Computing maps between Fukaya categories via Morse trees

Nathaniel Bottman

Princeton/IAS

September 2017

§1: context

The Fukaya category of a symplectic manifold

The Fukaya category of a symplectic manifold

A **symplectic manifold** is (M^{2n}, ω), with $\omega \in \Omega^2(M)$ closed, $\omega^{\wedge n}$ nonvanishing.

Eg: $(M, \omega) =$ (real surface, area form).

The Fukaya category of a symplectic manifold

A **symplectic manifold** is (M^{2n}, ω), with $\omega \in \Omega^2(M)$ closed, $\omega^{\wedge n}$ nonvanishing.

Eg: $(M, \omega) =$ (real surface, area form).

A **Lagrangian** is $L^n \subset M^{2n}$ with $\omega|_L = 0$.

Eg: $L =$ embedded curve.

The Fukaya category of a symplectic manifold

The Fukaya category of a symplectic manifold

Roughly, the **Fukaya category** $\mathrm{Fuk}(M, \omega)$ has:

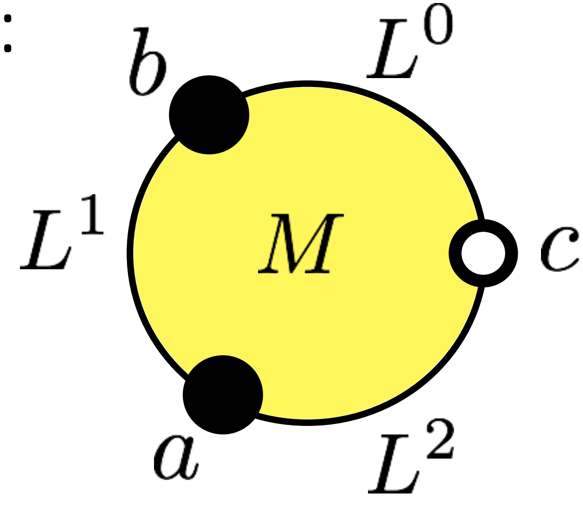
- objects are Lagrangians $L \subset M$,
- morphisms are $\hom(L, K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$.

The Fukaya category of a symplectic manifold

Roughly, the **Fukaya category** $\mathrm{Fuk}(M,\omega)$ has:

- objects are Lagrangians $L \subset M$,
- morphisms are $\hom(L,K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$.

Composition? Fix $a \in L^2 \cap L^1, b \in L^1 \cap L^0$; coefficient of $c \in L^2 \cap L^0$ in $a \circ b$ is a count of rigid pseudoholomorphic triangles:

