Motivations Vector-valued modular forms Metaplectic stacks Algebro-geometric theory Further results Further directions

# An algebro-geometric theory of modular forms taking values in the Weil representation

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## Theta functions

$$heta_{2,0}(q) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \ldots \hspace{0.5cm} \in \mathbb{Z}\llbracket q 
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• If 
$$q=e^{2\pi i au}$$
, for  $au\in\mathfrak{h}=\{z\in\mathbb{C}:\mathrm{Im}[z]>0\}$ , then

$$heta_{2,0}\left(rac{a au+b}{c au+d}
ight)=\epsilon(c,d)\sqrt{c au+d}\, heta_{2,0}( au)$$

for any 
$$\gamma=\left(egin{array}{c} a & b \ c & d \end{array}
ight)\in \mathsf{F}_0(4)$$
, where  $\epsilon(c,d)^4=1$ 

# Theta functions as modular forms

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Is there a way to express the fact that  $\theta_{2,0}(q)$  is a modular form of weight 1/2 *directly* as a formal power series in  $\mathbb{Z}\llbracket q \rrbracket$ ?

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## Modular forms of integral weight

 Analytic picture: modular forms of weight k ∈ Z are holomorphic functions f : h → C satisfying

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Algebro-geometric picture: modular forms of weight k ∈ Z are sections

$$f \in \Gamma(\mathcal{M}_1, \underline{\omega}^{\otimes k})$$

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- Analytic picture: the q-expansion of a modular form f of weight k is its Fourier expansion f(e<sup>2πiτ</sup>) at the cusp ∞.
- Algebro-geometric picture: let

$$\psi: \operatorname{Spec}(\mathbb{Z}((q))) \to \mathcal{M}_1$$

be the classifying map of the Tate elliptic curve Tate(q). Then

$$\psi^*(f) = f(q) \, \omega_{\operatorname{can}}^k, \qquad f(q) \in \mathbb{Z}(\!(q)\!)$$

The q-expansion of f is the formal power series f(q).

# q-expansions of theta-functions

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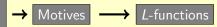
#### Question

Is there an algebro-geometric theory of modular forms of half-integral weight underlying these formal power series?

### Further motivations

For modular forms of integral weight:

 $\begin{array}{l} \mbox{Algebro-geometric theory} \\ \mbox{Hecke theory} \end{array}$ 



## Further motivations

For modular forms of integral weight:

Algebro-geometric theory + Hecke theory

#### Question

Given a Hecke theory and an algebro-geometric theory of modular forms of half-integral weight, can we construct motives and *L*-functions attached to them? How would they look like?

• Starting point: *On the equations defining abelian varieties I*,*II*,*III* (Mumford, Invent. math. 1966-67)

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There are several interesting topics which I have not gone into in this paper, but which can be investigated in the same spirit: for example, [...] a discussion of the transformation theory of theta-functions.

• For 
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 an elliptic curve,  $m \in 2\mathbb{Z}_{\geq 0}$ ,

$$\mathcal{L}_m := \mathcal{O}_E(m e) \otimes (\Omega^1_{E/S})^{\otimes m}$$

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• The rule

$$\{\pi: E \to S\} \longmapsto e^* \in \Gamma(S, (\pi_*\mathcal{L}_m)^*)$$

gives a section  $\theta_{\mathrm{null},m}$  of  $\mathcal{J}_m^*$ .

## The Weil representation

$$heta_{\mathrm{null},m}(q) = \left(\sum_{\substack{n \equiv 
u \mod m \\ n \in \mathbb{Z}}} q^{n^2/2m} = heta_{m,
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ho_m(\gamma)\, heta_{\mathrm{null},m}( au),\quad \phi^2=c au+d$$

for all  $\gamma = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi \right) \in Mp_2(\mathbb{Z})$ , where  $\rho_m$  is the Weil representation attached to the quadratic form  $x \mapsto mx^2/2$ .

