A tale of two conjectures: from Mahler to Viterbo

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Mahler's Conjecture

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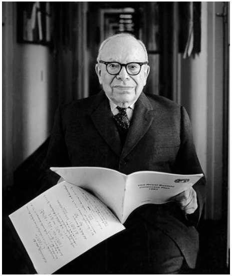

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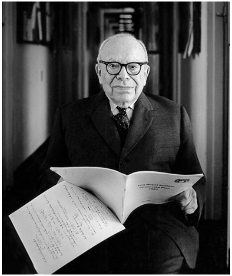

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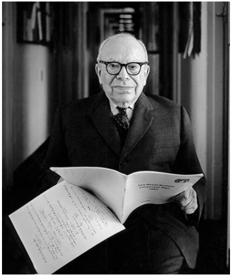

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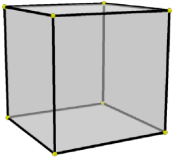

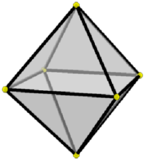

$K \subset \mathbb{R}^n$ convex

$$K^* = \{y \in \mathbb{R}^n \,|\, \langle y, x \rangle \leq 1, \ \forall x \in K\}$$

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Mahler's question: what are the least "round" or the "most pointed" centrally symmetric convex sets in $\mathbb{R}^n$?

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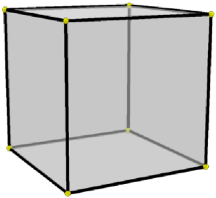

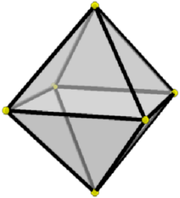

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Partial Results by: Ball, Barthe, Bourgain, Giannopoulos, Gordon, Kim, Kuperberg, Lutwak, Meyer, Milman, Nazarov, Paouris, Petrov, Pisier, Ryabogin, Reisner, Saint-Raymond, Schneider, Stancu, Vritsiou, Zvavitch, ...

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Currently, the best known constant is $c = \pi/4$ (Kuperberg, 2008).

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YOU SHOULD PUT ON YOUR SYMPELCTIC GLASSES!

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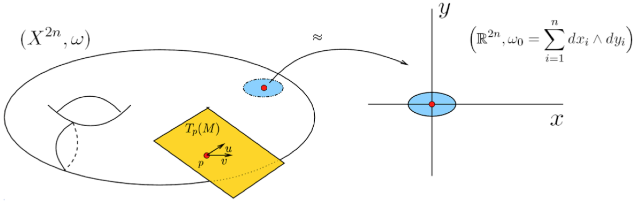
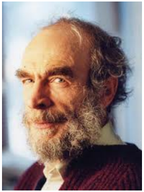
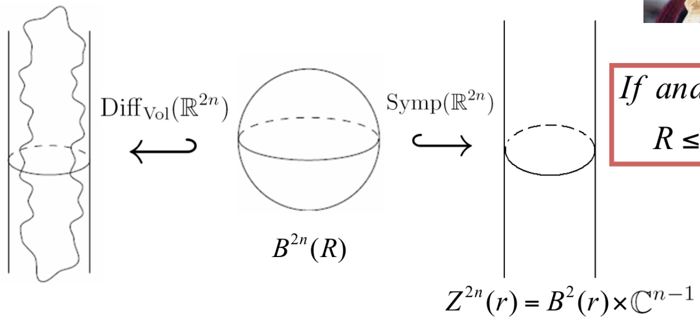
- **Important feature:** have infinitely many symmetries, $Symp(X, \omega) = \{f : X \to X \mid f^*\omega = \omega\}$ ∞-dim Lie group.
- **Bad News:** no local invariants (Darboux's theorem 1882), locally (X, ω) "looks like" $(\mathbb{R}^{2n}, \omega_{\text{std}} = dp \wedge dq)$



Existence of Global Invariants



Gromov's non squeezing theorem (1985):



