Act globally, Compute locally

Group actions, fixed points and localization

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& IAS

IAS Members's Seminar
20 October 2014

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Outline

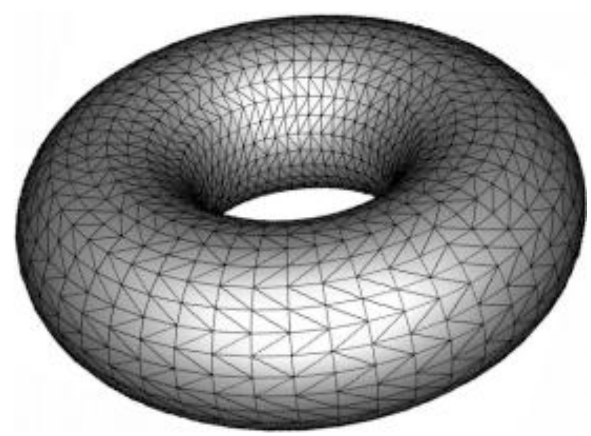
- Symplectic geometry (with lots of examples)
- Group actions & fixed points (with lots of examples)
- Localization (with lots of examples)
- Symplectic reduction (how to take a quotient)
  (with lots of examples)

Symplectic manifolds

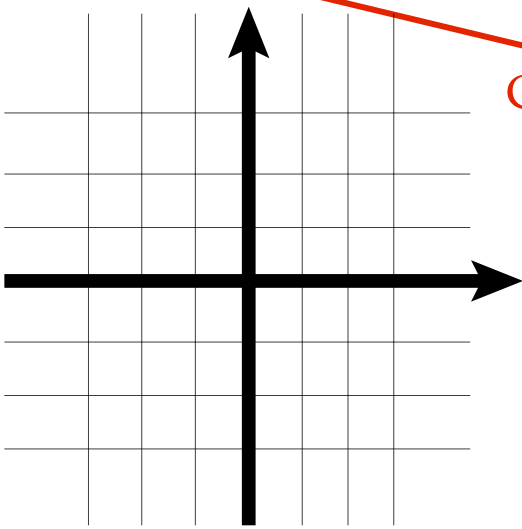
A symplectic manifold is a manifold with a two-form $\omega \in \Omega^2(M)$ that is:

- Closed: $d\omega = 0$
- Non-degenerate: $\omega^n = d\mathrm{Vol} \rightsquigarrow$ M is $2n$-dimensional & orientable





Symplectic manifolds



Calque of "complex" introduced by Weyl (1939)

Complex: Latin *com-plexus* "braided together"
Symplectic: Greek $\sigma\upsilon\mu$ - $\pi\lambda\varepsilon\kappa\tau\iota\kappa\acute{o}\varsigma$

$$(\mathbb{R}^2, \omega = dx \wedge dy) \rightsquigarrow \left(\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i\right)$$

Darboux's Theorem:

We may always choose coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on M so that locally

$$\omega = \sum dx_i \wedge dy_i.$$

$\rightsquigarrow$ There are no local invariants (like curvature).

