Flows of vector fields: classical and modern

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The classical theory I

Consider a time-dependent vector field u on $\mathbb{R}^n$ (or a domain of it, or an n-dimensional manifold) and the associated ODE

$$\dot\gamma(t) = u(t, \gamma(t))$$

Theorem (Cauchy-Lipschitz (Picard-Lindelöf))

If u is Lipschitz in space, i.e. $\exists C$ s.t.

$$|u(t,x) - u(t,y)| \leq C|x-y| \qquad \forall x, y \in \mathbb{R}^n ,$$

then for every $x \in \mathbb{R}^n$ there is a unique solution γ_x of the ODE $(\dagger)$ with

$$\gamma_x(0) = x .$$

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The classical theory II

The solution γ depends continuously on both the initial data x and the vector field u.

Furthermore, define the flow map

$$\Phi(t, x) := \gamma_x(t) .$$

Theorem (Cauchy-Lipschitz, continued)

$t \mapsto \Phi(t, \cdot)$ *is a continuous one-parameter family of biLipschitz homeomorphisms.*

Namely $\Phi(t, \cdot)$ is Lipschitz and bijective, with Lipschitz inverse. And Φ becomes more regular according to the regularity of u (C^1, C^k, C^∞, analytic, etc.).

Moreover $t \mapsto \Phi(t, \cdot)$ is an isotopy with the identity map $\Phi(0, \cdot)$.

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Main point

Problem (Pivotal for this talk!)

Can we go drop the Lipschitz regularity?

The classical theory III

Theorem (Peano)

Solutions exist if u is just continuous.

However... Uniqueness fails as soon as u is a tad below Lipschitz. The typical textbook example is

$$\dot\gamma(t) = |\gamma(t)|^\alpha \qquad \text{with } \alpha < 1.$$

In such examples nonuniqueness is "fatal": there is no selection principle which builds a natural global flow.

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The textbook example I

To fix ideas $t \geq 0$ and

$$\dot{\gamma}(t) = 2\sqrt{|\gamma(t)|}$$

Initial datum $x = 0$, many solutions

$$\gamma(t) = \begin{cases} 0 & \text{for } t \leq t_0 \\ (t - t_0)^2 & \text{for } t \geq t_0 \end{cases}$$

Initial datum $x > 0$ a unique solution for all $t > 0$:

$$\gamma(t) = (x + t)^2$$

Initial datum $x < 0$ a unique solution for small time until it its 0:

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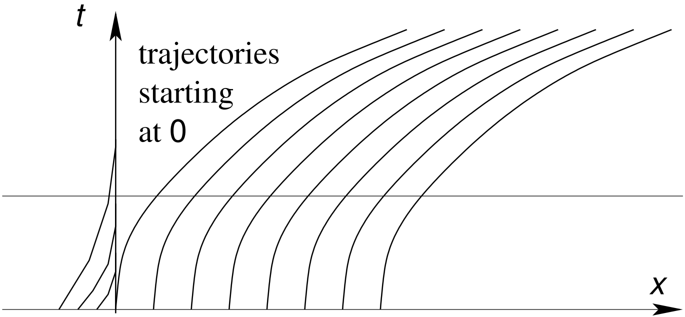
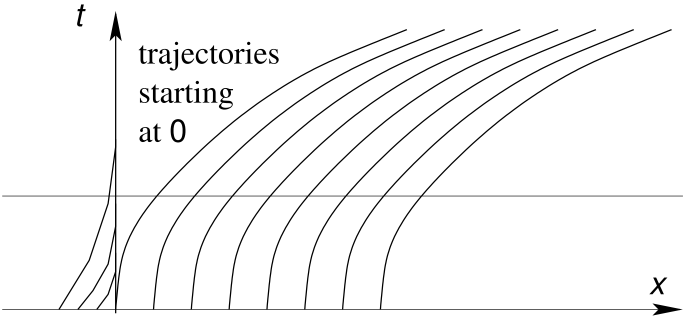


For this reason we will see below that, in some relevant situations, such textbook examples are *misleading*

Textbook example II



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OK, but who cares?

