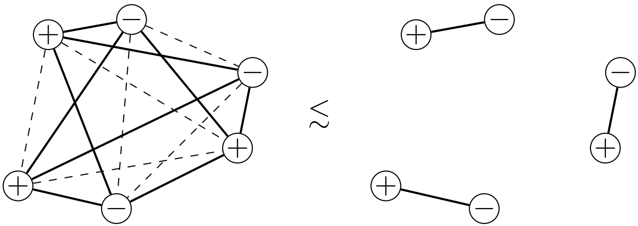
Da Prato - Debussche method: higher moments



——— $|z - z'|^{-\beta^2/(2\pi)}$

- - - - - $|z - z'|^{\beta^2/(2\pi)}$

Da Prato - Debussche method: higher moments



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Conclusion: $\partial_t v = \frac{1}{2}\Delta v + Im(\Psi)\cos(\beta v) + Re(\Psi)\sin(\beta v)$ is locally well-posed for $\beta^2 < 4\pi$. Solutions of

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \varepsilon^{-\beta^2/(4\pi)}\sin(\beta u_\varepsilon) + \xi_\varepsilon$$

converge to a limit u. (Recall $u = \Phi + v$)

Theory of regularity structure and $\beta^2 \geq 4\pi$

If $\Psi \in C^\gamma$ with $\gamma \leq -1$,

$$\partial_t v = \frac{1}{2}\Delta v + \Psi f(v)$$

"Young's theorem - Schauder's estimate" argument breaks down.

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A Stochastic ODE example:

$$dX_t = f(X_t)\, dB_t$$

- For B Brownian motion, $dB \in C^\gamma(\mathbb{R}_+)$ with $\gamma < -\frac{1}{2}$; the argument breaks down - one needs extra information to define the product $f(X_t)dB_t$.
- Extra information can be given by rough path theory.

Stochastic ODE example

For smooth function f

$$dX_t = f(X_t)\,dB_t$$

- X locally "looks like" Brownian motion

$$X_t - X_{t_0} = g_{t_0} \cdot (B_t - B_{t_0}) + \text{sth. smoother}$$

- If X is as above, then so is $f(X)$, and furthermore $\int_0^t f(X_s)\,dB_s$ is also as above.
- Only need to define one product $B\,dB$.

Theory of regularity structure and $\beta^2 \geq 4\pi$

$$dX_t = f(X_t)\, dB_t \qquad \partial_t v = \frac{1}{2}\Delta v + f(v)\,\Psi$$

$$dB \in C^{-1/2-\varepsilon} \qquad \Psi \in C^{-1-\varepsilon}$$

The solutions, at small scale, behave like

$$X \sim B = \int_0^t dB_s \qquad \text{analogous} \qquad v \sim K * \Psi$$

where $K = (\partial_t - \frac{1}{2}\Delta)^{-1}$.

- Only need to define one product $\Psi \cdot (K * \Psi)$.
- A whole theory (Theory of regularity structures recently developed by Martin Hairer) behind this "analogy".

Regularity structure and $\beta^2 \geq 4\pi$: moments of $\Psi\,(K * \Psi)$

- First moment:

$$\mathbb{E}\Big[\Psi(z)\int_{\mathbb{R}^{2+1}} K(z-w)\bar{\Psi}(w)dw\Big] = \int_{\mathbb{R}^{2+1}} K(z-w)\,|z-w|^{-\frac{\beta^2}{2\pi}}\,dw$$

For $\beta^2 \geq 4\pi$ non-integrable singularity at $z \approx w$.

- Renormalization: define the product to be

$$\Psi(K*\Psi) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\Big[\Psi_\epsilon(K*\bar{\Psi}_\epsilon) - C_\varepsilon\Big] \qquad \Big(C_\varepsilon = \int K(z)\,|z|^{-\frac{\beta^2}{2\pi}}\,dz\Big)$$

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- Need to analyze p-th moment of $\Psi(K * \Psi)$ for arbitrary p.
- This renormalization amounts to change of equation

$$\partial_t v_\epsilon = \frac{1}{2}\Delta v_\epsilon + \Big(Im(\Psi_\epsilon)\cos(\beta v_\epsilon) - C_\epsilon\cos(\beta v_\epsilon)\cos'(\beta v_\epsilon)\Big) + \Big(Re(\Psi_\epsilon)\sin(\beta v_\epsilon) - C_\epsilon\sin(\beta v_\epsilon)\sin'(\beta v_\epsilon)\Big)$$

The two red terms cancel.