Compact examples

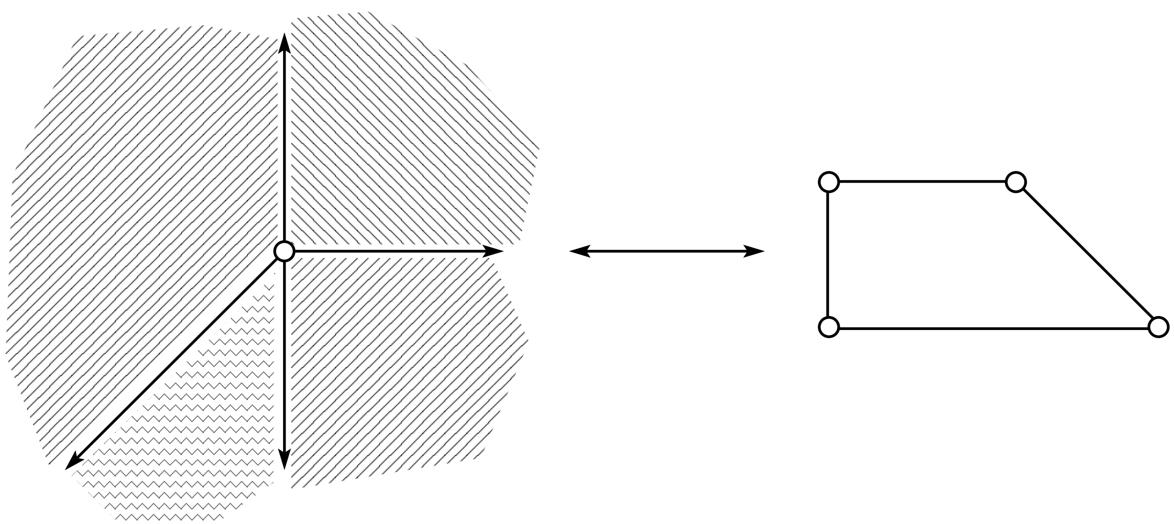
$(n \geq 2)$

Coadjoint orbits $\mathcal{O}_\lambda$

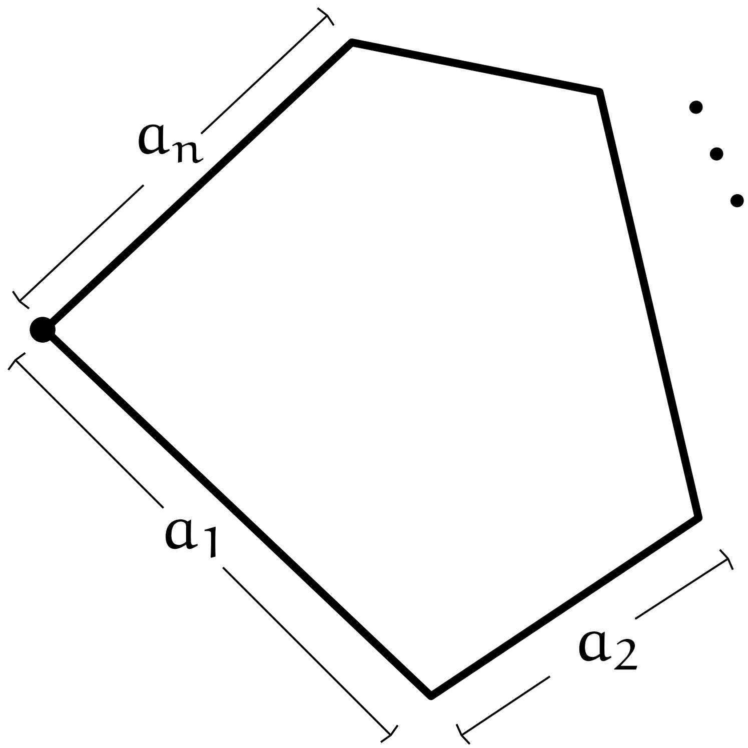
- ~~Even-dimensional spheres~~ $\cancel{S^{2n} = \{\vec{x} \subset \mathbb{R}^{2n+1} \mid \sum x_i^2 = 1\}}$
- Complex projective space $\mathbb{C}P^{n-1} = \{V \subseteq \mathbb{C}^n \mid \dim_{\mathbb{C}}(V) = 1\}$
- Grassmannian $\mathcal{G}r(k, \mathbb{C}^n) = \{V \subseteq \mathbb{C}^n \mid \dim_{\mathbb{C}}(V) = k\}$
- Flag varieties

$$\mathcal{F}\ell ags(\mathbb{C}^n) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}}(V_i) = i\}$$

- Smooth complex projective varieties
- Toric varieties
- Based loops $\Omega G = \{\gamma : S^1 \to G \mid \gamma(\mathrm{Id}) = \mathrm{Id}\}$



Example: $\mathcal{P}ol_d(a_1, \ldots, a_n)$



$\in \mathcal{P}ol_2(a_1, \ldots, a_5)$

$$\mathcal{P}ol_d(a_1, \ldots, a_n) = \frac{\left\{ (\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}) \;\middle|\; \overrightarrow{v_i} \in \mathbb{R}^d,\ |v_i| = a_i,\ \sum \overrightarrow{v_i} = \overrightarrow{0} \right\}}{SO(d)}$$