Typical example with some interest: the evolution of a fluid, or a gas, develops some singularity (shear flows, shock waves, detonations, cavitations).

The velocity field of the particles is not regular, yet we wish to make sense of the particles' trajectories.

Bad news: In such situations the vector field is typically discontinuous.
Good news: We want to track most particles.
More good news: Singular fields might be approximated with smooth fields, we are thus interested in the asymptotic behavior of the flows of the smooth approximations.

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Classical theory IV

$\Phi(t, \cdot)$ flow of u smooth. $J\Phi(t,x) := \det D_x\Phi(t,x)$.

Theorem (Liouville)

If u is C^1,

$$\frac{\partial J\Phi}{\partial t} = J\Phi\,[\operatorname{div} u](\Phi)\,.$$

$J\Phi$ stays bounded (and bounded away from zero): $|\operatorname{div} u| \leq C$ and $J\Phi(0,\cdot) \equiv 1$ (+ Gronwall's lemma) imply

$$e^{-Ct} \leq J\Phi(t,x) \leq e^{Ct}\,.$$

Corollary (Liouville for divergence-free fields)

If $\operatorname{div} u = 0$, then $\Phi(t,\cdot)$ is measure-preserving.

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Measure theory allows for an elegant formulation of Liouville's theorem and its corollary.

Let $\Phi(t,\cdot)_\sharp\mu$ be the push-forward measure

$$\int f(x) d(\Phi(t,\cdot)_\sharp\mu)(x) := \int f(\Phi(t,x))\, d\mu(x)\,.$$

Theorem (Measure-theoretic Liouville)

If u is Lipschitz, then $\Phi(t,\cdot)_\sharp\mathcal{L}^n = \rho(t,\cdot)\mathcal{L}^n$ and

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0 \\ \rho(0,\cdot) = 1\,. \end{cases} \qquad \textit{(continuity equation)}$$

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If $\operatorname{div} u = 0$ then $\operatorname{div}(u\rho) = u \cdot \nabla\rho + \rho \operatorname{div} u = u \cdot \nabla\rho$

$$\partial_t \rho + u \cdot \nabla \rho = 0 .$$

The latter is the *transport equation*. The scalar ρ is "transported along the flow":

$$\frac{\partial}{\partial t}(\rho(t, \Phi(t, x))) =$$

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Modern theory I

Note: the assumption $\operatorname{div} u$ bounded is equivalent to u Lipschitz in 1 space dimension and rules out all textbook examples.

$\operatorname{div} u$ bounded (and even $\operatorname{div} u = 0$) covers several (not all!) interesting singularities.

Typically interested in $u \in L^1([0,T], W^{1,p})$ or $u \in L^1([0,T], BV)$, where

$$BV = \{u \in L^1 : \; Du \text{ is a Radon measure}\}$$

To avoid technicalities about $+\infty$, let's assume the domain is the periodic torus $\mathbb{T}^n$.

Theorem (DiPerna-Lions 1988, Ambrosio 2002)

If u is Sobolev (BV) and $\operatorname{div} u$ is bounded, there exists a "reasonable" (but just measurable) flow Φ, which is unique and stable under approximations. In fact the space of such flows is locally compact.

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$$BV = \{u \in L^1 : \ Du \text{ is a Radon measure}\}$$

To avoid technicalities about $+\infty$, let's assume the domain is the periodic torus $\mathbb{T}^n$.

Theorem (DiPerna-Lions 1988, Ambrosio 2002)

If u is Sobolev (BV) and $\operatorname{div} u$ is bounded, there exists a "reasonable" (but just measurable) flow Φ, which is unique and stable under approximations. In fact the space of such flows is locally compact.

Modern theory II

Reasonable flows (Ambrosio's axiomatization):

(a) For a.e. x, $t \mapsto \gamma(x) = \Phi(t, x)$ is an absolutely continuous curve.