Larger values of β^2

- At $\beta^2 = 4\pi$, $\Psi \in C^{-1}$ - need $\Psi \cdot K\Psi$
- At $\beta^2 = 16\pi/3$, $\Psi \in C^{-4/3}$ - need $\Psi \cdot K(\Psi \cdot K\Psi)$
- At $\beta^2 = 6\pi$, $\Psi \in C^{-3/2}$ - need $\Psi \cdot K(\Psi \cdot K(\Psi \cdot K\Psi))$
- ......

Infinite thresholds:

$$0 < 4\pi < \frac{16\pi}{3} < 6\pi < \ldots < \frac{8(n-1)}{n}\pi < \ldots \to 8\pi$$

Problem: show well-posedness for all $\beta^2 < 8\pi$ (in progress).

- Sine-Gordon field theory: Kosterlitz-Thouless phase transition at 8π (Frohlich & Spencer '81)

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Φ^4: Da Prato - Debussche method

$$\partial_t \phi = \Delta \phi - \lambda \phi^3 + \xi$$

Write $\phi = \phi_0 + v$,

$$\partial_t \phi_0 = \Delta \phi_0 + \xi$$

$$\partial_t v = \Delta v - \lambda(\phi_0^3 + 3\phi_0^2 v + 3\phi_0 v^2 + v^3)$$

Two steps in $d = 2$:

- Renormalize:
  - Replace ϕ_0^2, ϕ_0^3 by Wick powers $:\phi_0^2:, :\phi_0^3: \in C^{-\delta}$ $(\delta > 0)$.
  - Nice fact: equivalence of moments.
- Solve v by fixed point argument (say, in space $v \in C^1$)

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This method breaks down in 3D, where $:\phi_0^2: \in C^{-1-\delta}$ $(\delta > 0)$

Φ^4 in 3D: regularity structures

Regularity structure theory:

- fixed point argument in space of functions/distributions with delicate descriptions of local behavior, i.e. locally behave like Brownian path or $K * \Psi$ etc.
- only need to define a few objects (often through renormalization procedure).

Φ^4 in 3D: regularity structures

Regularity structure theory:

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For Φ^4 equation in 3D, write $\phi = \phi_0 + \lambda\phi_1 + R$ where

$$\partial_t \phi_0 = \Delta\phi_0 + \xi$$

$$\partial_t \phi_1 = \Delta\phi_1 - \phi_0^3$$

with ansatz

$$R \sim K * (\phi_0^2) \in C^{1-\delta} \quad (\delta > 0)$$

Need to define $\phi_0^2 \cdot (K * (\phi_0^2))$ - another renormalization.

Remarks

Some other results:

- Global solution / solution on entire space: Φ^4 (Mourrat & Weber) (Hairer & Matetski)
- Weak universality results: KPZ (Hairer & Quastel), Φ^4 (Hairer & Xu) (Xu & S.)
- Non-Gaussian noises and central limit theorems: KPZ (Hairer & S.), Φ^4 (Xu & S.)
- Convergence from particle systems: Φ^4 (Mourrat & Weber)

Stochastic quantization of Maxwell Eq.

$d = 2$. Let (A_1, A_2) be a vector field. Let $F_A = \partial_1 A_2 - \partial_2 A_1$.

$$\exp\Big(-\frac{1}{2}\int F_A^2\,dx\Big)\mathcal{D}A = \exp\Big(-\frac{1}{2}\int(\partial_1 A_2 - \partial_2 A_1)^2\,dx\Big)\mathcal{D}A$$

The stochastic PDE:

$$\partial_t A_1 = -\partial_2 F_A + \xi_1 = \partial_2^2 A_1 - \partial_1\partial_2 A_2 + \xi_1$$
$$\partial_t A_2 = \partial_1 F_A + \xi_2 = \partial_1^2 A_2 - \partial_1\partial_2 A_1 + \xi_2$$

- Not parobolic.
- Gauge symmetry: For any function $f(x)$ let $\bar{A}_j = A_j + \partial_j f$. Then
$$F_{\bar{A}} = \partial_1\bar{A}_2 - \partial_2\bar{A}_1 = \partial_1 A_2 - \partial_2 A_1 = F_A$$
therefore $\bar{A}$ satisfies the same equation.

Stochastic quantization of Maxwell Eq.

