Recent advances on the inverse spectral problem

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April 28, 2020

Can one hear the shape of a drum?

A drum (in this talk) is a smooth plane domain $\Omega \subset \mathbb{R}^2$. A drum vibrates freely at its natural frequencies $\sqrt{\lambda_j}$, which are the square roots of the Dirichlet eigenvalues,

$$\begin{cases} \Delta\varphi = -\lambda\varphi, \\ \\ \varphi|_{\partial\Omega} = 0. \end{cases}$$

There exists an orthonormal basis $\{\varphi_j^\Omega\}_{j=1}^\infty$ of $L^2(\Omega)$ of eigenfunctions. The Laplace spectral map is the ordered list of eigenvalues,

$$\Lambda(\Omega) = (\lambda_1^\Omega, \lambda_2^\Omega, \cdots, \lambda_n^\Omega, \dots).$$

Kac's problem: Is Λ $1-1$ from the 'space of domains' to $\mathbb{R}_+^\infty$?

Classical and new results on Laplace spectral determination

- The only domain previously known to be spectrally determined is the disk **D** (M. Kac, 1965);
- The only domains known to be NOT spectrally determined are certain non-convex polygonal domains (hence, with corners) (Sunada, Buser, Gordon-Webb-Wolpert).
- The new Laplace spectral result is:

THEOREM

(Hamid Hezari and S. Zelditch (2019)) Ellipses of small eccentricity are spectrally determined among all smooth plane domains.

This builds on 'dynamical length inverse spectral results' of Avila, de Simoi, Kaloshin.

Proof that disks are spectrally determined

To say the disk $\mathbf{D}$ of area A is spectrally determined is to say that if

$$\Lambda(\mathbf{D}_A) = \Lambda(\Omega) \implies \mathbf{D}_A = \Omega.$$

Here,

$$\Lambda(\Omega) = (\lambda_1^\Omega, \lambda_2^\Omega, \cdots, \lambda_n^\Omega, \dots).$$

From $\Lambda(\Omega)$ one has the 'generating function'

$$H(t) := \mathrm{Tr}\ \mathrm{e}^{\mathrm{t}\Delta_\Omega} = \sum_{\mathrm{j}=1}^{\infty} \mathrm{e}^{-\mathrm{t}\lambda_\mathrm{j}^\Omega}.$$

Kac proved that

$$H(t) \sim t^{-1}\mathrm{Area}(\Omega) + \mathrm{t}^{-\frac{1}{2}}\mathrm{Length}(\partial\Omega) + \mathrm{O}(1), \quad \mathrm{t} \to 0^+.$$

Hence both the area and perimeter are *spectral invariants*. But for any domain $L^2 \geq 4\pi A$ with equality if and only if the domain is a disk.

Quantum versus classical mechanics

Let E_e be an ellipse of eccentricity e. The proof of our main result,

$$\Lambda(\Omega) = \Lambda(E_e) \implies \Omega = E_e$$

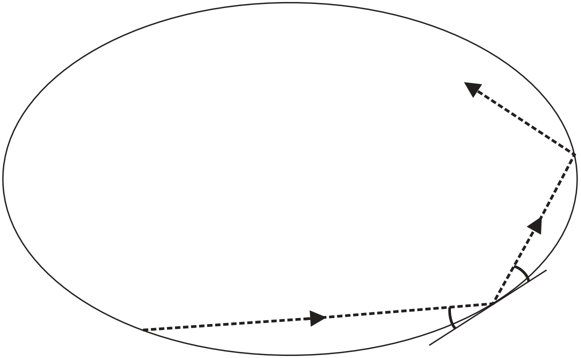
is very different from Kac'. It is partly based on new results on billiard dynamics, due to Avila, Kaloshin, de Simoi, Sorrentino, Wei and others on billiard dynamics.

The link between eigenvalues of Δ (quantum mechanics) and billiard dynamics is through the *wave trace formula* for $\mathrm{Tr} e^{it\sqrt{-\Delta}}$.

