An algebro-geometric theory of modular forms taking values in the Weil representation

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Theta functions

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- If $q = e^{2\pi i\tau}$, for $\tau \in \mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, then

$$\theta_{2,0}\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(c,d)\sqrt{c\tau + d}\,\theta_{2,0}(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, where $\epsilon(c,d)^4 = 1$.

Theta functions as modular forms

Question

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Modular forms of integral weight

- Analytic picture: modular forms of weight $k \in \mathbb{Z}$ are holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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- Algebro-geometric picture: modular forms of weight $k \in \mathbb{Z}$ are sections

$$f \in \Gamma(\mathcal{M}_1, \underline{\omega}^{\otimes k})$$

q-expansions of classical modular forms

- Analytic picture: the q-expansion of a modular form f of weight k is its Fourier expansion $f(e^{2\pi i\tau})$ at the cusp ∞.

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- Algebro-geometric picture: let

$$\psi : \mathrm{Spec}(\mathbb{Z}((q))) \to \mathcal{M}_1$$

be the classifying map of the Tate elliptic curve $\mathrm{Tate}(q)$. Then

$$\psi^*(f) = f(q)\,\omega_{\mathrm{can}}^k, \qquad f(q) \in \mathbb{Z}((q))$$

The q-expansion of f is the formal power series $f(q)$.

q-expansions of theta-functions

Definition

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Is there an algebro-geometric theory of modular forms of half-integral weight underlying these formal power series?

Further motivations

For modular forms of integral weight:

Algebro-geometric theory + Hecke theory → Motives ⟶ L-functions

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Question

Given a Hecke theory and an algebro-geometric theory of modular forms of half-integral weight, can we construct motives and L-functions attached to them? How would they look like?

Mumford's algebraic theta functions

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- Mumford writes:

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> *There are several interesting topics which I have not gone into in this paper, but which can be investigated in the same spirit: for example, [...] a discussion of the transformation theory of theta-functions.*

Mumford's algebraic theta functions

- For $(\pi : E \to S, e)$ an elliptic curve, $m \in 2\mathbb{Z}_{\geq 0}$,

$$\mathcal{L}_m := \mathcal{O}_E(m\, e) \otimes (\Omega^1_{E/S})^{\otimes m}$$

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- Over $\mathcal{M}_1$, let $\mathcal{J}_m$ be the sheaf

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- Over $\mathcal{M}_1$, let $\mathcal{J}_m$ be the sheaf
$$\{\pi : E \to S\} \longmapsto \Gamma(S, \pi_* \mathcal{L}_m)$$
- The rule
$$\{\pi : E \to S\} \longmapsto e^* \in \Gamma(S, (\pi_* \mathcal{L}_m)^*)$$
gives a section $\theta_{\mathrm{null},m}$ of $\mathcal{J}_m^*$.

The Weil representation

$$\theta_{\mathrm{null},m}(q) = \left(\sum_{\substack{n \equiv \nu \mod m \\ n \in \mathbb{Z}}} q^{n^2/2m} = \theta_{m,\nu}(q) \right)_{\nu \in \mathbb{Z}/m\mathbb{Z}}$$

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- If $q = e^{2\pi i \tau}$

$$\theta_{\mathrm{null},m}\left(\frac{a\tau + b}{c\tau + d}\right) = \phi\, \rho_m(\gamma)\, \theta_{\mathrm{null},m}(\tau), \quad \phi^2 = c\tau + d$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi \right) \in \mathrm{Mp}_2(\mathbb{Z})$, where ρ_m is the Weil representation attached to the quadratic form $x \mapsto mx^2/2$.

Vector-valued modular forms

Definition (Eichler-Zagier, Borcherds)

A vector-valued modular form of weight $k/2$ and index m is a holomorphic function

$$f : \mathfrak{h} \to \mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$$

such that

$$f(M\tau) = \phi^k \rho_m(\gamma) f(\tau)$$

for every $\gamma = (M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

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Fact: there are no non-zero vector-valued modular forms of weight $k/2$ unless k is odd.

Metaplectic orbifolds and vector-valued modular forms

Vector-valued modular forms of weight $k/2$ and index m are global sections of the vector bundle

$$\mathcal{W}_m \otimes \underline{\omega}^{k/2}$$

over the metaplectic orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, where:

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- $\underline{\omega}^{k/2}$ is the line bundle corresponding to the 1-cocycle $\mathrm{Mp}_2(\mathbb{Z}) \to \mathcal{O}_{\mathfrak{h}}^*$ given by

$$\begin{aligned} \mathrm{Mp}_2(\mathbb{Z}) &\to \mathcal{O}_{\mathfrak{h}}^* \\ (M, \phi) &\longmapsto \phi^k. \end{aligned}$$

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$$(M, \phi) \longmapsto \phi^k.$$

- $\mathcal{W}_m$ is the local system of rank m given by the Weil representation

$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

Algebro-geometric theory

To do list:

- Give an algebraic analog of the metaplectic orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$.
- Give an algebraic analog of $\underline{\omega}^{k/2}$.
- Give an algebraic analog of $\mathcal{W}_m$.

Metaplectic stacks

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- $\mathcal{M}_1^{\mathrm{an}} \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ and any metaplectic stack over an analytic base is equivalent to either $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ or $(\mathrm{SL}_2(\mathbb{Z}) \times \mu_2)\backslash\backslash\mathfrak{h}$.

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- Inspired by 'metaplectic correction' in geometric quantization theory.

The metaplectic stack $\mathcal{M}_{1/2}$

All schemes are over $\mathbb{Z}[1/m]$, $m \in 2\mathbb{Z}_{>0}$.