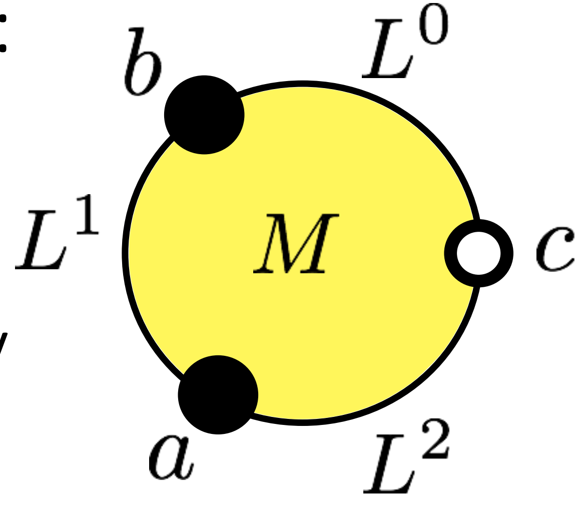


The Fukaya category of a symplectic manifold

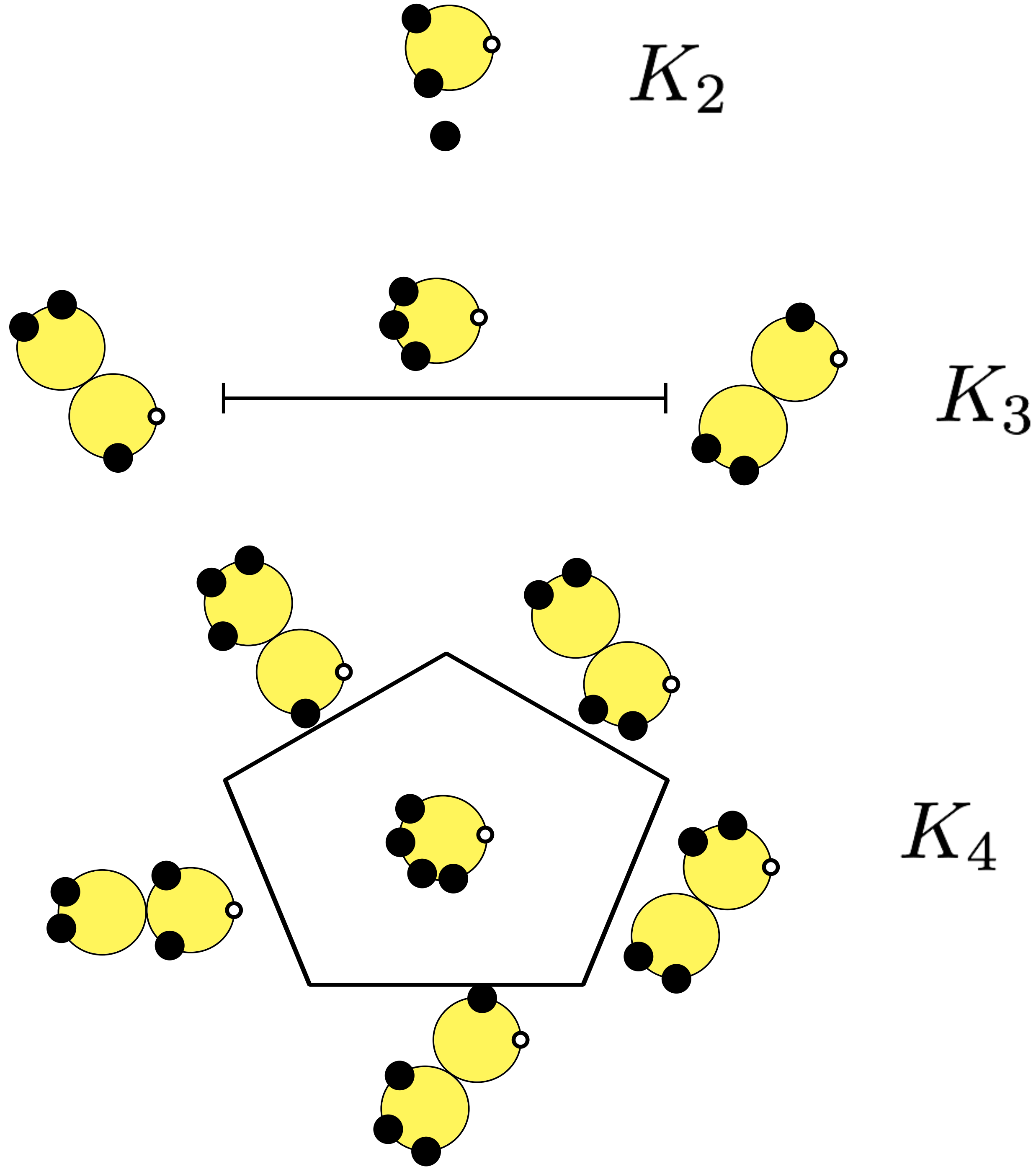
Roughly, the **Fukaya category** $\mathrm{Fuk}(M, \omega)$ has:

- objects are Lagrangians $L \subset M$,
- morphisms are $\hom(L, K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$.

Composition? Fix $a \in L^2 \cap L^1, b \in L^1 \cap L^0$; coefficient of $c \in L^2 \cap L^0$ in $a \circ b$ is a count of rigid pseudoholomorphic triangles:



...composition **not** associative!
but can make into an A_∞-category by counting rigid polygons.


K_2
K_3
K_4

Functoriality for Fuk?

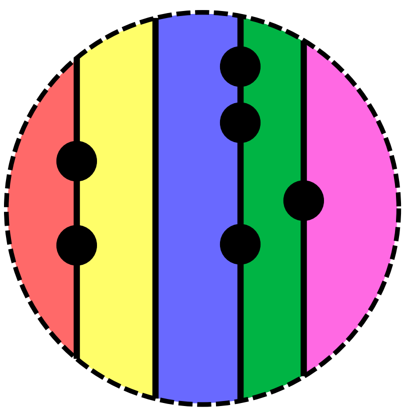
Idea (Bottman, building on MWW+Weinstein): build an $(A_\infty, 2)$-category, Symp, whose objects are *M*'s and $\hom(M,N) := \mathrm{Fuk}(M^- \times N)$. E.g., need:

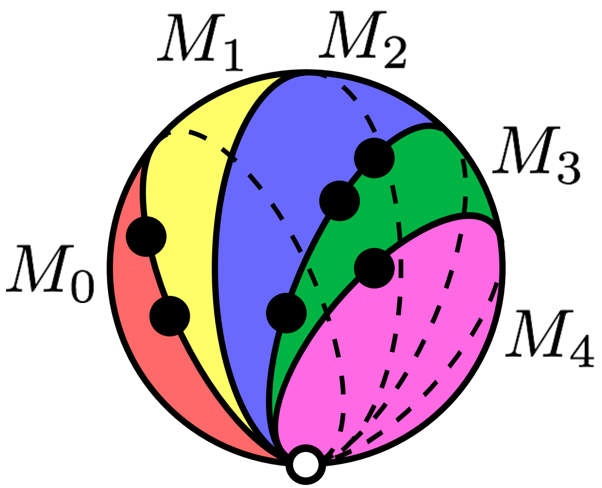
$$\mathrm{Fuk}(M_0^- \times M_1) \otimes \mathrm{Fuk}(M_1^- \times M_2) \to \mathrm{Fuk}(M_0^- \times M_2)$$

Idea (Bottman, building on MWW+Weinstein): build an $(A_\infty, 2)$-**category**, **Symp**, whose objects are *M*'s and $\mathrm{hom}(M, N) := \mathrm{Fuk}(M^- \times N)$. E.g., need:

$$\mathrm{Fuk}(M_0^- \times M_1) \otimes \mathrm{Fuk}(M_1^- \times M_2) \to \mathrm{Fuk}(M_0^- \times M_2)$$

Do so by counting **witch balls** – pseudoholomorphic maps from the colored patches to symplectic manifolds, with "seam conditions" given by Lagrangian correspondences.