If and only if

$R \leq r$

Symplectic Measurements

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A **symplectic capacity** is a map $c : \mathcal{P}(\mathbb{R}^{2n}) \to [0, \infty]$, such that

- $U \subseteq V \Rrightarrow c(U) \leq c(V)$ (Monotonicity)
- $c(\psi(U)) = |\alpha| c(U),\ \text{for } \psi^*\omega = \alpha\omega$ (Conformality)
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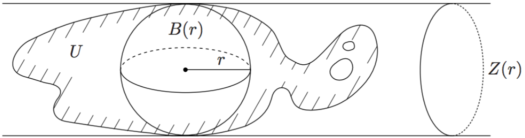
Note:

1. Scales like a 2-dimensional invariant.
2. Last property disqualifies any volume-related invariant.
3. Existence of a single capacity implies Gromov's NST.

Two Examples (Symplectic Embeddings)

- $c_G(U) = \sup\{\pi r^2 \mid B^{2n}(r) \overset{\text{symp}}{\hookrightarrow} U\}$ (Gromov width)
- $c^Z(U) = \inf\{\pi r^2 \mid U \overset{\text{symp}}{\hookrightarrow} Z^{2n}(r)\}$ (Cylindrical capacity)



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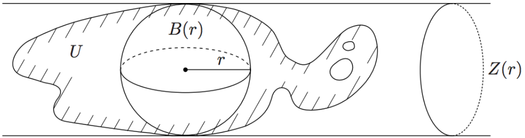

Note: for every (normalized) symplectic capacity c one has

$$c_G \leq c \leq c^Z$$

Some Other Examples (partial list)

Capacity	**"Technology"**
Gromov's width	J-holomorphic curves (1985)
Hofer-Zehnder	∞-dim functional analysis (1990)
Hofer's displacement energy	∞-dim functional analysis (1991)
Viterbo's capacity	generating functions (1992)
Floer-Hofer capacity	Floer homology (1994)
homological capacity	symplectic homology (1994)
Hutching's ECH capacities	embedded contact homology (2011)
Cieliebak-Mohnke capacity	punctured holomorphic curves (2014)
Tamarkin's "sheaf capacity"	microlocal theory of sheaves (2015)

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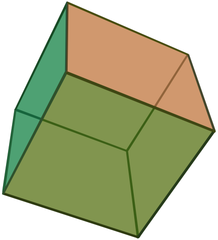

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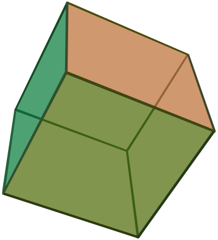

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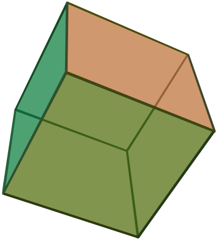

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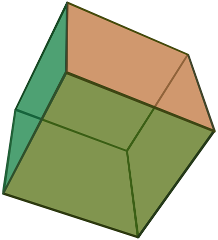

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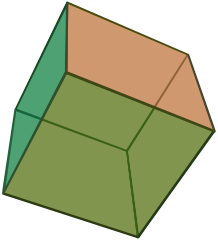

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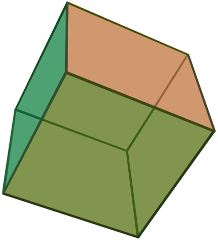

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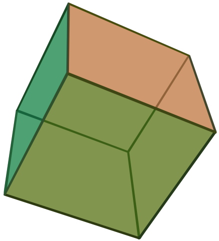

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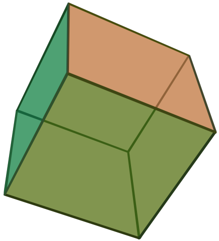

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Viterbo's Systolic Conjecture

Conjecture (Viterbo 2001): For every convex body $K \subset \mathbb{R}^{2n}$, and every symplectic capacity c, one has

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- Equivalent formulation: $c(K) \leq \big(n! \,\mathrm{Vol}(K)\big)^{\frac{1}{n}}$.