# Vector-valued modular forms

#### Definition (Eichler-Zagier, Borcherds)

A vector-valued modular form of weight k/2 and index m is a holomorphic function

 $f:\mathfrak{h}\to\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ 

such that

$$f(M\tau) = \phi^k \rho_m(\gamma) f(\tau)$$

for every  $\gamma = (M, \phi) \in Mp_2(\mathbb{Z})$ .

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Fact: there are no non-zero vector-valued modular forms of weight k/2 unless k is odd.

### Metaplectic orbifolds and vector-valued modular forms

Vector-valued modular forms of weight k/2 and index m are global sections of the vector bundle

$$\mathcal{W}_m \otimes \underline{\omega}^{k/2}$$

over the metaplectic orbifold  $Mp_2(\mathbb{Z}) \setminus \mathfrak{h}$ , where:

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•  $\mathcal{W}_m$  is the local system of rank m given by the Weil representation

$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

## Algebro-geometric theory

To do list:

- Give an algebraic analog of the metaplectic orbifold  $\operatorname{Mp}_2(\mathbb{Z}) \backslash\!\!\backslash \mathfrak{h}.$
- Give an algebraic analog of  $\underline{\omega}^{k/2}$ .
- Give an algebraic analog of  $\mathcal{W}_m$ .

#### Definition

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- *M*<sub>1</sub><sup>an</sup> ≃ SL<sub>2</sub>(ℤ) \\ h and any metaplectic stack over an analytic base is equivalent to either Mp<sub>2</sub>(ℤ) \\ h or (SL<sub>2</sub>(ℤ) × μ<sub>2</sub>) \\ h.

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- Inspired by 'metaplectic correction' in geometric quantization theory.

# The metaplectic stack $\mathcal{M}_{1/2}$

All schemes are over  $\mathbb{Z}[1/m]$ ,  $m \in 2\mathbb{Z}_{>0}$ .

- $\mathcal{M}_{1/2}$  = the category of pairs  $(E/S, (Q, \iota))$ 
  - E/S is an elliptic curve
  - Q is an invertible  $\mathcal{O}_S$ -module with

$$\iota: \mathcal{Q}^{\otimes 2} \xrightarrow{\simeq} \underline{\omega}_{E/S}.$$

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•  $\mathcal{M}_{1/2}$  is a DM stack, and a  $\mu_2$ -gerbe  $\mathcal{M}_{1/2} \rightarrow \mathcal{M}_1$ .

### The square root of $\underline{\omega}$

•  $\mathcal{M}_{1/2}$  is canonically endowed with an invertible sheaf  $\underline{\omega}^{1/2}$  defined by

 $\{(E/S,(\mathcal{Q},\iota))\} \longmapsto \Gamma(S,\mathcal{Q}).$ 

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• If  $p: \mathcal{M}_{1/2} \to \mathcal{M}_1$  is the 'forget the quadratic form' functor, then

$$p^*\underline{\omega} \simeq \left(\underline{\omega}^{1/2}\right)^{\otimes 2}$$

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$$0 \to \mathbb{G}_m \to \mathscr{G}(\mathcal{L}_m) = \mathcal{L}_m|_{E[m]} \to E[m] \to 0$$

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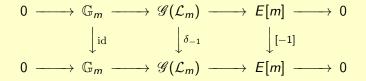
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 The O<sub>S</sub>-module π<sub>\*</sub>L<sub>m</sub> is an irreducible representation of *G*(L<sub>m</sub>), locally free of rank m over S.

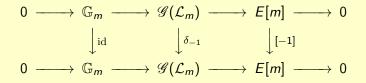
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•  $\mathscr{G}(\mathcal{L}_m)$  is a  $\mu_2$ -torsor over E[m].