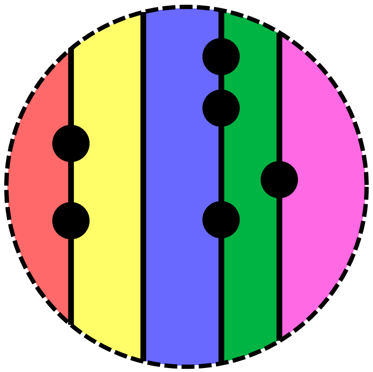


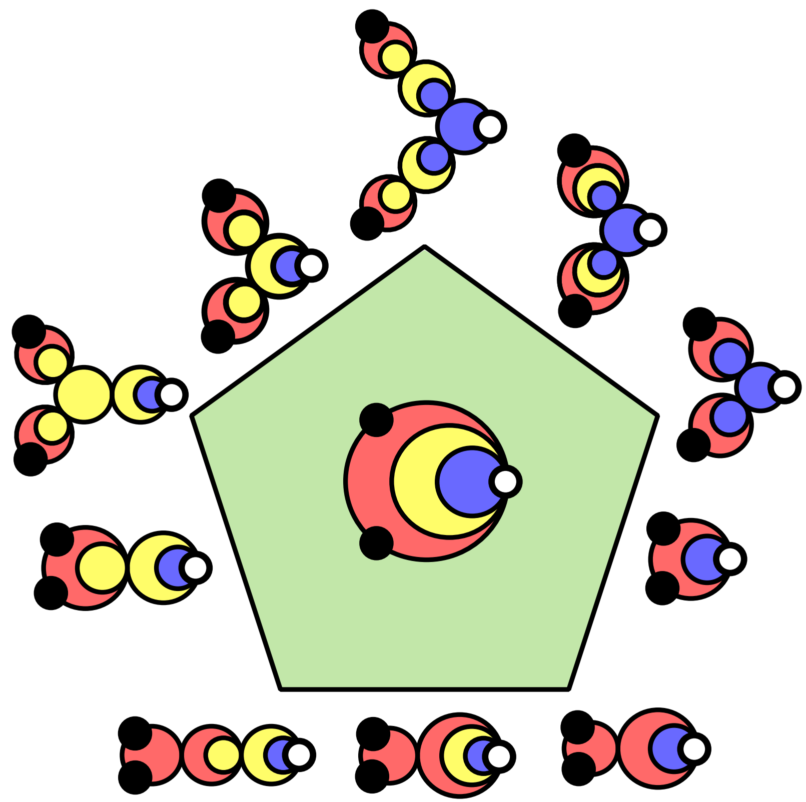
§2: 2-associahedra

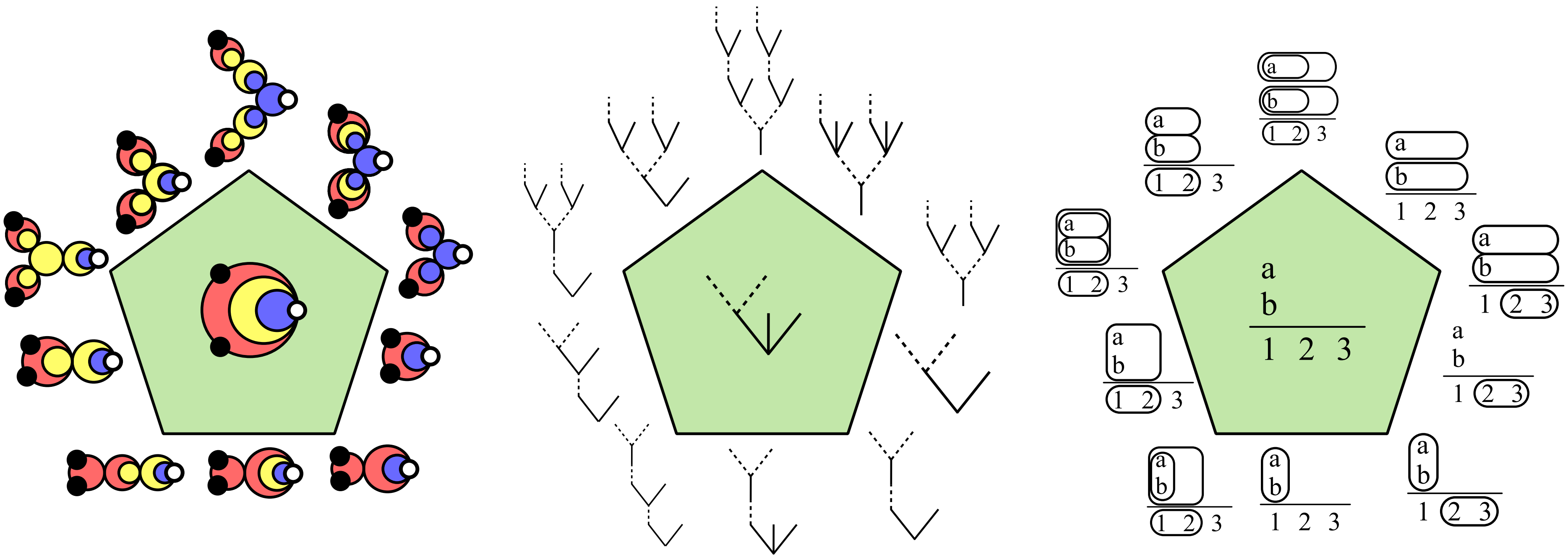
Defining the 2-associahedra

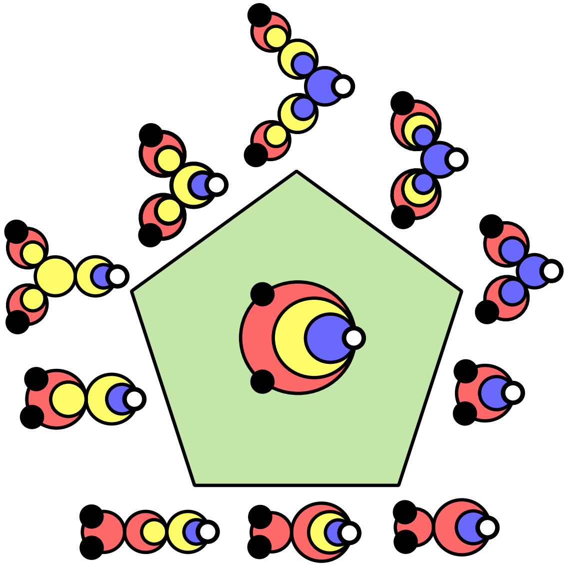
To understand the algebraic structure of **Symp**, need to understand the degenerations that can take place in the domain moduli space $\overline{2\mathcal{M}_{\mathbf{n}}}$, where:

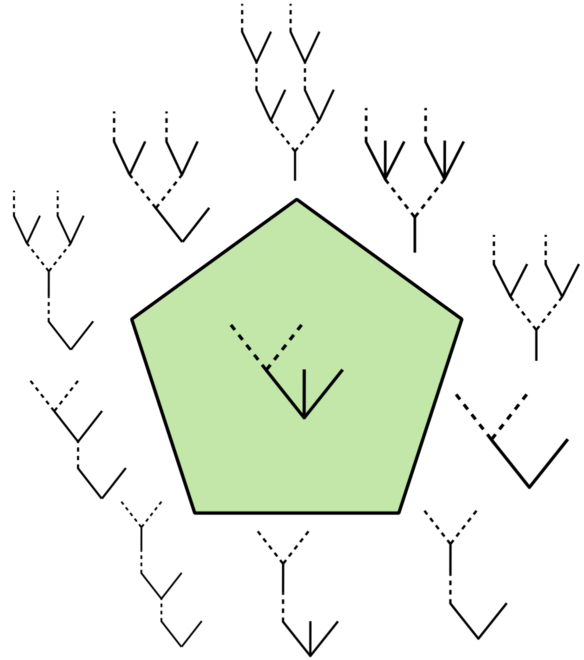