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$$\partial_t A_2 = \partial_1 F_A + \xi_2 = \partial_1^2 A_2 - \partial_1 \partial_2 A_1 + \xi_2$$

"De Turck trick": Let B solves

$$\partial_t B_j = \Delta B_j + \xi_j \qquad (j = 1, 2)$$

Then let

$$A_1(t) = B_1(t) - \int_0^t \partial_1 \nabla \cdot B(s)\, ds$$
$$A_2(t) = B_2(t) - \int_0^t \partial_2 \nabla \cdot B(s)\, ds$$

We can check:

$$\partial_t A_1 = -\partial_2 F_B + \xi_1, \quad \partial_t A_2 = \partial_1 F_B + \xi_2$$

and, $\qquad F_A = \partial_1 A_2 - \partial_2 A_1 = \partial_1 B_2 - \partial_2 B_1 = F_B$

Nonlinear SPDE with gauge

Consider quantum field theory for vector A and complex valued Φ

$$\exp\Big(-\frac{1}{2}\int \big(F_A^2+\sum_{j=1,2}|D_j^A\Phi|^2\big)\,dx\Big)\mathcal{D}A\mathcal{D}\Phi$$

The stochastic PDE:

$$\begin{aligned}
\partial_t A_1 &= \partial_2^2 A_1 - \partial_1\partial_2 A_2 - i\lambda\Big(\bar\Phi D_1^A\Phi - \Phi\overline{D_1^A\Phi}\Big) + \xi_1 \\
\partial_t A_2 &= \partial_1^2 A_2 - \partial_1\partial_2 A_1 - i\lambda\Big(\bar\Phi D_2^A\Phi - \Phi\overline{D_2^A\Phi}\Big) + \xi_2 \\
\partial_t \Phi &= \sum_j D_j^A D_j^A \Phi + \zeta
\end{aligned}$$

where D_j^A is gauge covariant derivative:

$$D_j^A\Phi = \partial_j\Phi - i\lambda A_j\Phi \qquad (j=1,2)$$

- Let $(A^{(\varepsilon)},\Phi^{(\varepsilon)})$ solve the smooth noise Eq, show limit $\varepsilon\to 0$.

Nonlinear SPDE with gauge

$$\partial_t A_1 = \partial_2^2 A_1 - \partial_1 \partial_2 A_2 - i\lambda\Big(\bar{\Phi} D_1^A \Phi - \Phi \overline{D_1^A \Phi}\Big) + \xi_1$$

$$\partial_t A_2 = \partial_1^2 A_2 - \partial_1 \partial_2 A_1 - i\lambda\Big(\bar{\Phi} D_2^A \Phi - \Phi \overline{D_2^A \Phi}\Big) + \xi_2$$

$$\partial_t \Phi = \sum_j D_j^A D_j^A \Phi + \zeta$$

De Turck trick:

$$\partial_t B_j^{(\varepsilon)} = \Delta B_j^{(\varepsilon)} - i\lambda\Big(\overline{\Psi^{(\varepsilon)}} D_j^{B^{(\varepsilon)}} \Psi^{(\varepsilon)} - \Psi^{(\varepsilon)} \overline{D_j^{B^{(\varepsilon)}} \Psi^{(\varepsilon)}}\Big) + \xi_j^{(\varepsilon)}$$

$$\partial_t \Psi^{(\varepsilon)} = \sum_j D_j^{B^{(\varepsilon)}} D_j^{B^{(\varepsilon)}} \Psi^{(\varepsilon)} + i\lambda(\nabla \cdot B^{(\varepsilon)})\Psi^{(\varepsilon)} + e^{i\lambda \int_0^t \nabla \cdot B^{(\varepsilon)}(s)\, ds} \zeta^{(\varepsilon)}$$

Then define

$$A_j^{(\varepsilon)}(t) = B_j^{(\varepsilon)}(t) - \int_0^t \partial_j \nabla \cdot B^{(\varepsilon)}(s)\, ds$$

$$\Phi^{(\varepsilon)}(t) = e^{-i\lambda \int_0^t \nabla \cdot B^{(\varepsilon)}(s)\, ds} \Psi^{(\varepsilon)}(t)$$

Nonlinear SPDE with gauge

$$\partial_t A_1 = \partial_2^2 A_1 - \partial_1 \partial_2 A_2 - i\lambda\Big(\bar{\Phi} D_1^A \Phi - \Phi \overline{D_1^A \Phi}\Big) + \xi_1$$
$$\partial_t A_2 = \partial_1^2 A_2 - \partial_1 \partial_2 A_1 - i\lambda\Big(\bar{\Phi} D_2^A \Phi - \Phi \overline{D_2^A \Phi}\Big) + \xi_2$$
$$\partial_t \Phi = \sum_j D_j^A D_j^A \Phi + \zeta$$

Expected result: Let $(A^{(\varepsilon)}, \Phi^{(\varepsilon)})$ solve the smooth noise Eq. There exists a family (parametrized by time) of gauge transformations $\mathcal{G}_t^{(\varepsilon)}$ such that $\mathcal{G}_t^{(\varepsilon)}(A^{(\varepsilon)}, \Phi^{(\varepsilon)})$ converges to a nontrivial limit.