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Thm (Abbondandolo, Bramham, Hryniewicz, Salomão, 2017) There exists a C^3-neighborhood $\mathcal{E}$ of the Euclidean ball within the set of all convex smooth domains in $\mathbb{R}^4$ such that Viterbo's conjecture holds for every $K \in \mathcal{E}$

From Viterbo's Conjecture to Mahler's (and back)

Conjecture (Viterbo): Let $K \subset \mathbb{R}^{2n}$ be a convex body. Then for any symplectic capacity c one has

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Conjecture (Mahler): Let X be an n-dim normed space. Then,

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MAIN IDEA OF THE PROOF: there is a symplectic capacity for which $c(B_X \times B_{X^*}) = 4$ for any convex body $B_X \subset \mathbb{R}^n$.

From Viterbo's Conjecture to Mahler's (and back)

Conjecture (Viterbo): Let $K \subset \mathbb{R}^{2n}$ be a convex body. Then for any symplectic capacity c one has

$$c(K) \leq \big(n!\mathrm{Vol}(K)\big)^{\frac{1}{n}}$$

Conjecture (Mahler): Let X be an n-dim normed space. Then,

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This is closely related with **Finsler billiard dynamics**!

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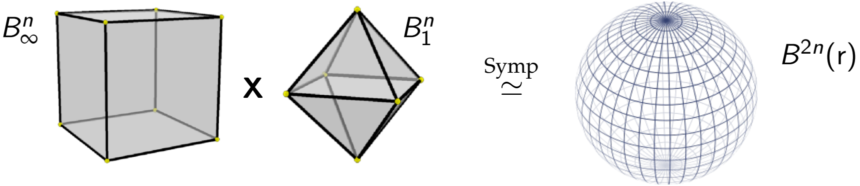

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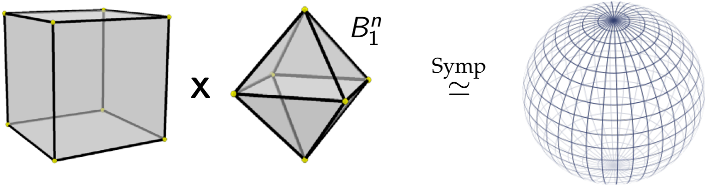

Thm [Karasev, Schlenk (in progress)]: every Hanner-Lima polytope is symplectomorphic to a Euclidean ball with the same volume

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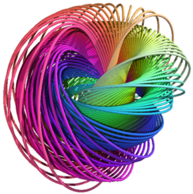

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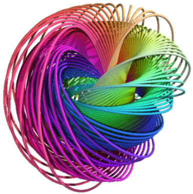

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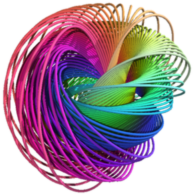

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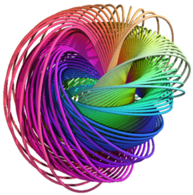

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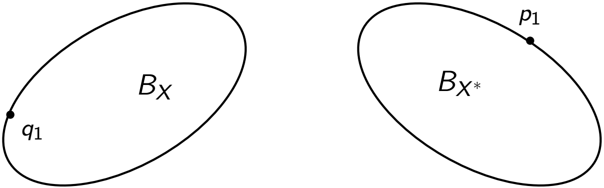

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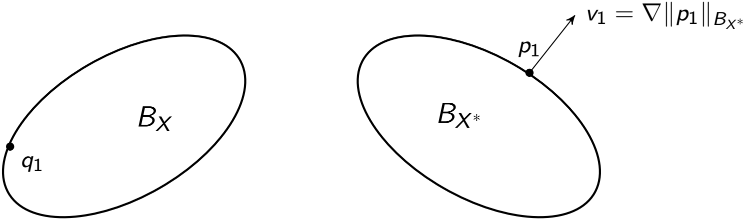