(b) $\gamma(0) = x$ and $\dot\gamma(t) = u(t, \gamma(t))$ for a.e. t.

(c) $\Phi(t, \cdot)_\sharp \mathcal{L}^n \leq C(t)\mathcal{L}^n$

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Natural question: is this axiom really needed?

Modern III

PDE-ODE relations and Liouville still hold

Theorem

$\rho \in L^\infty([0,T], L^{p'})$ solves the transport equation

$$\partial_t \rho + u \cdot \nabla \rho = 0$$

iff ρ is constant along a.a. curves $t \mapsto \Phi(t,x)$
ρ (locally) solves the continuity equation

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0$$

iff $\rho(t,\cdot)\mathcal{L}^n = \Phi(t,\cdot)_\sharp(\rho(0,\cdot)\mathcal{L}^n)$.

Corollary

If $\mathrm{div}\, u = 0$, then $\Phi(t,\cdot)$ is measure preserving.

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understood “distributionally”.

Note: it needs $\rho \in L^\infty([0,T], L^{p'})$ and $u \in L^1([0,T], L^p)$. For $u \in L^1([0,T], W^{1,p})$ it needs $\rho \in L^\infty([0,T], L^{(p^*)'}$ (DiPerna-Lions assumption suboptimal...).

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$$\partial_t \rho + \operatorname{div}(\rho u) - \rho \operatorname{div} u = 0\,.$$

Solutions in the distributional sense.

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Strong convergence:

Theorem

If u Sobolev (BV), u_k Sobolev (BV), $\|u_k - u\|_{L^1} \to 0$ then

$$\|\Phi_k - \Phi\|_{L^1} \to 0$$

for the corresponding regular Lagrangian flows.

Corollary

If

$$\sup_k \|u_k\|_{L^1([0,T],W^{1,p})} + \|\mathrm{div}\, u_k\|_{L^1([0,T],L^\infty)} < \infty\,,$$

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The DiPerna-Lions approach

The "DiPerna-Lions" theory proves first well-posedness for bounded solutions of the transport and continuity equations.

Hence it concludes the existence, uniqueness and stability of regular Lagrangian flows.

A sketch when $\operatorname{div} u = 0$. **Existence:**

- Regularize u as $u_\varepsilon := u * \varphi_\varepsilon$ and solve the corresponding transport-continuity equation:

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \rho_\varepsilon(0, \cdot) = \rho_0 * \varepsilon \,. \end{cases} .$$

- Use classical theory to infer $\sup_x |\rho_\varepsilon(t,x)| \leq \sup_x |\rho_0(x)|$;
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Uniqueness:

- Prove ρ solution $\Longrightarrow \beta(\rho)$ solution (renormalization property) through a regularization scheme; this is the "hard analytic part" with a "commutator estimate";
- Use linearity to reduce to the case $\rho_0 = 0$;
- To show $\rho \equiv 0$, observe $|\rho|$ is a solution and integrate in space domains, formally
$$\frac{d}{dt}\int |\rho|(x,t)\,dx = 0\,.$$

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Show that the left hand side vanishes as $\varepsilon \downarrow 0$.

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Show that the left hand side vanishes as $\varepsilon \downarrow 0$. Rather simple for Sobolev, quite hard for BV.

The DiPerna-Lions approach III

Stability:

(i) Weak* compactness as in existence proof;

(ii) Uniqueness implies weak* continuity, i.e. if

$$\partial_t \rho_k + u_k \cdot \nabla \rho_k = 0$$

and $u_k \to u$, then $\rho_k \rightharpoonup^* \rho$ with

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(iii) Renormalization property + (ii) $\Longrightarrow \beta(\rho_k) \rightharpoonup^* \beta(\rho)$ for any test $\beta \in C^1$;

(iv) (iii) $\Longrightarrow$ strong converge of solutions to PDE;

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(v) Compactness of solutions to PDEs;

(vi) Compactness of flows ($\Longrightarrow$ existence and uniqueness of flows).

