Nonuniqueness of weak solutions to the Navier-Stokes equation

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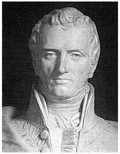
November 29, 2017

The Incompressible Navier–Stokes Equations

The pair (v, p) solves the *incompressible Navier–Stokes equations* if



$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \nu \Delta v &= 0 \\ \operatorname{div} v &= 0 \end{aligned}$$

for kinematic viscosity $\nu > 0$, velocity $v : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3$ and pressure $p : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$.

Weak solutions to the Navier–Stokes equations

We say $v \in C_t^0 L_x^2$ is a weak solution of NSE if for any $t \in \mathbb{R}$ the vector field $v(\cdot, t)$ is weakly divergence free, has zero mean, and

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} v \cdot (\partial_t \varphi + (v \cdot \nabla)\varphi + \nu \Delta \varphi) dxdt = 0\,,$$

for any divergence free test function φ. Fabes-Jones-Riviere '72, implies such a solutions satisfies the integral equation

$$v(t) = e^{\nu\Delta(t)} v(\cdot, 0) + \int_0^t e^{\nu\Delta(t-s)} \mathbb{P}\, \mathrm{div}(v(\cdot, s) \otimes v(\cdot, s)) ds\,.$$

Based on the natural scaling of the equations $v(x,t) \mapsto v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t)$:

- A number of partial regularity results have been established: Scheffer '76, Cafarelli-Kohn-Nirenberg '82, Lin '98, Ladyzhenskaya-Seregin '99, Vasseur '07, Kukavica '08, ...
- Local existence for the Cauchy problem has been proven in scaling-invariant spaces Kato '84, Giga-Miyakawa '85, Koch-Tataru '01, Jia-Sv̌erák '14, ...
- Conditional regularity has been established under geometric structure assumptions (Constantin-Fefferman '93), or assuming a signed pressure (Seregin-Sv̌erák '02).
- The conditional regularity and weak-strong uniqueness results known under the umbrella of Ladyzhenskaya-Prodi-Serrin conditions: Kiselev-Ladyzhenskaya '57, Prodi '59, Serrin '62, Escauriaza-Seregin-Šverák '03, ...
- For the class of weak solutions defined above, if $v \in C_t^0 L_x^3$ then such a solution is unique: Furioli–Lemarié-Rieusset–Terraneo '00, Lions-Masmoudi '01.

Nonuniqueness of weak solutions

Theorem 1 (B-Vicol '17)

There exists $\beta > 0$*, such that for any smooth* $e(t)\colon [0, T] \to \mathbb{R}_{\geq 0}$*, there exists a weak solution* $v \in C^0_t([0, T]; H^\beta_x(\mathbb{T}^3))$ *of the Navier-Stokes equations, such that*

$$\int_{\mathbb{T}^3} |v(x, t)|^2 \, dx = e(t) \,,$$

for all $t \in [0, T]$.

Dissipative Euler solutions arise in the inviscid limit

Theorem 2 (B-Vicol '17)

Let $u \in C^{\bar\beta}_{t,x}(\mathbb{T}^3 \times [-2T, 2T])$, *for* $\bar\beta > 0$, *is a weak solution of the Euler equations:*

$$\partial_t u + (\operatorname{div} u \otimes u) + \nabla p = 0 \quad \text{and} \quad \operatorname{div} u = 0$$

Then there exists $\beta > 0$, *a sequence* $\nu_n \to 0$, *and a uniformly bounded sequence* $v^{(\nu_n)} \in C^0_t([0, T]; H^\beta_x(\mathbb{T}^3))$ *of weak solutions to the Navier-Stokes equations:*

$$\partial_t v^{(\nu_n)} + \operatorname{div}\left(v^{(\nu_n)} \otimes v^{(\nu_n)}\right) + \nabla p - \nu_n \Delta v^{(\nu_n)} = 0 \quad \text{and} \quad \operatorname{div} v^{(\nu_n)} = 0$$

with $v^{(\nu_n)} \to u$ *strongly in* $C^0_t([0, T]; L^2_x(\mathbb{T}^3))$.

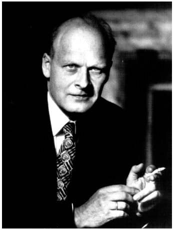
Onsager's Conjecture

Lars Onsager, in his famous note on statistical hydrodynamics [Onsager '49]), conjectured the following dichotomy:

(a) Any weak solution v belonging to Hölder spaces C^β for $\beta > \frac{1}{3}$ conserves the kinetic energy.