# Schrödinger representations

#### Definition

A symmetric lagrangian subgroup  $\mathscr{H} \subseteq \mathscr{G}(\mathcal{L}_m)$  is a subgroup scheme of rank *m* such that  $\mathscr{H} \cap \mathbb{G}_m \simeq \{1\}$ ,  $\delta_{-1}(h) = h^{-1}$  for all  $h \in \mathscr{H}$ , and

$${m E}[m]\simeq {m H} imes \widehat{m H}$$
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#### Definition

The Schrödinger representation  $\mathcal{W}_{\mathscr{H}}$  is the locally free  $\mathcal{O}_{S}$ -module of functions  $f : \mathscr{G}(\mathcal{L}_{m}) \to \mathcal{O}_{S}$  such that: (i)  $f(hg) = f(g), \quad \forall h \in \mathscr{H} \subseteq \mathscr{G}(\mathcal{L}_{m}),$ (ii)  $f(\lambda g) = \lambda f(g), \quad \forall \lambda \in \mathbb{G}_{m} \subseteq \mathscr{G}(\mathcal{L}_{m}).$ 

### Morphisms and the Weil representation

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- Morphisms of Schrödinger representations:

 $\mathscr{G}$ -module isomorphism  $\mathcal{W}_{\mathscr{H}} \to \mathcal{W}_{\mathscr{H}'}$ 

induced by a  $\mu_2$ -torsor isomorphism

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• 
$$\operatorname{Aut}(\mathcal{W}_m) = \mu_2$$
 (compare with Weil).

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•  $\mathcal{K}_m$  is canonically endowed with a locally free module  $\mathcal{W}_m$  or rank m

$$\{(E/S, \mathcal{W}_{\mathscr{H}})\} \longmapsto \Gamma(S, \mathcal{W}_{\mathscr{H}}).$$

#### Geometric vector-valued modular forms

#### Definition

Let *m* be a positive even integer and let  $k \in \mathbb{Z}$ . A  $\mathcal{W}_m$ -valued modular form of weight k/2 is a global section of the sheaf

 $\mathcal{W}_m \otimes \underline{\omega}^{k/2}$ 

over the metaplectic stack  $\mathcal{K}_m \times_{\mathcal{M}_1}^{\mu_2} \mathcal{M}_{1/2} \to \mathcal{M}_1$ .

## Analytic picture

• Over an analytic base,

$$\mathcal{K}_m = \mathrm{Mp}_2(\mathbb{Z}) \mathbb{h}, \quad \mathcal{M}_{1/2} = \mathrm{Mp}_2(\mathbb{Z}) \mathbb{h}$$

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• 
$$\underline{\omega}^{k/2}$$
 is given by the 1-cocycle  
 $Mp_2(\mathbb{Z}) o \mathcal{O}^*_{\mathfrak{h}}$   
 $(M, \phi) \longmapsto \phi^k$ 

•  $\mathcal{W}_m$  is given by the Weil representation

$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

## Vector-valued modular forms are 'modular'

#### Theorem

Let  $k \in \mathbb{Z}$  be odd. Then the sheaf

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of  $W_m$ -valued modular forms of weight k/2 descends to a locally free sheaf of rank m over  $M_1$ .

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• Fact: when k is even, there are no nonzero  $\mathcal{W}_m$ -valued modular forms of weight k/2.

### The algebraic Eichler-Zagier Theorem

 $\mathcal{J}_m$  is the sheaf

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#### Theorem

There is a canonical isomorphism

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#### Proof.

Use Stone-Von Neumann-Mackey.

#### The transformation law of theta constants

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• Thus:

$$\theta_{\operatorname{null},m} \in \Gamma(\mathcal{M}_1, \mathcal{W}_m^* \otimes \underline{\omega}^{1/2}),$$

a  $\mathcal{W}_m^*$ -valued modular form of weight 1/2.

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- We can define *holomorphic*  $\mathcal{W}_m$ -valued modular forms and compute the dimension of these spaces.
- From  $\theta_{2,0}(q)$ , we can construct a geometric theory of modular forms of half-integral weight, in the sense of Shimura.

### What to do with this theory

• Expand the theory to include vector-valued modular forms attached to any quadratic form (applications: classical modular forms, Shimura lifts, Borcherds products...).

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- Hecke theory, 'motives' and *L*-functions for vector-valued modular forms (applications: Galois representations?).
- A theory of *p*-adic vector-valued modular forms (applications: *p*-adic generating series for *L*-values, *p*-adic Gross-Kohnen-Zagier...)