In addition to the billiard dynamics, the proof is based on very special properties of the wave trace for any 'nearly circular' domain Ω. Only at the last step in the proof do we use that Ω is iso-spectral to E_e. There is potential for more general results.

Billiards

The classical mechanics underlying the Laplace eigenvalue problem is billiards on the table Ω: hit a ball at a point p with velocity v, and it travels along straight lines in Ω and bounces off the boundary by Snell's law of equal angles. This defines a broken geodesic flow $G^t(p, v)$ on the set $S^*\Omega$ of unit tangent vectors on Ω.



Billiard map

Billards are entirely determined by the discrete time billiard map: $\beta : S^*_{in}(\partial\Omega) \to S^*_{in}(\partial\Omega)$ on inward pointing unit vectors. The billiard map $\beta(x, \theta)$ follows an inward pointing unit vector along the straight line from its initial point to its next impact with $\partial\Omega$, then reflects the tangent vector inward. We identify $S^*_{in}(\partial\Omega)$ with $B^*\partial\Omega$ (sub-unit tangent vectors to $\partial\Omega$) by projection, and with (s, φ) in the annulus,

$$\Pi = \mathbb{R}/\ell\mathbb{Z} \times [0, \pi], \quad \ell = |\partial\Omega|,$$

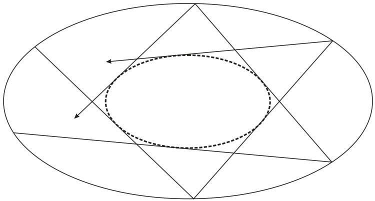
where φ represents the angle that the inward unit vector at s makes with the positive unit tangent vector at s. We write the billiard map as

$$\begin{cases} \beta : \Pi \to \Pi, \\ \beta(s, \varphi) = (s_1(s, \varphi), \varphi_1(s, \varphi)) . \end{cases}$$

β is an area-preserving diffeomorphism ('twist map').

Caustics and Invariant curves

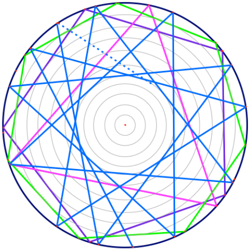
A curve $\mathcal{C}$ lying in Ω is called a caustic if any tangent line drawn to $\mathcal{C}$ remains a tangent to $\mathcal{C}$ after reflection at the boundary of Ω.

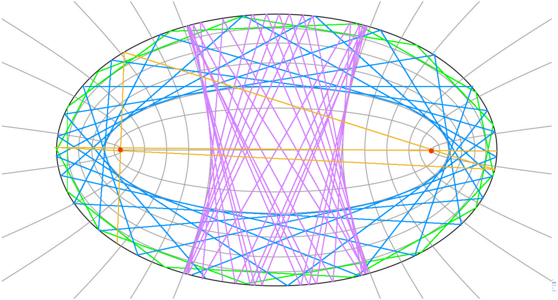


A caustic $\mathcal{C}$ gives rise to a smooth closed invariant curve of β in the phase space $B^*\partial\Omega$ (corresponding to tangents to the caustic).

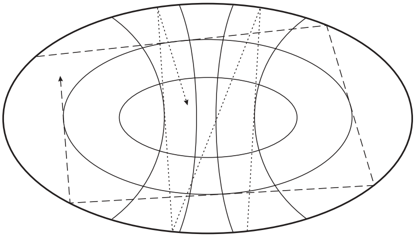
In the picture, the caustic is a convex closed curve (hence, called a convex caustic). There may also be non-convex caustics.

Elliptic and hyperbolic caustics of ellipses; circles only have circular caustics

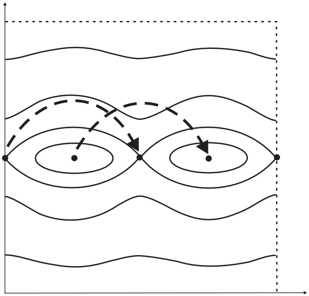




Billiard inside ellipses E: foliation of E by convex caustics; foliation of B^*E by invariant curves