(b) For any $\beta < \frac{1}{3}$ there exist weak solutions $v \in C^\beta$ which do not conserve the kinetic energy.

Part (a) of this conjecture was proven by [Constantin, E and Titi '94], (cf. [Eyink '94], [Duchon-Robert '00], [Cheskidov-Constantin-Friedlander-Shvydkoy '08])



Part (b): Existence of non-conservative solutions

Part b) was recently resolved: $L^2_{x,t}$ [Scheffer '93]; $L^\infty_t L^2_x$ [Shnirelman '00]; $L^\infty_{x,t}$ [De Lellis-Székelyhidi Jr. '09-'11]; $C^0_{x,t}$ [De Lellis-Székelyhidi Jr. '12]; $C^{1/10-}_{x,t}$ [De Lellis-Székelyhidi Jr. '12]; $C^{1/5-}_{x,t}$ [Isett '13]; $C^{1/5-}_{x,t}$ [B.-De Lellis-Székelyhidi Jr. '13]; $C^{1/3-}_x$ a.e. in time; [B. '15]; $L^1_t C^{1/3-}_x$ [B.-De Lellis-Székelyhidi Jr. '16].

Theorem 1 (Isett '16)

For every $\beta < 1/3$, there exists weak solutions $v \in C^\beta_{x,t}$ to the Euler equations with compact support in time.

Theorem 2 (B-De Lellis-Székelyhidi Jr.-Vicol '17)

For every smooth strictly positive energy profile $e : [0, T] \to \mathbb{R}$ and $\beta < 1/3$, there exists weak solutions $v \in C^\beta_{x,t}$ such that $\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = e(t)$.

Structure functions

Define the structure functions for homogeneous, isotropic turbulence by

$$S_p(\ell) := \langle [\delta v(\ell)]^p \rangle \, ,$$

where $\langle \cdot \rangle$ denotes an ensemble average. Kolmogorov's famous four-fifths law can be stated as

$$S_3(\ell) \sim -\frac{4}{5}\varepsilon \ell \ ,$$

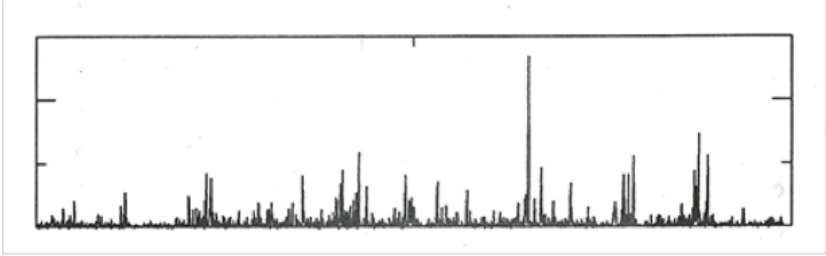
More generally, Kolmogorov's scaling laws can be stated as

$$S_p(\ell) = C_p \varepsilon^{\zeta_p} \ell^{\zeta_p} \ ,$$

for any positive integer p, for $\zeta_p = {}^p/_3$.

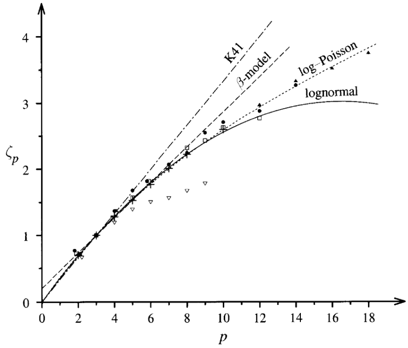
Intermittency

[Landau '59]: The rate of energy dissipation is intermittent, i.e., spatially inhomogeneous.



Intermittency Corrections

- lognormal model of [Kolmogorov '62]: $\zeta_2 = 0.694444$.
- β-model [Frisch-Sulem-Nelkin '78]: $\zeta_2 = 0.733333$.
- log-Poisson model of [She-Leveque '94]: $\zeta_2 = 0.695937$.
- mean-field theory of [Yakhot '01]: $\zeta_2 = 0.700758$.



Intermittent Euler result

Theorem 3 (B.- Masmoudi - Vicol *(in preparation)*)

Fix any $\alpha < 5/14$. There exist infinitely many weak solutions

$$u \in C_t^0 H_x^\alpha$$

of the 3D Euler equations which have compact support in time.