$$2\mathcal{M}_{\mathbf{n}} := \left\{ \begin{array}{rcl|c} (x_1, \ldots, x_r) & \in & \mathbf{R}^r & x_1 < \cdots < x_r \\ (y_{11}, \ldots, y_{1n_1}) & \in & \mathbf{R}^{n_1} & y_{11} < \cdots < y_{1n_1} \\ & \vdots & & \vdots \\ (y_{r1}, \ldots, y_{rn_r}) & \in & \mathbf{R}^{n_r} & y_{r1} < \cdots < y_{rn_r} \end{array} \right\} \Big/ \mathbf{R}^2 \rtimes \mathbf{R}_{>0}$$

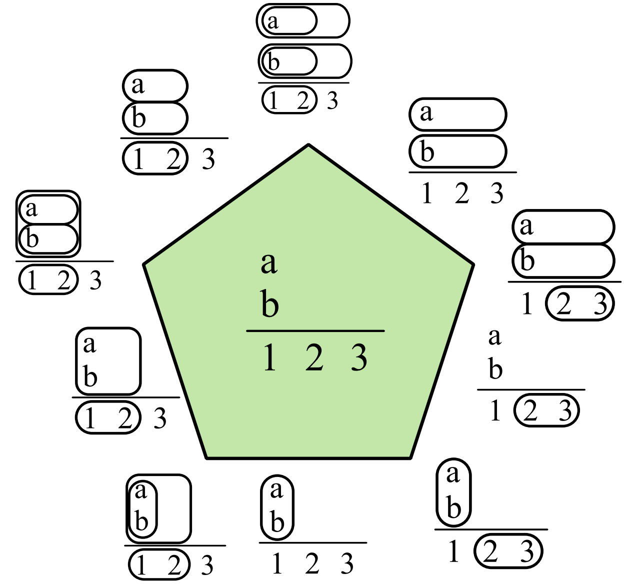
$$= \left\{ \right. $$  $$\left. \right\} \Big/ \sim$$