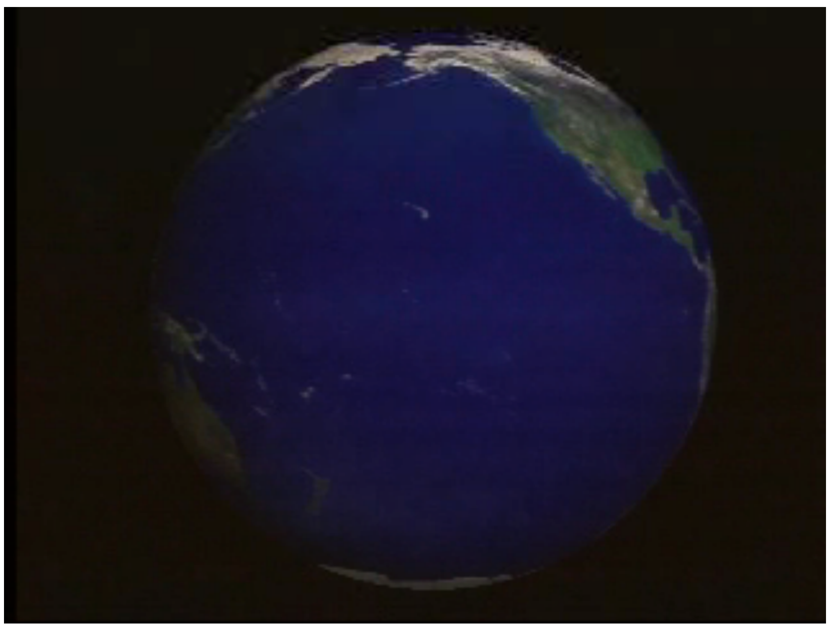
$\mathcal{P}ol_3(a_1, \ldots, a_n)$ is symplectic! N.B. d=3!!

Symplectic actions

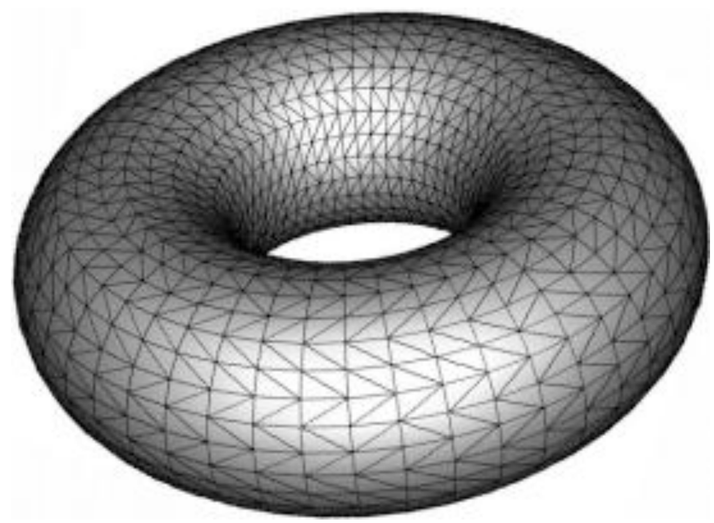
Symplectic manifolds often exhibit symmetries, encoded by a group action. (It's a hard topological question, "How many manifolds do or do not have symmetries?" ...)

DEF: A group action $G \circlearrowright M$ is **symplectic** if it preserves ω ; that is,

$$\tau_g^* \omega = \omega \quad \forall\, g \in G.$$



$S^1 \circlearrowright S^2$ by rotation



$S^1 \circlearrowright T^2$ by rotation

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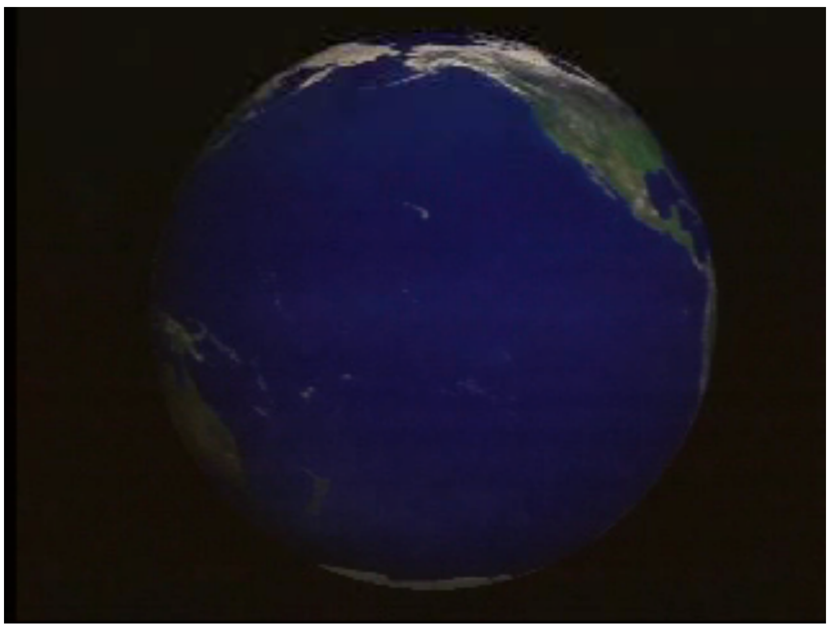