Stability:

(i) Weak* compactness as in existence proof;

(ii) Uniqueness implies weak* continuity, i.e. if

$$\partial_t \rho_k + u_k \cdot \nabla \rho_k = 0$$

and $u_k \to u$, then $\rho_k \rightharpoonup^* \rho$ with

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The DiPerna-Lions approach III

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Postmodern?

Bressan, 2002: can we quantify the compactness of flows? Conjecture, explicit rate.

Back to classical, uniqueness of flow:

$$\dot\gamma(t) = u(t,\gamma(t))$$
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$$\frac{d}{dt}|\gamma(t) - \bar\gamma(t)|$$

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When u is Lipschitz. Gronwall $\Longrightarrow |\gamma-\bar{\gamma}|\equiv 0$

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When u is Sobolev we can use the maximal function *MDu*.

[Crippa-De Lellis 2007] This heuristic can be made rigorous.
Interesting: this approach is neither a subset nor a superset of the DiPerna-Lions theory.

Postmodern II

[Crippa-De Lellis 2007] Bressan's conjectured rate correct for $W^{1,p}$, $p > 1$.

Conjecture still open for BV!

Theorem

$p > 1$, $\exists C(p, n)$ *s.t.*
If $u \in C^\infty$ *and* Φ *is the corresponding flow,* $\forall \varepsilon > 0$ $\exists K$ *with* $|K| < C\varepsilon$ *such that*

$$\mathrm{Lip}(\Phi|_K) \leq C \exp \left(\frac{C \|Du\|_{L^p}}{\varepsilon^{1/p}} \right) .$$

[Bresch-Jabin 2015], [Léger 2018] $|\hat{\Phi}(\xi)|^2 \log(1 + |\xi|) \in L^1$,
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Well-posedness "almost everywhere"

A nonrigorous interpretation of the DiPerna-Lions theory: there is a unique solution of the ODE for almost every initial point x.

[Ambrosio 2002, Alberti, Crippa] Can this interpretation be made rigorous?

Let u be a Sobolev vector field on $\mathbb{R}^n$. Is it true that for almost every $x \in \mathbb{R}^n$ there is a unique absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ such that

$$\begin{cases} \dot{\gamma}(t) &= u(t, \gamma(t)) \qquad \text{for a.e. } t \\ \gamma(0) &= x\,. \end{cases}$$

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Let us call an absolutely continuous curve as in $(\dagger)$ a trajectory for u with initial point x.

Well-posedness “almost everywhere”, answers

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The example can be made continuous [Giri, work in progress], falling into Peano’s existence theory.

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What happens in the critical case $p = n$?

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A.e. uniqueness holds when $Du \in L^{n,1}$ (Lorentz space).

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BY ACCIDENT ☺

Back to the DiPerna-Lions theory I

Recall, all that counts to define solutions ρ of the continuity equation is the (local) summability of ρu. $\rho u \in L^1_{loc}$ guaranteed by $\rho \in L^q$ and $u \in L^p$ with

$$\frac{1}{q} + \frac{1}{p} \leq 1 \qquad (\dagger).$$

Theorem (DiPerna-Lions 88)

Continuity and transport equations are well posed for $u \in W^{1,p}$ and $\rho \in L^q$ satisfying $(\dagger)$.

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Note, by Sobolev embedding $u \in W^{1,p}$ guarantees $u \in L^{p^*}$ for an exponent $p^* > p$. Indeed $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ when $p < n$, while for $p > n$, u is bounded!

Q1: Is $(\dagger)$ a technical condition, i.e. does the Sobolev improved summability of u allows less summability of ρ?

Q2: If we know, independently of the Sobolev property, some extra summability (for instance u bounded), can we just require the bare minimum for ρ?

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Sharp summability, $p > n$

Corollary (Caravenna-Crippa 2018)

$u \in W^{1,p}$, $p > n$. Positive solutions of the continuity equations are well-posed under the minimal summability requirement $\rho \in L^1$.

