Liouville Equations and Functional Determinants

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• While physicists may like these formulas, mathematicians usually have problems with infinite products of diverging numbers.

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If ζ is regular near $s = 0$ one can define the *regularized determinant* $\det'(-\Delta_g)$ via the following formula

$$\det{}'(-\Delta_g) = e^{-\zeta'(0)}.$$

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$$\zeta(s) = \frac{1}{\Gamma(s)} \left\{ \frac{A(\Sigma)}{4\pi(s-1)} + \left(\frac{\chi(\Sigma)}{6} - 1 \right) + \text{holom. in } s \right\},$$

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Existence of extremals is easy for positive genus. On spheres it can be achieved via a *balancing condition* and Möbius invariance, ([Aubin, '76], [Osgood-Phillips-Sarnak, '88], [Gui-Moradifam, '18]).

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$$\sum_{j=1}^{\infty} e^{-\lambda_j t} =: Tr(e^{\Delta t}) = \frac{1}{t} \sum_{j=0}^{l} t^j \int_{\Sigma} \Omega_j(x) dV + o(t^l),$$

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therefore one gets bounds even in higher Sobolev norms.

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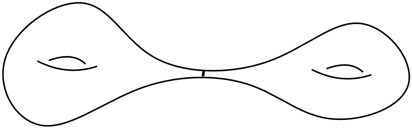

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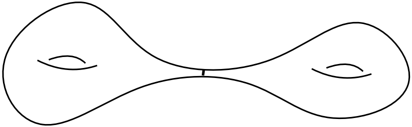

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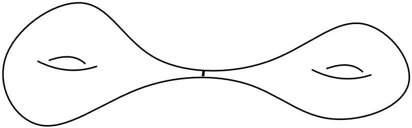

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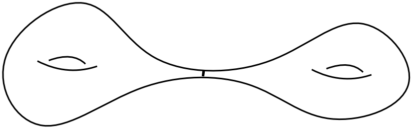

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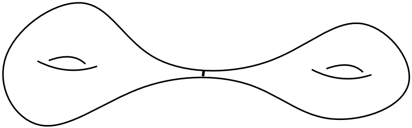

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3. The *Dirac operator* $\mathcal{D}$ for $n \geq 2$: $(a,b) = \left(\frac{n-1}{2}, \frac{n+1}{2} \right)$.

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where

$$I[w] = 4 \int_M w |W_g|^2 \, dv - \Big(\int_M |W_g|^2 \, dv \Big) \log \fint_M e^{4w} \, dv,$$

$$II[w] = \int_M w P_g w \, dv - \Big(\int_M Q_g \, dv \Big) \log \fint_M e^{4(w - \overline{w})} \, dv,$$

$$III[w] = 12 \int_M (\Delta_g w + |\nabla w|^2)^2 \, dv - 4 \int_M (w \Delta_g R_g + R_g |\nabla w|^2) \, dv.$$

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• Extremal metrics for linear combinations of the functionals I, II, III were useful in studying rigidity of K-E metrics in 4D ([Gursky, '98]).

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• Sometimes we will reverse signs to get coercivity/convexity.

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• We aim to discuss here the situations when either *(ii)* fails (e.g. in negative curvature), or when *(i)* fails (as for the Paneitz operator). The latter case is indeed much harder.

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Compactness and quantization for Q-curvature

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The topological structure of the energy (joint with a *monotonicity argument* by Struwe) allows to produce solutions of *perturbed equations*

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If blow-up occurs, use Green's formula to show that e^{4u_n} accumulates at finitely-many points ([Brezis-Merle', 91]), so $u_n - \bar{u}_n \to u_s$, with u_s s.t.

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Finally *bubbling analysis*, shows that $\beta_i = 8\pi^2$ for all i ([Li-Shafrir, '93], [Druet-Robert, '06], [M., '06]), a contradiction to $k_Q \notin 8\pi^2\mathbb{N}$.

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Then either $(u_n)_n$ stays bounded in $C^{4,\alpha}(M)$ or $\mu_n\, e^{4u_n} \rightharpoonup 8\pi^2 \sum_{i=1}^{l} \delta_{p_i}$, for distinct points $p_1, \ldots, p_l$.

Compactness of extremal metrics for F_L

We focus next on the log-determinant of the conformal Laplacian L (some results apply to more general F_A's). Extremal metrics satisfy

$$\mathcal{N}(u) + U_g = \mu\, e^{4u}, \qquad \text{with } \mathcal{N}(u) \text{ in divergence form.}$$

$$\mathcal{N}_L(u) \simeq \Delta^2 u - \Delta_4 u; \qquad \Delta_4 u = \operatorname{div}(|\nabla u|^2 \nabla u),$$

$$U_g = \gamma_1 |W_g|_g^2 + \gamma_2 Q_g - \gamma_3 \Delta_g R_g; \qquad \mu = \int_M U_g\, dv.$$

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Consequence. At blow-up points concentrates at least ε_0 volume, so the set of blow-up points is finite.

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Proposition 1

There exists a distributional solution of $\mathcal{N}(u_s) + U_g = \Sigma_{i=1}^{l} \beta_i \delta_{p_i}$ such that $u_s = \alpha_i \log d(x, p_i) + w$ near p_i, with $\alpha_i = \alpha_i(\beta_i) < 0$ (explicit), and

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For the p-Laplacian see [Serrin, '64], [Kichenassamy-Veron, '86]: in this case one has homogeneity of the operator and the maximum principle.

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- The argument works for any (finite) measure data.

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• For general coefficients, it would be enough to know the uniqueness of the *singular profile* of u_s, without knowing global uniqueness.

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One could try to understand them defining and studying a suitable *mass* for the blown-up manifold via the fundamental solution.

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On S^4 instead one has

$$\begin{aligned} F_P[w] &= \int_{S^4} \left[18(\Delta w)^2 + 64|\nabla w|^2 \Delta w + 32|\nabla w|^4 - 60|\nabla w|^2\right] dv \\ &+ 112\pi^2 \log\left(\fint_{S^4} e^{4(w-\overline{w})}\, dv\right). \end{aligned}$$

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- It goes similarly for compact hyperbolic manifolds.

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(d) A similar result holds in $\mathbb{R}^4$, much easier to prove.

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It is an interesting question to characterize extremals of this quotient in $\mathbb{R}^4$, vaguely related to the above problem.

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Apart from the compactness issues, new sharp Moser-Trudinger inequalities would be expected.

Thanks for your attention