$S^1 \circlearrowright S^2$ by rotation

$SO(3) \circlearrowright S^2$ by multiplication

~~$O(3) \circlearrowright S^2$ by multiplication~~

Symplectic actions

DEF: A group action $G \circlearrowright M$ is **symplectic** if it preserves ω ; that is,

$$\tau_g^* \omega = \omega \quad \forall\, g \in G.$$

DEF: Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $G \circlearrowright M$. For any $\xi \in \mathfrak{g}$, we may define a vector field on M by,

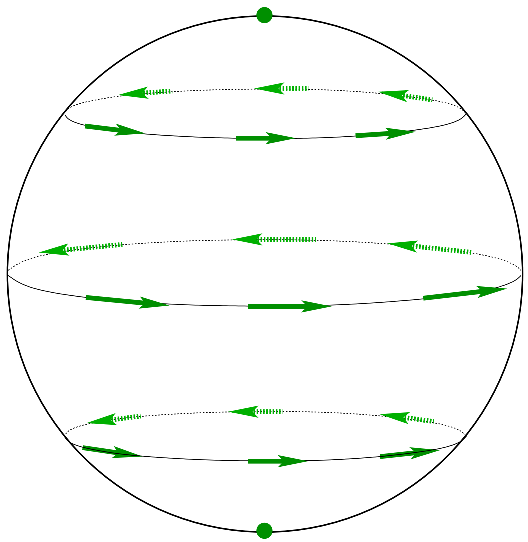
$$\mathcal{X}_\xi(p) = \frac{d}{dt}\Big[\exp(t\xi)\cdot p\Big]\Big|_{t=0}.$$

Infinitesimally

$$\mathcal{L}_{\mathcal{X}_\xi}\omega = 0$$

$$\mathcal{L}_{\mathcal{X}_\xi}\omega = d\iota_{\mathcal{X}_\xi}\omega + \iota_{\mathcal{X}_\xi}\underbrace{d\omega}_{=0}$$

$$\implies \quad d\Big(\omega(\mathcal{X}_\xi,\cdot)\Big) = 0$$



$S^1 \circlearrowright S^2$ by rotation $\rightsquigarrow$ Vector field parallel to latitude lines

Hamiltonian actions

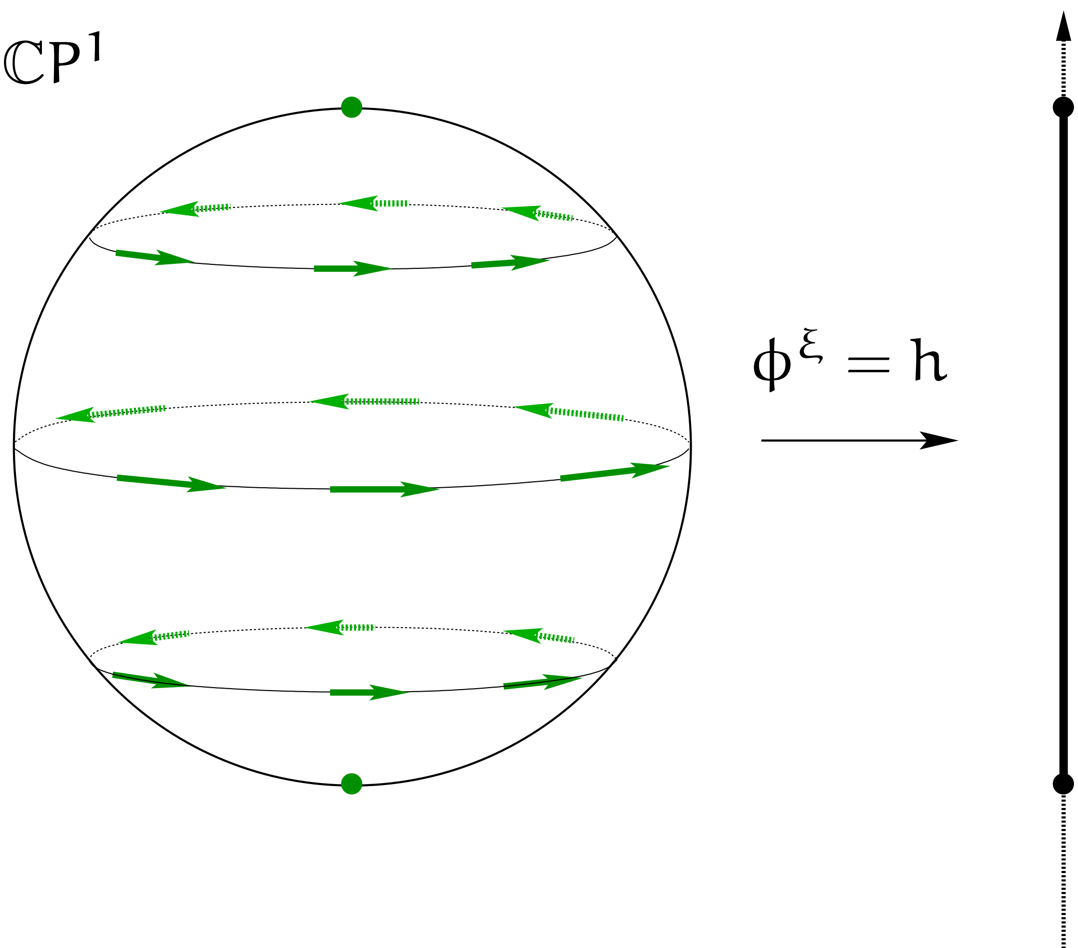
$$d\Big(\omega(\mathcal{X}_\xi, \cdot)\Big) = 0$$