Positive solutions are nicer because of Ambrosio's superposition principle

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Linearity and superposition

Assume for the moment you had “two flows” Φ_1 and Φ_2 for the same vector field $u = 0$. $\rho_1 = (\Phi_1)_\sharp(\rho_0 \mathcal{L}^n)$ and $\rho_2 = (\Phi_2)_\sharp(\rho_0 \mathcal{L}^n)$ solve

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(0, \cdot) = \rho_0 \end{cases}$$

But $\lambda\rho_1 + (1-\lambda)\rho_2$ is a solution too.

Ambrosio's interpretation: you choose Φ_1 with probability λ and Φ_2 with probability $1-\lambda$.

In an appropriate sense, *all positive* solutions can be build by “choosing trajectories” at random.

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Theorem (Ambrosio 2002)

Let $u \in L^q$ and $\rho \in L^p$, with $\frac{1}{q} + \frac{1}{p} \leq 1$ and ρ positive such that

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$$\int f(x)\rho(t,x)\,dx = \int \int f(\gamma(t))\,d\eta_x(\gamma)\rho_0(x)\,dx$$

for every continuous f.

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The formula

$$\rho(t,\cdot)\mathcal{L}^n = \Phi(t,\cdot)_\sharp(\rho_0 \mathcal{L}^n)$$

holds and determines the solution.

Jabin, Caravenna-Crippa a.e. uniqueness

Recall, if $u \in W^{1,p}$ then $\exists f \in L^p$ such that

$$|u(x) - u(y)| \leq (f(x) + f(y))|x - y| \qquad \forall x, y$$

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u field, γ and $\bar{\gamma}$ trajectories

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Interpolating I

Is there a family of inequalities (depending on p) which interpolates between the two extreme situations

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Remark

The range of α is optimal.

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This could be just a technical limitation... but what happens otherwise?

Ambrosio's superposition principle III

Ambrosio's superposition principle holds in the full range of summability where the equations make sense.

From it we infer:

A.e. uniqueness of trajectories

$\Longrightarrow$ Uniqueness for positive solutions of the continuity equation.

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Remark

Since $\operatorname{div} u = 0$, the function $\bar{\rho} \equiv 1$ solves $(\dagger)$

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$u \in W^{1,p} \cap L^{q'}$ *divergence free*

$\rho \in L^q$ *positive* *SUCH THAT*

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0 \\ \rho(0, \cdot) = 1 \end{cases} \qquad (\dagger)$$

$\{\rho \neq 1\}$ *has positive measure.* $(\star)$

Remark

Since $\operatorname{div} u = 0$, the function $\bar{\rho} \equiv 1$ solves $(\dagger)$
By $(\star)$ ρ is a second distinct solution!

Convex integration generates monsters II

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because we did not discover that uniqueness cannot hold in the full range of possible exponents:

[Modena-Székelyhidi 2018] proved the previous theorem for $p < n - 1$ and sign-changing solutions

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Getting to positive solutions is highly nontrivial for $n - 1 \leq p < n$. Feature: our argument is considerably simpler than [Modena-Sattig] (especially when $n \geq 3$; some tricky combinatorics is needed when $n = 2$.)

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[De Lellis - Székelyhidi 2007] + [De Lellis - Székelyhidi 2012] invented "convex integration type methods" to generate irregular solutions of the incompressible Euler equations.

Inspired by the literature on differential inclusions (Bressan, Cellina, Dacorogna-Marcellini, Kirchheim, Müller-Šverak), by Nash's C^1 isometric embedding theory and by Gromov's h-principle.

These ideas were greatly improved in several aspects in the last 13 years (De Lellis -Székelyhidi, Cordoba-Faraco-Gancedo, Shvidkoy, Isett, Buckmaster, Vicol, Shkoller, Daneri, Colombo, De Rosa, . . .)

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The two most striking achievements:

[Isett 2016] Proof of the Onsager conjecture in fully developed turbulence.

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Thank you

for your attention!