- $\mathcal{M}_{1/2} =$ the category of pairs $(E/S, (\mathcal{Q}, \iota))$
  - E/S is an elliptic curve
  - $\mathcal{Q}$ is an invertible $\mathcal{O}_S$-module with

$$\iota : \mathcal{Q}^{\otimes 2} \xrightarrow{\simeq} \underline{\omega}_{E/S}.$$

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- $\mathcal{M}_{1/2}$ is a DM stack, and a μ_2-gerbe $\mathcal{M}_{1/2} \to \mathcal{M}_1$.

The square root of $\underline{\omega}$

- $\mathcal{M}_{1/2}$ is canonically endowed with an invertible sheaf $\underline{\omega}^{1/2}$ defined by

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- If $p : \mathcal{M}_{1/2} \to \mathcal{M}_1$ is the 'forget the quadratic form' functor, then
$$p^*\underline{\omega} \simeq \left(\underline{\omega}^{1/2}\right)^{\otimes 2}.$$

Finite Heisenberg groups

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$$0 \to \mathbb{G}_m \to \mathscr{G}(\mathcal{L}_m) = \mathcal{L}_m|_{E[m]} \to E[m] \to 0$$
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- The $\mathcal{O}_S$-module $\pi_*\mathcal{L}_m$ is an irreducible representation of $\mathscr{G}(\mathcal{L}_m)$, locally free of rank m over S.

Symmetric Heisenberg groups

- Since $\mathcal{L}_m$ is symmetric, there is an involution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathscr{G}(\mathcal{L}_m) & \longrightarrow & E[m] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \delta_{-1}} & & \downarrow{\scriptstyle [-1]} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathscr{G}(\mathcal{L}_m) & \longrightarrow & E[m] & \longrightarrow & 0 \end{array}$$

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- $\mathscr{G}(\mathcal{L}_m)$ is a μ_2-torsor over $E[m]$.

Schrödinger representations

Definition

A symmetric lagrangian subgroup $\mathscr{H} \subseteq \mathscr{G}(\mathcal{L}_m)$ is a subgroup scheme of rank m such that $\mathscr{H} \cap \mathbb{G}_m \simeq \{1\}$, $\delta_{-1}(h) = h^{-1}$ for all $h \in \mathscr{H}$, and

$$E[m] \simeq H \times \widehat{H} \quad \text{(projection onto } E[m]\text{)}$$

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Definition

The Schrödinger representation $\mathcal{W}_{\mathscr{H}}$ is the locally free $\mathcal{O}_S$-module of functions $f : \mathscr{G}(\mathcal{L}_m) \to \mathcal{O}_S$ such that:

(i) $f(hg) = f(g), \quad \forall\, h \in \mathscr{H} \subseteq \mathscr{G}(\mathcal{L}_m)$,

(ii) $f(\lambda g) = \lambda f(g), \quad \forall\, \lambda \in \mathbb{G}_m \subseteq \mathscr{G}(\mathcal{L}_m)$.

Morphisms and the Weil representation

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- Morphisms of Schrödinger representations:

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induced by a μ_2-torsor isomorphism

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- $\mathrm{Aut}(\mathcal{W}_m) = \mu_2$ (compare with Weil).

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$$\{(E/S, \mathcal{W}_{\mathscr{H}})\} \longmapsto \Gamma(S, \mathcal{W}_{\mathscr{H}}).$$

Geometric vector-valued modular forms

Definition

Let m be a positive even integer and let $k \in \mathbb{Z}$. A $\mathcal{W}_m$-valued modular form of weight $k/2$ is a global section of the sheaf

$$\mathcal{W}_m \otimes \underline{\omega}^{k/2}$$

over the metaplectic stack $\mathcal{K}_m \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2} \to \mathcal{M}_1$.

Analytic picture

- Over an analytic base,

$$\mathcal{K}_m = \mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}, \quad \mathcal{M}_{1/2} = \mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$$

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$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

Vector-valued modular forms are ‘modular’

Theorem

Let $k \in \mathbb{Z}$ be odd. Then the sheaf

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of $\mathcal{W}_m$-valued modular forms of weight $k/2$ descends to a locally free sheaf of rank m over $\mathcal{M}_1$.

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- Fact: when k is even, there are no nonzero $\mathcal{W}_m$-valued modular forms of weight $k/2$.

The algebraic Eichler-Zagier Theorem

$\mathcal{J}_m$ is the sheaf

$$\{\pi : E \to S\} \longmapsto \Gamma(S, \pi_* \mathcal{L}_m)$$

Theorem

There is a canonical isomorphism

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Proof.

Use Stone-Von Neumann-Mackey. □

The transformation law of theta constants

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- Thus:

$$\theta_{\mathrm{null},m} \in \Gamma(\mathcal{M}_1, \mathcal{W}_m^* \otimes \underline{\omega}^{1/2}),$$

a $\mathcal{W}_m^*$-valued modular form of weight 1/2.

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- We can define *holomorphic* $\mathcal{W}_m$-valued modular forms and compute the dimension of these spaces.
- From $\theta_{2,0}(q)$, we can construct a geometric theory of modular forms of half-integral weight, in the sense of Shimura.

What to do with this theory

- Expand the theory to include vector-valued modular forms attached to any quadratic form (applications: classical modular forms, Shimura lifts, Borcherds products...).

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- Hecke theory, 'motives' and L-functions for vector-valued modular forms (applications: Galois representations?).

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- Expand the theory to include vector-valued modular forms attached to any quadratic form (applications: classical modular forms, Shimura lifts, Borcherds products...).
- Hecke theory, 'motives' and L-functions for vector-valued modular forms (applications: Galois representations?).
- A theory of p-adic vector-valued modular forms (applications: p-adic generating series for L-values, p-adic Gross-Kohnen-Zagier...)