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

Example: $S^1 \circlearrowright M = S^2 = \mathbb{C}P^1$

$$\begin{aligned} \omega &= d\theta \wedge dh \\ \mathcal{X}_\xi &= \frac{\partial}{\partial\theta} \\ \omega(\mathcal{X}_\xi, \cdot) &= dh \\ &\Longrightarrow \phi^\xi = h \end{aligned}$$



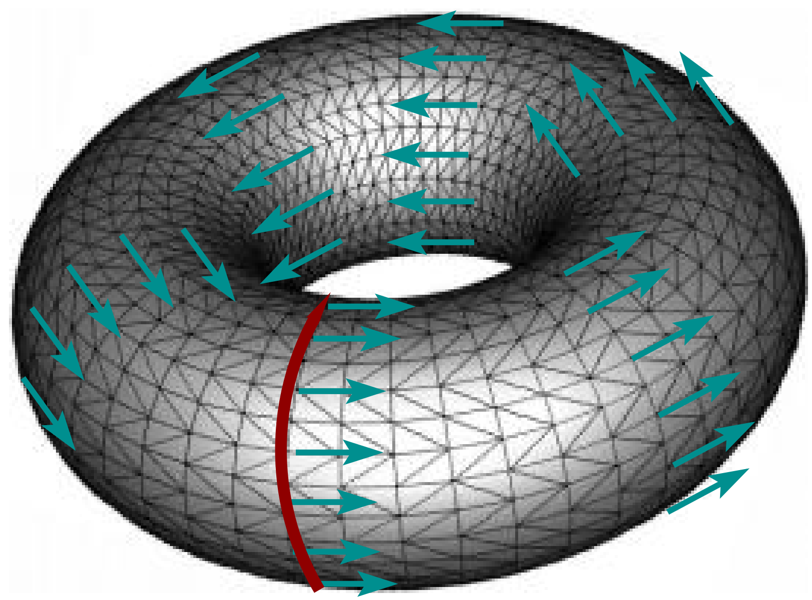
A non-Hamiltonian action

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

Non-Example: $S^1 \circlearrowright T^2 = S^1 \times S^1$ rotating the first factor.

$$\begin{aligned} \omega &= dx \wedge dy \\ \mathcal{X}_\xi &= \frac{\partial}{\partial x} \\ \omega(\mathcal{X}_\xi, \cdot) &= dy \end{aligned}$$



But $\quad dy \in H^1(T^2; \mathbb{Z})$ is certainly not exact!

Hamiltonian actions

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

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Frankel's Theorem:
A symplectic circle action $S^1 \circlearrowright (M, \omega)$ which preserves the compatible complex structure on a compact Kähler manifold M is

$$\text{Hamiltonian} \iff M^{S^1} \neq \emptyset.$$

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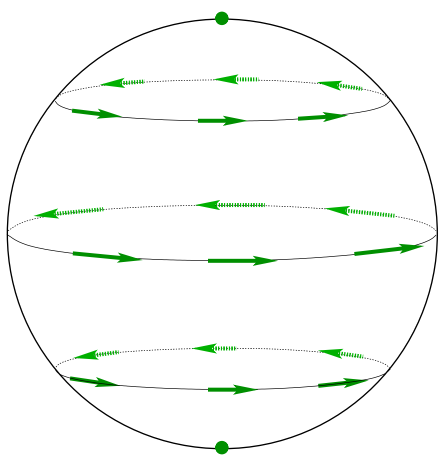