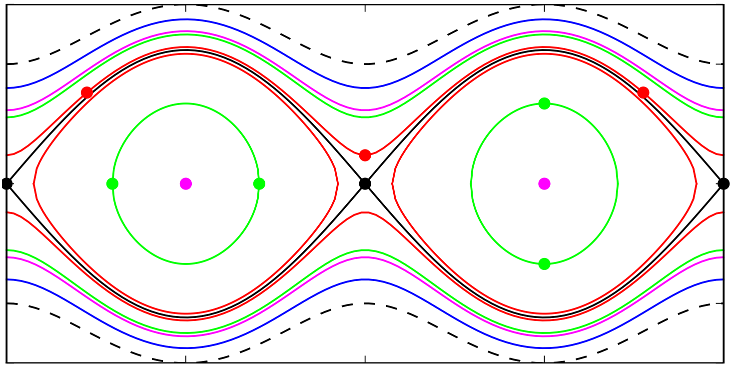
Confocal ellipses and hyperbolas



Phase space foliation

Birkhoff Conjecture: The only integrable strictly convex billiards are ellipses. Integrable means that the union of all convex caustics has a non-empty interior in $\mathbb{R}^2$.

Phase space portrait



Phase space foliation of the ellipse by invariants curves

Purple dots correspond to the bouncing ball orbit on the minor axis. **Black** dots correspond to the bouncing ball orbit on the major axis. Green dots correspond to a 4-periodic orbit tangent to a confocal hyperbola. Red dots correspond to a 3-periodic orbit tangent to a confocal ellipse.

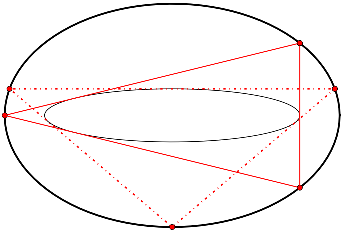
Periodic orbits inside convex domains

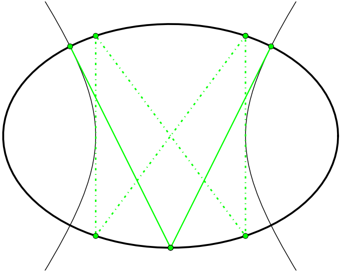
A periodic billiard trajectory in Ω is a trajectory which smoothly closes up (its terminal position and velocity equal the initial position and velocity.)

A periodic billiard orbit γ has a length $L(\gamma)$, a winding number p and a bounce number q: p is the number of times it winds around the boundary, and q is the number of bounces off the boundary. We call the set of such periodic orbits $\Gamma(p, q)$. The rotation number of a $\Gamma(p, q)$ orbit is $\frac{p}{q}$.
Birkhoff: For each $\frac{p}{q} \in (0, \frac{1}{2}]$ in lowest terms, there exist at least two distinct periodic orbits with rotation number $\frac{p}{q}$.

$\zeta = (s, \varphi) \in S^*_{in}(\partial\Omega)$ lies on a periodic orbit with q bounces if $\beta^q(\zeta) = \zeta$.

Periodic orbits in an ellipse





Poncelet[1822]: Periodic orbits come in 1-parameter families: If a primitive periodic billiard trajectory is tangent to a caustic of an ellipse then all orbits tangent to that caustic are also periodic, have the same periods, and have the same lengths.

Circular domains (disk)

A billiard trajectory of the disk is determined by the angle θ it makes with the circle. The angle remains the same after each reflection.If $\theta = \frac{p}{q}$ then the billiard orbit with this angle is periodic of period q and makes p turns around the boundary.

- A $\Gamma(1, q)$ orbit is a regular q-gon (The angles are the same at all q bounce points).
- The length of a $\Gamma(1, q)$ orbit is $2q \sin \frac{\pi}{q}$.
- The rotate of a periodic orbit is periodic: periodic orbits come in circular families.

Question: What happens if we perturb the disk?

General elliptical billiards are integrable

Billiards on an ellipse E_e of any eccentricity e,

$$E_e = \left\{(x, y) \in \mathbb{R}^2;\ x^2 + \frac{y^2}{1-e^2} \leq 1\right\}$$

are ‘completely integrable’.