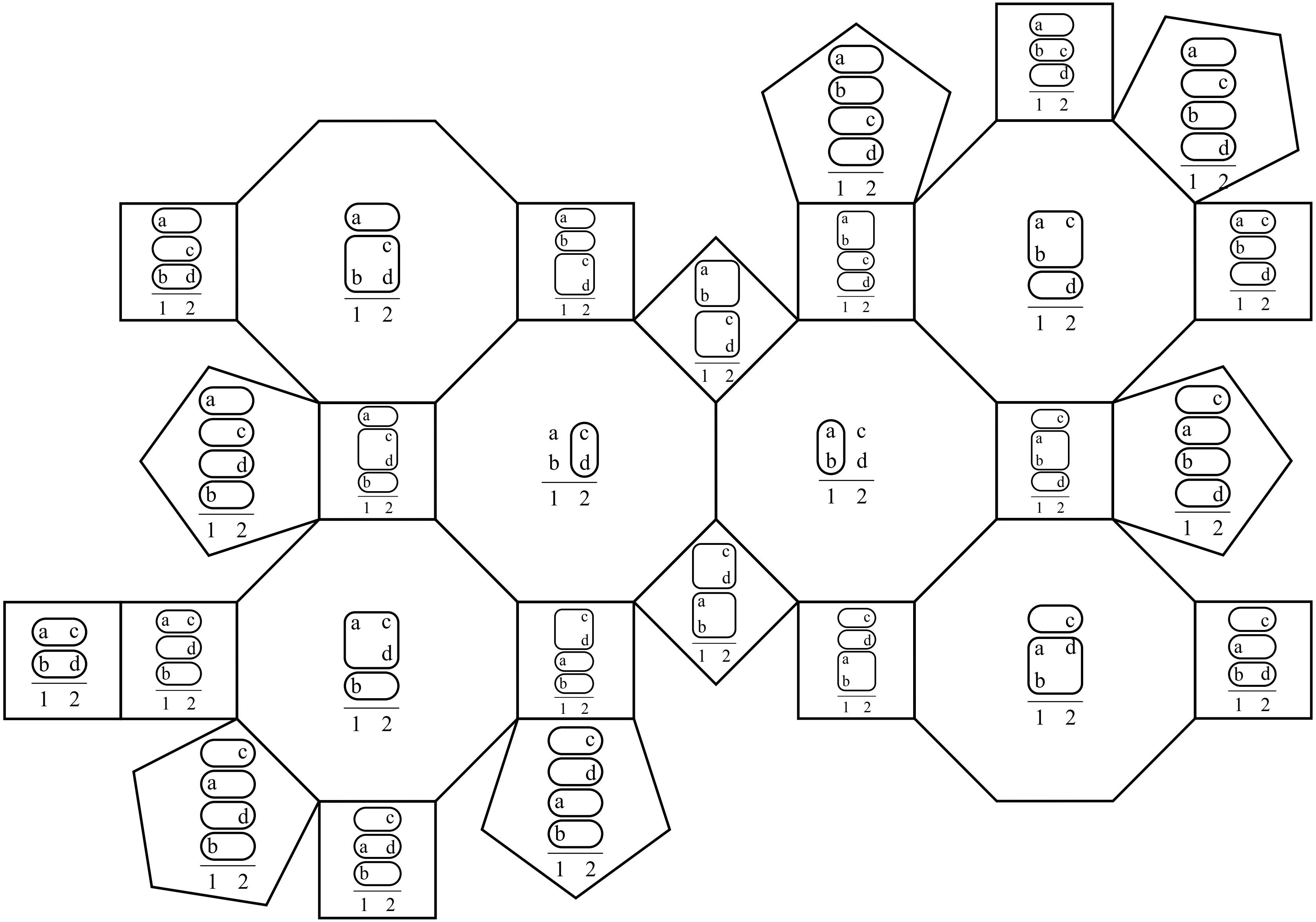

a
b
1 2 3







Theorem (B, arXiv: 1709.00119): For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^{r}$, the 2-associahedron $W_{\mathbf{n}}$ is a poset which is an abstract polytope.