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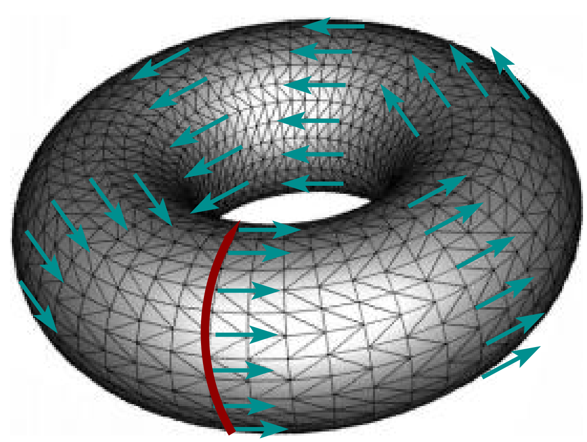

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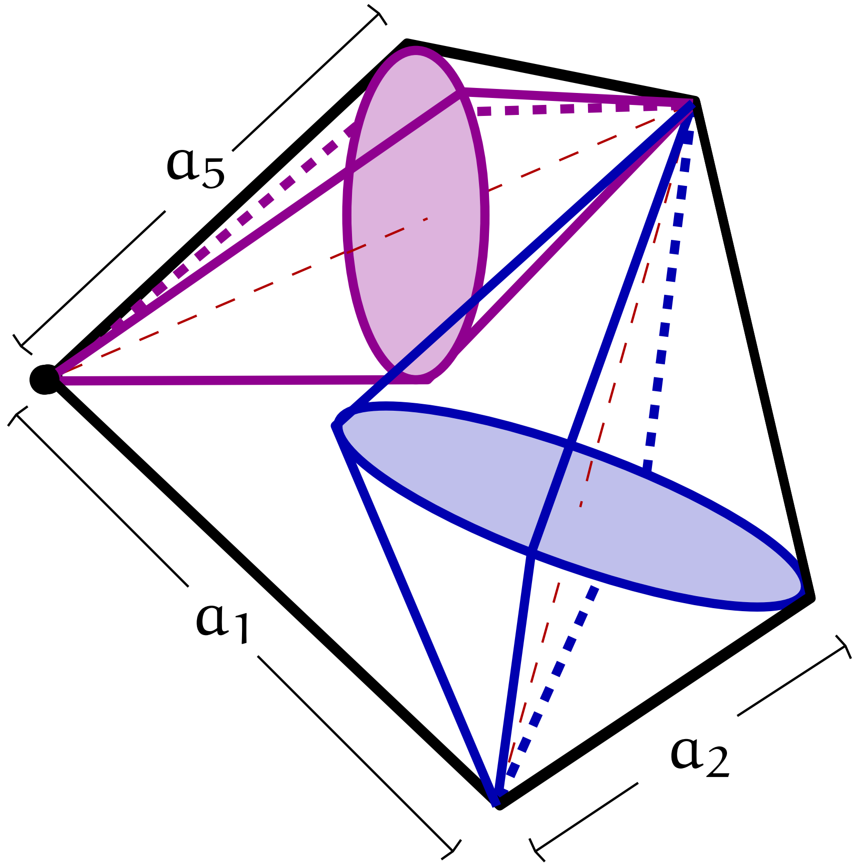

$T^2 \circlearrowright \mathcal{P}ol_3(a_1, \ldots, a_5)$ is Hamiltonian.

Hamiltonian actions

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<u>McDuff's Theorem</u>: M is compact.

(a) $S^1 \circlearrowright (M^4, \omega) \Longrightarrow$

$$\text{Hamiltonian} \iff M^{S^1} \neq \emptyset.$$

(b) $\exists\, S^1 \circlearrowright (M^6, \omega)$ that has fixed points but is not Hamiltonian.

<u>Questions:</u>

- Are there examples of (b) where the fixed points are isolated?
- Can circle-valued ϕ^{ξ} play an analogous role?