- The ellipse E_e is foliated by caustics (with a singular ‘leaf’ through the foci). There is no singular leaf for the unit disk.
- The phase space $S^*_{in}\partial E_e = B^*\partial E_e$ is foliated by invariant smooth curves for the billiard map β;

Birkhoff conjecture: The ellipse is the only domain with integrable billiards. Avila-de Simoi-Kaloshin, Kaloshin-Sorrentino have proved ‘local versions’.

Rationally integrable

Definition:
(i) An integrable rational caustic $\Gamma \subset \Omega$ *is a caustic such that tangential orbits are periodic. The corresponding (non-contractible) invariant curve in* $B^*\partial\Omega = \Pi$ *consists of periodic points (hence the rotation number is rational).*

(ii) Ω *is rationally integrable if the billiard map of* Ω *admits integrable rational caustics of rotation number* $\frac{1}{q}$ *for all* $q \geq 3$.

Rationally integrable is weaker than integrability: one does not use a foliation by caustics or invariant curves, but only a 'partial, rational, foliation' by invariant curves of periodic points.

Rigidity of ellipses

Birkhoff Conjecture: The only integrable strictly convex billiards are ellipses. Integrable means that the union of all convex caustics has a non-empty interior in $\mathbb{R}^2$.

Avila-De Simoi-Kaloshin[15]: A rationally integrable billiard table which is nearly circular must be an ellipse.

Kaloshin-Sorrentino[17]: A rationally integrable billiard table which is sufficiently close to an ellipse must be an ellipse.

This is good news for Δ-spectral analysts, because the Δ-spectrum only 'sees' periodic orbits.

Deformations (perturbations) of a domain

Inverse results are often based on perturbation theory of domains. If Ω is a domain, we perturb it by deforming the boundary:

$$\partial\Omega \to \partial\Omega + \epsilon\vec{n}$$

where $\vec{n}(x) = \rho(x)\nu(x)$ is a multiple $\rho(x)$ of the unit normal $\nu(x)$ at $x \in \partial\Omega$.

- (Ramirez-Ros) If the foliation by caustics of $\mathbf{D}$ persists under perturbation $\partial\mathbf{D} + \epsilon\vec{n}$, then $\vec{n}$ is trivial. Proof: study the Fourier coefficients of $\vec{n}$; they must be trivial.
- (Avila-de Simoi-Kaloshin) For ellipses of small eccentricity, if rational caustics persist under perturbation, then the perturbation is trivial. Proof: construct a good basis like sines and cosines on $\partial\Omega$ and prove orthogonality relations for $\vec{n}$.
- A domain is δ-*nearly circular* if $\partial\Omega = \partial\mathbf{D} + \vec{n}$ where $\vec{n}$ and 39 of its derivatives are $< \delta$.

Laplace spectral rigidity of ellipses

One can also study perturbations of eigenvalues.
Hezari.- Z[2012]: Ellipses are infinitesimally spectrally rigid among smooth domains with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ reflectional symmetries of the ellipse.

Popov-Topalov[2016]: Analogous result holds for ellipsoids in $\mathbb{R}^3$.

The new result uses geometric perturbation theory:

Theorem

(Hamid Hezari and S. Zelditch (2019)) Ellipses of small eccentricity are spectrally determined among all smooth plane domains.

The proof uses very special properties of the 'length spectrum' and the 'wave trace' of a nearly circular domain.

Length spectrum for nearly circular billiards

The length spectrum is the set of lengths of closed billiard trajectories,

$$Lsp(\Omega) = \{L \in \mathbb{R} : \exists \text{ periodic } \gamma : \mathrm{L}(\gamma) = \mathrm{L}\}.$$

Let $\Gamma(p, q)$ be the periodic orbits with winding number p and bounce number q. Let $\mathcal{L}\Gamma(p, q)$ be the lengths $L(\gamma)$ of the periodic orbits $\gamma \in \Gamma(p, q)$.