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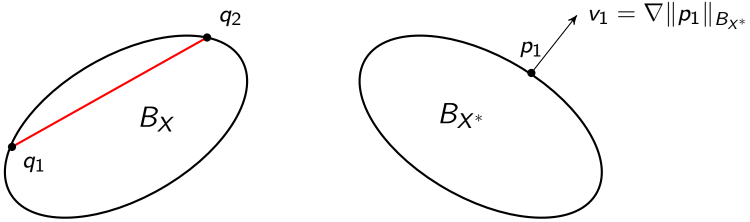

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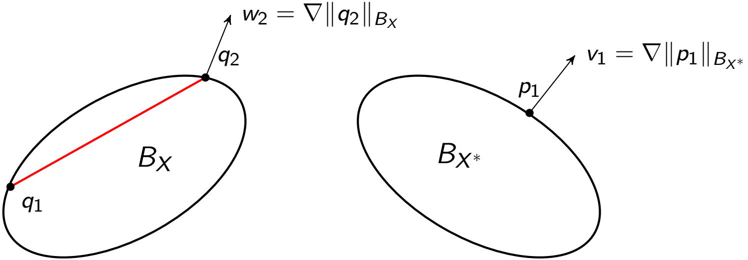

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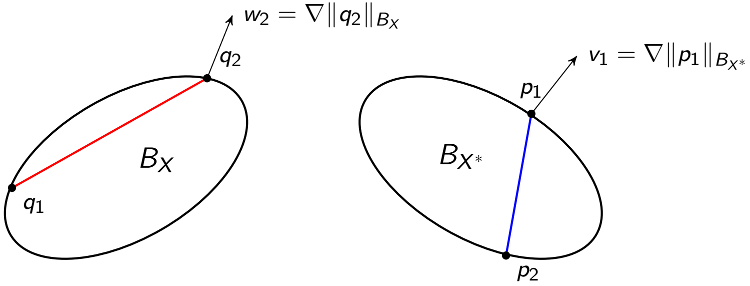

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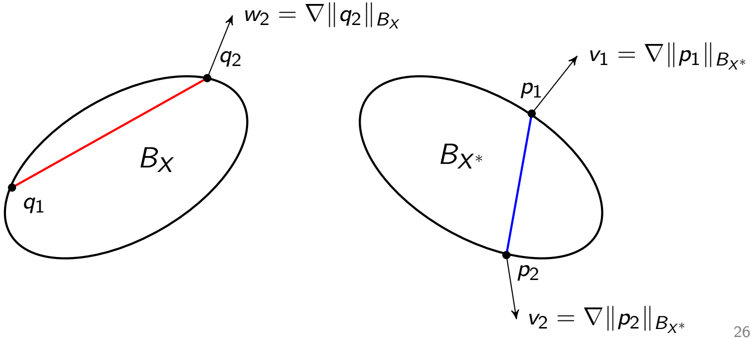

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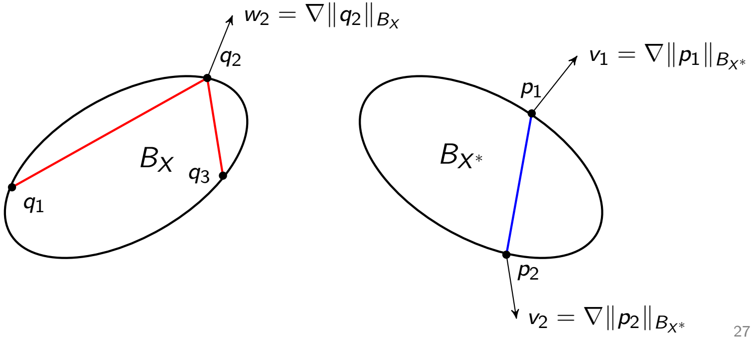