The number 5/14 is not sharp. Arguments of [C-C-F-S '08]: for $\alpha > 5/6$ energy is conserved.

The convex integration scheme

The proof proceeds via induction, for each $q \geq 0$ we assume we are given a solution $(v_q, p_q, \mathring{R}_q)$ to the Navier-Stokes-Reynolds system.

$$\begin{aligned} \partial_t v_q + \mathrm{div}(v_q \otimes v_q) + \nabla p_q - \nu \Delta v_q &= \mathrm{div}\, \mathring{R}_q \\ \mathrm{div}\, v_q &= 0\,. \end{aligned}$$

where the stress $\mathring{R}_q$ is assumed to be a trace-free symmetric matrix.

The perturbation

As part of the induction step, the perturbation $w_{q+1} = v_{q+1} - v_q$ is designed such that the new velocity v_{q+1} solves the Navier-Stokes-Reynolds system

$$\begin{aligned} \partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} - \nu \Delta v_{q+1} &= \operatorname{div} \mathring{R}_{q+1} \\ \operatorname{div} v_{q+1} &= 0\,. \end{aligned}$$

with a smaller Reynolds stress R_{q+1}. Writing $v_{q+1} = w_{q+1} + v_q$ and using the equation for v_q we may write

$$\begin{aligned} \operatorname{div} R_{q+1} &= (-\nu \Delta w_{q+1} + \partial_t w_{q+1}) + \operatorname{div}(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q) \\ &\quad + \operatorname{div}(w_{q+1} \otimes w_{q+1} - \mathbb{R}_q) + \nabla(p_{q+1} - p_q) \\ &=: \operatorname{div}\left(\widetilde{R}_{\text{linear}} + \widetilde{R}_{\text{quadratic}} + \widetilde{R}_{\text{oscillation}}\right) + \nabla(p_{q+1} - p_q). \end{aligned}$$

The perturbation $w_{q+1} = v_{q+1} - v_q$ is constructed as a superposition of intermittent Beltrami waves at frequency λ_{q+1}:

$$\lambda_q = a^{(b^q)}$$

for $a, b \gg 1$. The perturbation will be of the form

$$w_{q+1} \sim \sum_{\xi \in \Lambda} a_{\overline{\xi}}(\mathring{R}_q)\mathbb{W}_{\overline{\xi}}$$

in order to cancel the low frequency $(\approx \lambda_q)$ error of $\mathring{R}_q$ of size given

$$\left\| \mathring{R}_q \right\|_{L^1} \leq \lambda_{q+1}^{-2\beta}$$

for $0 < \beta \ll 1$. From scaling considerations we expect

$$\| w_{q+1} \|_{L^2} \leq \lambda_{q+1}^{-\beta} .$$

Beltrami waves

A stationary divergence free vector field v is called a *Beltrami flow* if it satisfies the *Beltrami condition*:

$$\lambda v = \operatorname{curl} v \,, \quad \lambda > 0 \quad .$$

Given a Beltrami flow v, we have the following identity

$$\operatorname{div}(v \otimes v) = v \cdot \nabla v = \nabla \frac{|v|^2}{2} - v \times (\operatorname{curl} v) = \nabla \frac{|v|^2}{2} - \lambda v \times v = \nabla \frac{|v|^2}{2}.$$

Setting $p := \frac{|v|^2}{2}$, then (v, p) is a stationary solution to the Euler equations.

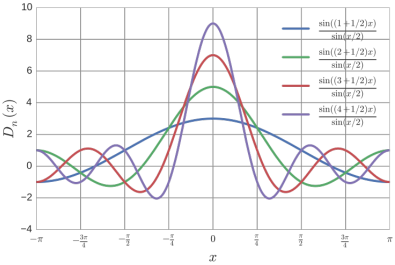
Intermittent Beltrami waves

- Gain if can build a version of the Beltrami waves $\mathbb{W}_{\bar{\xi}}$ such that

$$\|\mathbb{W}_{\bar{\xi}}(\lambda_{q+1}\cdot)\|_{L^2} \approx 1, \qquad \|\mathbb{W}_{\bar{\xi}}(\lambda_{q+1}\cdot)\|_{L^1} \ll_{\lambda_{q+1}} 1$$