We are mainly concerned with $\Gamma(1, q)$ orbits. For a nearly-circular domain, only these orbits have lengths $< |\partial\Omega|$. Let

$$t_q = \inf_{\gamma \in \Gamma(1,q)} L(\gamma), \quad T_q = \sup_{\gamma \in \Gamma(1,q)} L(\gamma).$$

Length spectrum and the band-gap property for nearly circular billiards

The length spectrum is the set of lengths of closed billiard trajectories,

$$Lsp(\Omega) = \{L \in \mathbb{R} : \exists \text{ periodic } \gamma : \mathrm{L}_\gamma = \mathrm{L}\}.$$

BAND-GAP property of nearly-circular domains ($\ell = |\partial\Omega|$):

$$Lsp(\Omega) \cap [0, \ell] = \bigcup_{q \geq 2} \mathcal{L}\Gamma(1, q) \subset \bigcup_{q \geq 2} [t_q, T_q]$$

where the band = interval $[t_q, T_q]$ contains the lengths of $\Gamma(1, q)$ periodic orbits with q bounces. The gaps $T_{q+1} - t_q \sim q^{-3}$ are MUCH bigger than the bands $T_q - t_q \sim q^{-\infty}$: the bounce number q is determined by the length of the orbit.
Prior results: Marvizi-Melrose (did $q \geq q_0(\Omega)$).

More details on Band-gap property for nearly circular billiards

Let $\partial\Omega_\tau = \partial E_0 + \tau f N_0$ be a curve of nearly circular domains. We need dependence of estimates $[t_q, T_q] = O(q^{-\infty}) >> $ gaps $t_{q+1} - T_q \sim q^3$, on f:

Assume $\|f\|_{C^8} \leq 1$ and $\|f\|_{C^2}$ is sufficiently small so that $\kappa_\tau = 1 + O(\|f\|_{C^2}) \geq \frac{1}{2}$. Then,

$$T_q - t_q = q^{-3} O(\|f\|_{C^6}) + O(q^{-4}), \tag{1}$$

$$T_q = \ell_\tau - \frac{1}{4}\left(\int_0^{\ell_\tau} \kappa_\tau^{2/3}(s)\, ds\right)^3 q^{-2} + q^{-3} O(\|f\|_{C^6}) + O(q^{-4}). \tag{2}$$

Wave trace approach (Poisson relation)

The inverse spectral results are based on the trace of the wave group $U(t) = e^{it\sqrt{-\Delta}}$ of on $L^2(\Omega)$. The wave trace is,

$$TrU(t) = \sum_{\lambda_j \in Sp(-\Delta)} e^{it\sqrt{\lambda_j}}.$$

It is a tempered distribution on $\mathbb{R}$, which is smooth except when $t \in \pm Lsp(\Omega) \cup \{0\}$, where

$$Lsp(\Omega) = \{L \in \mathbb{R} : \exists \text{ periodic } \gamma : \mathrm{L}(\gamma) = \mathrm{L}\}.$$

Note:

$$\bigcup_{q \geq 2} \mathcal{L}\Gamma(1, q) = Lsp(\Omega) \cap [0, |\partial\Omega|].$$

THEOREM (Hezari-Z)

For a nearly circular domain, $\mathrm{TrU(t)}$ *determines the entire* $\bigcup_{q \geq 2} \mathcal{L}\Gamma(1, q)$. *Moreover, it determines bounce numbers for each length.*

Ideas of Proof (I): Singularity expansions

On a length interval where all closed geodesics are Bott-Morse non-degenerate, $TrU(t)$ has a singularity expansion at each $L \in Lsp(\Omega)$:

$$TrU(t) \equiv e_0(t) + \sum_{L \in Lsp(\Omega)} e_L(t) \quad mod \quad C^\infty, \tag{3}$$

where e_0, e_L are Lagrangean distributions with singularities at just one point, i.e. $singsuppe_0 = \{0\}, singsuppe_L = \{L\}$.

At $t = 0$ the wave trace is essentially equivalent to the heat trace:

$$e_0(t) = a_{0,-n}(t + i0)^{-n} + a_{0,-n+1}(t + i0)^{-n+1} + \cdots \tag{4}$$

The wave coefficients $a_{0,k}$ at $t = 0$ are essentially the same as the singular heat coefficients, and are integrals of curvature invariants.