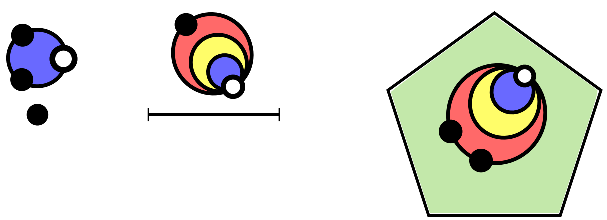


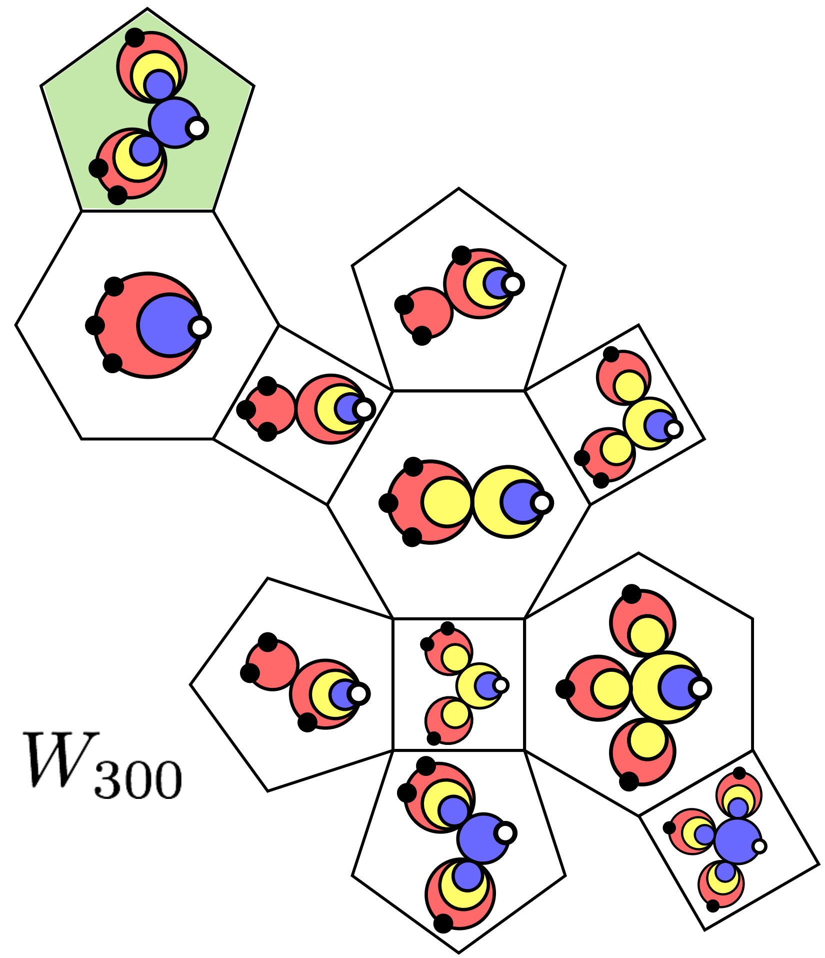
Theorem (B, 2017): The 2-associahedra form a **relative 2-operad over the associahedra**.

Corollary: Can finally define the notion of $(A_\infty, 2)$-category!

Theorem (B, 2017): The 2-associahedra form a **relative 2-operad over the associahedra**.

Corollary: Can finally define the notion of $(A_\infty, 2)$-category!





$$W_2 \times W_{100} \times_{K_3} W_{200} \hookrightarrow W_{300}$$

§3: computation via Morse trees?

Polygons in T^*B

Polygons in T^*B

Fix a metric g on B; get $g\colon TB \to T^*B$.

Identify $T(T^*B) \simeq TB \otimes T^*B$ and define:

$$J_\epsilon \in \mathrm{End}\big(T(T^*B)\big),\ J_\epsilon := \begin{pmatrix} 0 & \epsilon g^{-1} \\ -\epsilon^{-1} g & 0 \end{pmatrix}$$

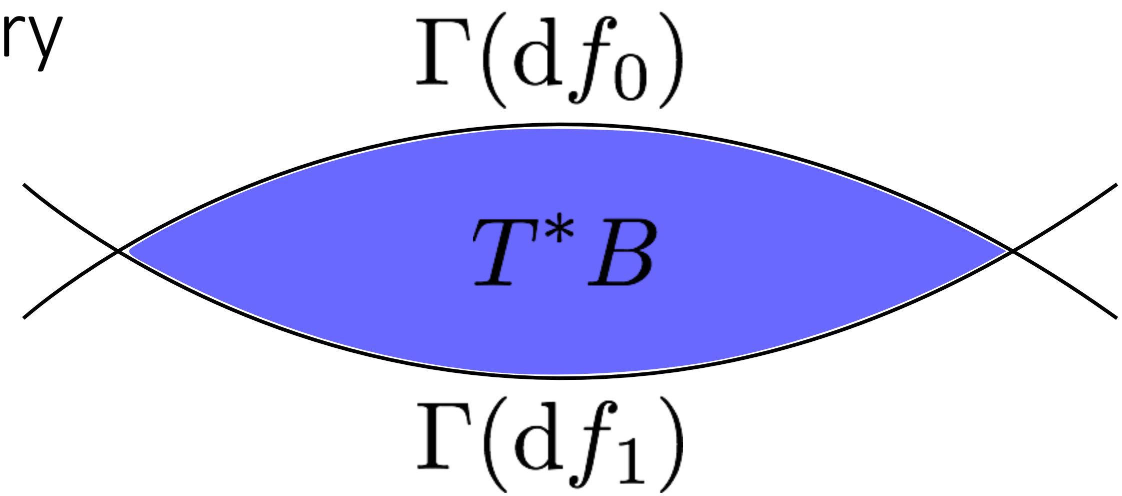
Polygons in T^*B

Fix a metric g on B; get $g\colon TB \to T^*B$.