- Recall, in 1D the normalized Dirichlet kernel obeys:

$$\left\| \frac{1}{\sqrt{r}} \sum_{-r\leq k\leq r} e^{ikx} \right\|_{L^2} \approx 1, \quad \left\| \frac{1}{\sqrt{r}} \sum_{-r\leq k\leq r} e^{ikx} \right\|_{L^1} \approx \frac{\log r}{\sqrt{r}} \ll 1.$$



Heuristic estimate on dissipation error

Each intermittent Beltrami wave $\mathbb{W}_{\overline{\xi}}$ will be made up of

$$\left(\frac{\lambda_{q+1}}{\lambda_q}\right)^p = \lambda_{q+1}^{p'}$$

distinct frequencies, for some $2 < p' < p < 3$. By setting $\nu = 1$ and writing

$$\begin{aligned}\Delta w_{q+1} &= \operatorname{div}(\nabla w_{q+1}) \\ &= \operatorname{div}\left(\nabla \sum_{\overline{\xi}} a_{\overline{\xi}} \mathbb{W}_{\overline{\xi}}\right)\end{aligned}$$

The dissipation error's contribution to $\mathring{R}_{q+1}$ can be heuristically estimated by

$$\|\nabla w_{q+1}\|_{L^1} \lesssim \sum_{\overline{\xi}} \left\| a_{\overline{\xi}} \mathbb{W}_{\overline{\xi}} \right\|_{W^{1,1}} \lesssim \lambda_{q+1}^{1-p'/2} .$$

Estimate on the perturbation

A naïve estimate of the perturbation would give

$$\|w_{q+1}\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\| a_{\overline{\xi}} \mathbb{W}_{\overline{\xi}} \right\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\| a_{\overline{\xi}} \right\|_{L^\infty} \left\| \mathbb{W}_{\overline{\xi}} \right\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\| a_{\overline{\xi}} \right\|_{L^\infty}$$

However, we have no control on $\left\| a_{\overline{\xi}} \right\|_{L^\infty} \approx \left\| \mathring{R}_q \right\|_{L^\infty}^{1/2}$!

Lemma 4

Assume f is supported in a ball of radius λ in frequency, and that g is a $(\mathbb{T}/\sigma)^3$-periodic function. If $\lambda \ll \sigma$, then

$$\|f\, g\|_{L^p(\mathbb{T}^3)} \lesssim \|f\|_{L^p(\mathbb{T}^3)} \|g\|_{L^p(\mathbb{T}^3)}.$$

Then heuristically we obtain

$$\|v_{q+1} - v_q\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\|a_{\overline{\xi}} \mathbb{W}_{\overline{\xi}}\right\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\|a_{\overline{\xi}}\right\|_{L^2} \left\|\mathbb{W}_{\overline{\xi}}\right\|_{L^2} \lesssim \sum_{\overline{\xi}} \left\|a_{\overline{\xi}}\right\|_{L^2}$$

which gives us the correct estimate since

$$\left\|a_{\overline{\xi}}\right\|_{L^2} \approx \left\|\mathring{R}_q\right\|_{L^1}^{1/2} \lesssim \lambda_{q+1}^{-\beta}.$$

Future directions

Given a weak solution $v \in C_t^0 L_x^2 \cap L_t^2 H_x^1$ to the Navier-Stokes equation, we say that v is a Leray-Hopf solution if in addition it satisfies the energy inequality

$$\frac{1}{2}\int_{\mathbb{T}^3} \left|v(x,t)^2\right| \, dx + \int_{\mathbb{T}^3 \times [0,t]} \left|\nabla v(x,s)\right|^2 \, dxds \le \frac{1}{2}\int_{\mathbb{T}^3} \left|v(x,0)\right|^2 \, dx \, .$$

In Jia-Šverák '15 proved that non-uniqueness of Leray-Hopf weak solutions in the regularity class $L_t^\infty L_x^{3,\infty}$ is implied if a certain spectral assumption holds for a linearized Navier-Stokes operator. Very recently Guillod-Šverák '17 have provided compelling numerical evidence that the spectral condition is satisfied.

We conjecture that non-uniqueness of Leray-Hopf solutions can be proven via convex integration. This is known in the case where the Laplacian $-\Delta$ is replaced by the fractional laplacian $(-\Delta)^s$ for $s \in (0, 1/5)$, Colombo-De Lellis-De Rosa '17.

Questions?