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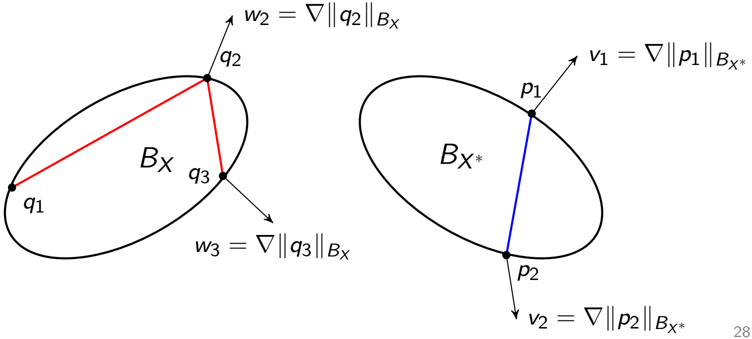

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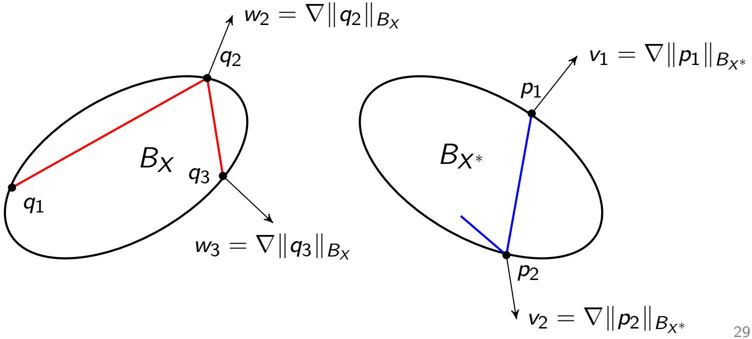

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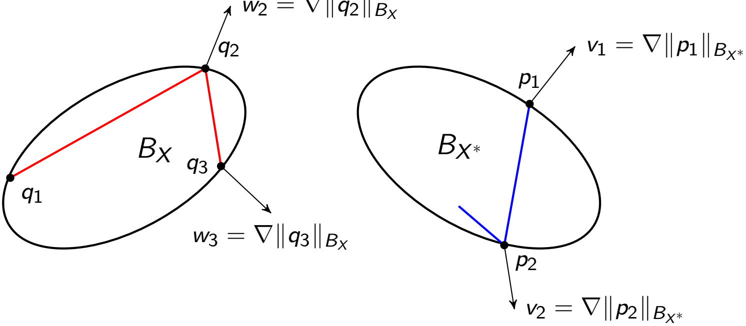

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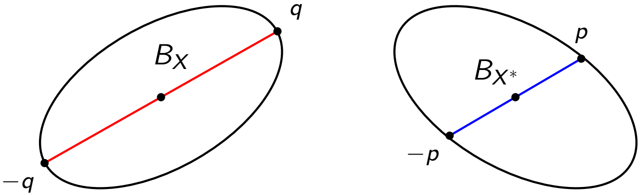

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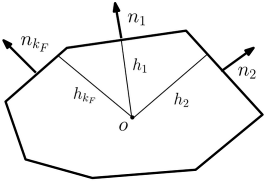

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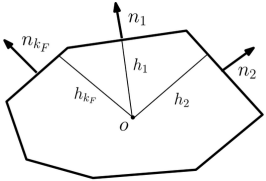

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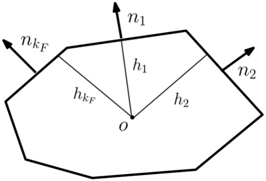

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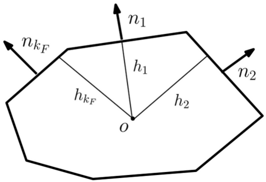

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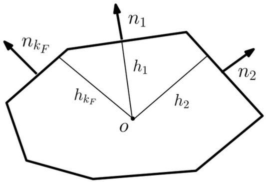

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CAN THIS BE COMPARED WITH THE VOLUME OF K?

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Theorem (P. Kislev, 2018): Subadditivity holds for hyperplane cuts of convex domains.

THANK YOU VERY MUCH!