Identify $T(T^*B) \simeq TB \oplus T^*B$ and define:

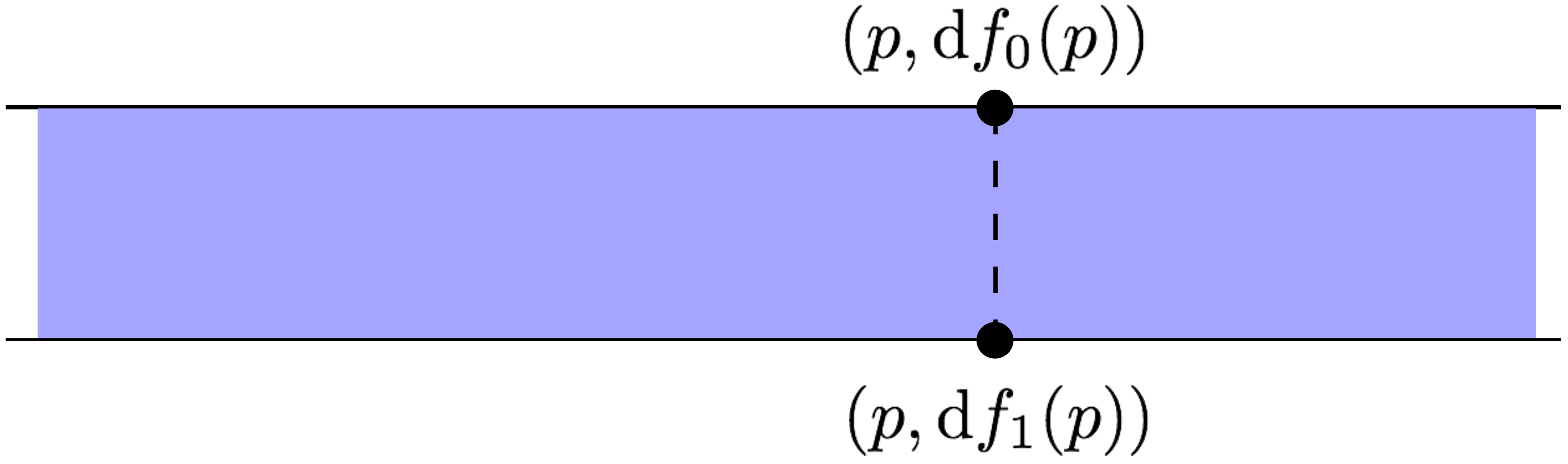
$$J_\epsilon \in \mathrm{End}(T(T^*B)),\ J_\epsilon := \begin{pmatrix} 0 & \epsilon g^{-1} \\ -\epsilon^{-1} g & 0 \end{pmatrix}$$

Question (Fukaya—Oh): Characterize J_ϵ-hol. strips (polygons) with bdry on $\Gamma(\mathrm{d}f)$'s?



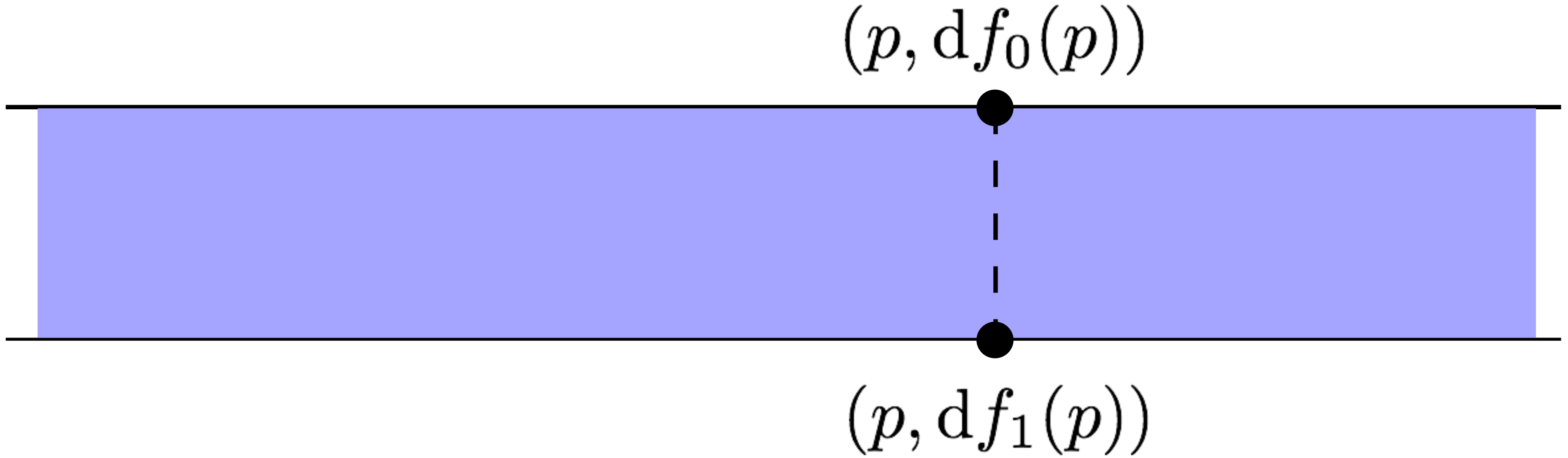
Polygons in T^*B

For small ϵ, fibers of T^*B look small and strips become linear in the fibers:



Polygons in T^*B

For small ϵ, fibers of T^*B look small and strips become linear in the fibers:



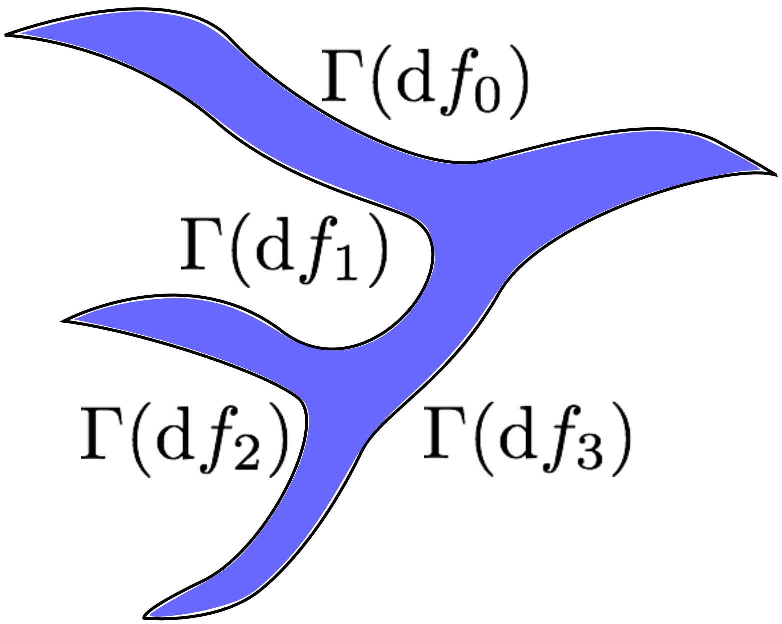
strip is J_ϵ-holomorphic $\Longrightarrow \dot{p}(t) = \epsilon\, \mathrm{d}(f_1 - f_0)(p(t))$

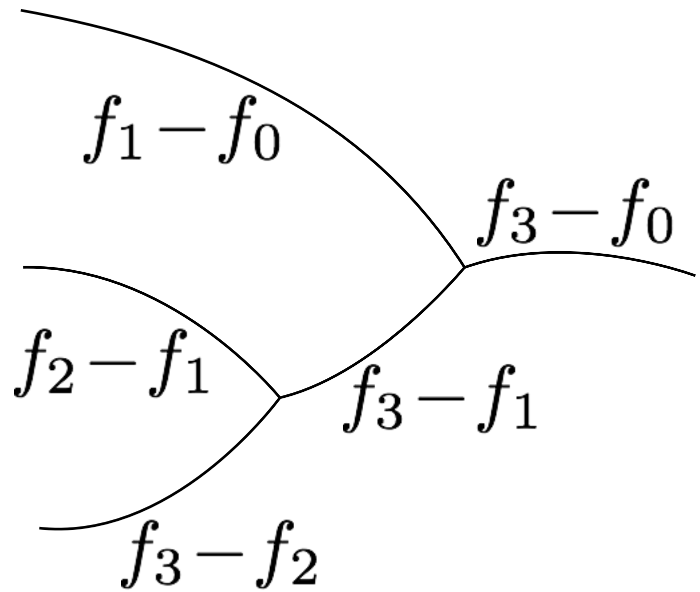
$\Longrightarrow p(\epsilon^{-1}t)$ Morse flowline

$$\Longrightarrow \Big(\hom\big(\Gamma(\mathrm{d}f_0), \Gamma(\mathrm{d}f_1)\big), \mu^1\Big) \simeq \big(CM(f_1 - f_0), \mathrm{d}_{\mathrm{Morse}}\big)$$

Polygons in T^*B

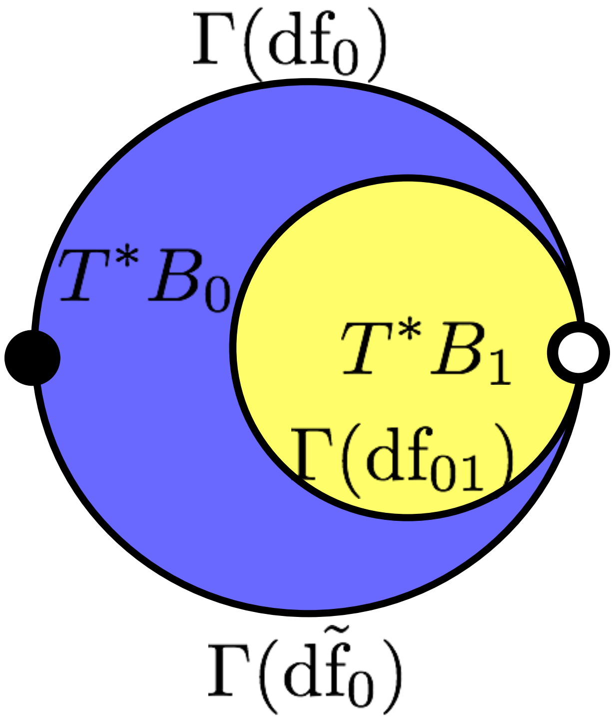
And similarly for polygons in T^*B:





...how about witch balls?

Question: How about witch balls in cotangent bundles?



thanks!