Domains isospectral to a nearly circular ellipse are convex and nearly circular

Using the wave trace coefficients at $t = 0$ (and many prior results), one has:

Proposition

Let E_e denote an ellipse of eccentricity e. Suppose Ω and E_e are isospectral. Then after a rigid motion, the domain Ω is $\mathcal{O}_n(e)$-close in C^n to the unit disk E_0 for all $n \in \mathbb{N}$. In particular, for sufficiently small e, Ω is convex.

Proof: Use the wave invariants at $t = 0$ to show, ($e =$ eccentricity)

$$\int_0^{\ell_e} |\kappa(\Omega) - \kappa(E_e)|^2 ds \leq C\ e, \quad \kappa = \text{curvature}$$

$$\int_0^{\ell_e} (\kappa^{(n)})^2(\Omega)\, ds \leq C_n\ e, \quad n \geq 1.$$

C, C_n are universal constants.

Singularities at non-degenerate closed geodesics

When the closed orbits are Bott-Morse, the wave invariants for $t \neq 0$ have the form:

$$e_L(t) = \sum_{\gamma : L(\gamma) = L} e_\gamma(t), \tag{5}$$

where $e_\gamma(t)$ is the contribution of the periodic orbit γ. If γ is *non-degenerate*,

$$\begin{aligned} e_L(t) &= a_{L,-1}(t - L + i0)^{-1} + a_{L,0} \log(t - L + i0) \\ &+ a_{L,1}(t - L + i0) \log(t - L + i0) + \cdots \quad , \end{aligned} \tag{6}$$

where $\cdots$ refers to homogeneous terms of ever higher integral degrees.

All periodic orbits with $p = 1, q > 2$ of E_e are degenerate but Bott-Morse non-degenerate.

q-bounce decomposition of the wave trace

The trace of the wave group has a q-bounce decomposition in $(0, \ell), \ell = |\partial\Omega|$:

$$\mathrm{TrU(t)}|_{(0,\ell)} = \sum_{\mathrm{q}\geq 2} \hat{\sigma}_{1,\mathrm{q}}(\mathrm{t}) \tag{7}$$

where the qth term corresponds to q-bounce periodic orbits. For discrete length spectra, $Tr\ U(t)|_{[t_q, T_q]} = \hat{\sigma}_{1,q}(t)$

$$\hat{\sigma}_{1,q}(t) = \sum_{L \in \mathcal{L}\Gamma_{1,q}} e_{1,q,L}(t), \quad \text{with} \quad \mathrm{e}_{1,\mathrm{q},\mathrm{L}}(\mathrm{t}) = \sum_{\gamma \in \Gamma(1,\mathrm{q}):\mathrm{L}(\gamma)=\mathrm{L}} \mathrm{e}_\gamma(\mathrm{t}). \tag{8}$$

THEOREM (Hezari -Z)

For a nearly circular domain, the singularities of $\hat{\sigma}_{1,q}$ can be 'heard' – they are spectral invariants for each $q \geq 3$.

Melrose-Marvizi parametrix

Proposition

If Ω is a C^∞ bounded domain with $\ell = |\partial\Omega|$, and nearly circular strictly convex planar region, then for $q \geq 3$, up to a (known) Maslov phase factor, Tr $U(t)|_{[t_q, T_q]} = \hat{\sigma}_{1,q}(t)$,

$$Tr\ U(t)|_{[-t_q, T_q]} = \frac{1}{2\pi} \int_0^\infty \int_0^{|\partial\Omega|} e^{i\tau(t-\mu_q(s))} a_q(s,\tau) ds d\tau,$$

where μ_q is the qth loop length function:

$$\mu_q(s) = \psi_q(s, s')|_{s=s'},$$

where $\psi_q(s,s') = L(\mathfrak{g})$, where $\mathfrak{g}$ is q-fold reflected geodesic from s to s'. $\mu_q(s)$ is the length of the unique broken geodesic loop at s with q bounces. The amplitude $a_q(s,\tau)$ is a symbol, with positive leading order coefficient $a_q(s) > 0$.

Length spectrum multiplicity and audibility

The perennial problem in inverse spectral theory is multiplicity in the length spectrum – the possibility that the 'wave trace singularity invariants' at two closed geodesics of the same length may cancel each other and be 'inaudible.'

- Cancellation can only occur for non-degenerate orbits (due to Soga for 1D oscillatory integrals, e.g. the Marvizi-Melrose integrals);
- Cancellation can only occur if the Maslov indices σ in the phases i^{σ} of two orbits have opposite signs. But all $\Gamma(1, q)$ orbits with $q \geq 3$ and lengths $L < |\partial\Omega|$ cannot have opposite sign Maslov indices.

Completion of proof

Since one can 'hear' $\hat{\sigma}_{1,q}$, for $q \geq 3$,

$$\hat{\sigma}^{\Omega}_{1,q}(t) = \hat{\sigma}^{E_e}_{1,q}(t),$$

Proposition

Assume that Ω is isospectral to an ellipse E_e of small eccentricity. Let $\ell = |\partial\Omega| = |\partial E_e|$. Let $L^{E_e}_q$ be the loop length function for E_e (a constant). Then,

$$\int_0^{\ell} e^{i\lambda\mu_q(s)} a_q(s,\lambda) ds = e^{i\lambda L^E_q} \int_0^{\ell} a^{E_e}_q(s,\lambda) ds + O(\lambda^{-\infty}).$$

For an ellipse the right side is a symbol of order 0.

We now show that this equality forces $\mu_q(s)$ to be constant.

$\mu_q(s)$ must be constant

Critical points of $\mu_q(s)$ are points $s \in \partial\Omega$ where there exists a closed orbit of the Billiard flow in the class $\Gamma(1, q)$ starting at s. In the case of the ellipse, the loop length function is a constant.
The final point is to prove:

Lemma

μ_q has exactly one critical value. Hence, μ_q is constant. Thus, every loop is a closed billiard trajectory and a loop exists at every $s \in \partial\Omega$.

NOTE: the loop length function $\mu_q(s)$ is one of the main objects we study. Its variation is the 'Melnikov function', the key thing studied by Avila-de Simoi-Kaloshin.

Final remarks

- Until the last step, where we use that the ‘bands’ collapse to points, the results hold for all nearly circular domains. In particular, the Melrose-Marvizi integrals are spectral invariants of nearly circular domains. The goal is to ‘recover’ the phase of the integrals from the asymptotics.
- All periodic orbits of E_e are degenerate except the bouncing ball orbits of the major/minor axes. The same would be true for any isospectral domain if one can rule out cancellations of wave invariants.
- The critical point set of the phase μ_q of a general C^∞ (even, nearly circular) domain can be an arbitrary closed set, e.g. a Cantor set. But for real analytic domains, either the phase is constant or it has only finitely many isolated critical points with finite vanishing order. We can show that it must be rationally integrable ‘close to the boundary.’

The four levels of spectral determinacy

For either the Laplace spectral map Λ, or the length spectral map $\mathcal{L}$, we say that a domain Ω is:

- spectrally determined when for any $\Omega' : \Lambda(\Omega) = \Lambda(\Omega')$ if and only if $\Omega = \Omega'$;
- locally spectrally determined if there exists a neighborhood $\mathcal{U} \subset M$ of Ω so that forany $\Omega' \in \mathcal{U}, \Lambda(\Omega) = \Lambda(\Omega')$ if and only if $\Omega = \Omega'$
- spectrally rigid if any smooth one-parameter isospectral family Ω_t with $t \in (-\epsilon, \epsilon)$ of domains $\Lambda(\Omega_t) = \Lambda(\Omega_0)$ is isometric;
- infinitesimally spectrally rigid if any smooth one-parameter isospectral family Ω_t with $t \in (-\epsilon, \epsilon)$ of domains with $\Lambda(\Omega_t) = \Lambda(\Omega)$ is so that Ω_t is tangent to an isometric family at $t = 0$.