The momentum map

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

DEF: Combining these for all $\xi \in \mathfrak{g}$, we define the momentum map

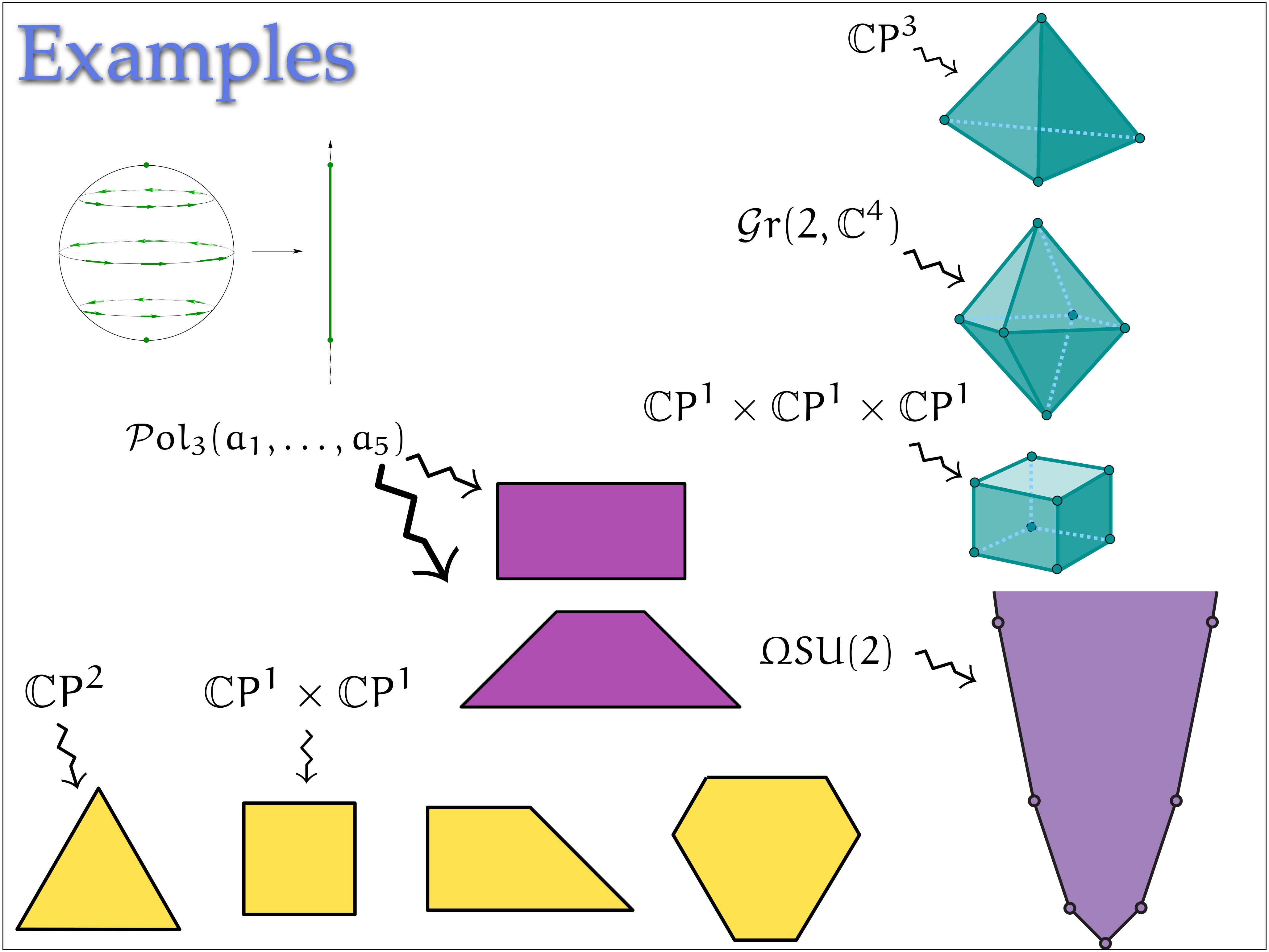
$$\begin{array}{rcl} \Phi : M & \to & \mathfrak{g}^* \\ p & \mapsto & \left(\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^\xi(p) \end{array} \right). \end{array}$$

<u>Convexity Theorem</u> [Atiyah, Guillemin-Sternberg]:

If $T = (S^1)^d \circlearrowright (M, \omega)$ is Hamiltonian, $\Phi(M)$ is a convex polytope.

$$\Phi(M) = \mathrm{Conv}(\Phi(M^T)).$$

Examples



$\mathbb{C}P^3$
$\mathcal{G}r(2, \mathbb{C}^4)$
$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$
$\mathcal{P}ol_3(a_1, \ldots, a_5)$
$\Omega SU(2)$
$\mathbb{C}P^2$
$\mathbb{C}P^1 \times \mathbb{C}P^1$

Localization

A **localization phenomenon** is a global feature of $T \circlearrowright M$ that can be described by the evidence of that feature at the T-fixed points.

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In terms of topology, we use localization to make global equivariant computations in terms of local computations at fixed points.

Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category.

- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$

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$$G \circlearrowright M \rightsquigarrow H^*_G(M;\mathbb{Z}) \text{ or } H^*_G(M;R)$$

$$f : M \to N \implies f^* : H^*_G(N) \to H^*_G(M)$$

Mayer-Vietoris

Et cetera

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$$T = T^d = S^1 \times \cdots \times S^1 \rightsquigarrow$$

$$H_T^*(pt; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_d]$$

$$\deg(x_i) = 2$$

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Equivariant cohomology

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- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$
- Equivariant cohomology of a point is not $\mathbb{Z}$
- Spaces, maps should be equivariant
- If $G \circlearrowright M$ is a free action, then $H^*_G(M) = H^*(M/G)$
- Abelianization trick: $H^*_G(M;\mathbb{Q}) \cong H^*_T(M;\mathbb{Q})^W$

  (False over $\mathbb{Z}$ -- joint work with Sjamaar)

Cohomological Localization

$$M^T \hookrightarrow M \quad \rightsquigarrow \quad H_T^*(M;R) \longrightarrow H_T^*(M^T;R)$$

Theorem [Frankel; Atiyah; Kirwan]:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, then

$$H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(M^T;\mathbb{Q})$$

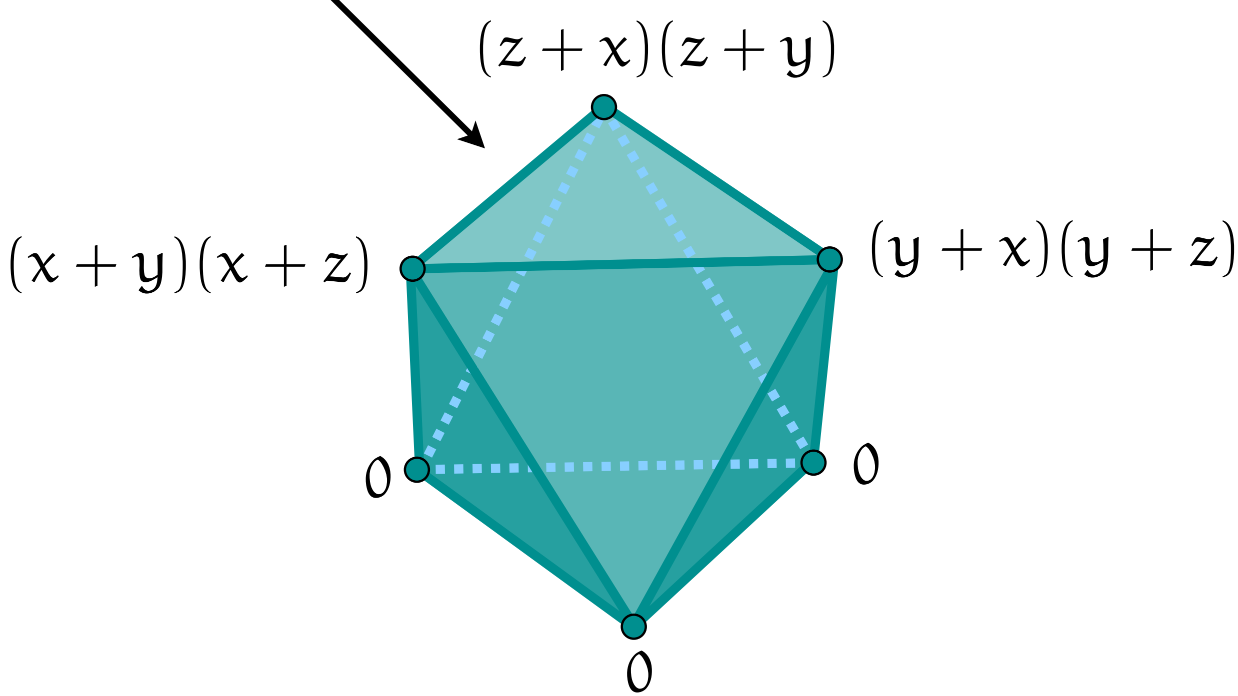
is an injection. (The statement sometimes holds over $\mathbb{Z}$.)

Example

$$T^3 \circlearrowright \mathcal{G}r(2,\mathbb{C}^4) \rightsquigarrow H_T^*(\mathcal{G}r(2,\mathbb{C}^4);\mathbb{Z}) \subseteq \bigoplus_{i=1}^{6} \mathbb{Z}[x,y,z]$$

$$\begin{array}{c} \Psi \\ \alpha = [\mathbb{P}^2] \end{array}$$

Φ



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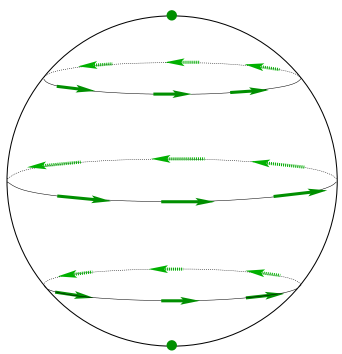

$$\alpha \in H_T^*(M;R) \Longrightarrow (\alpha|_N, \alpha|_S) \in \mathbb{R}[x] \oplus \mathbb{R}[x]$$

<u>Fact:</u> $(\alpha|_N, \alpha|_S) \in \mathrm{Im} \iff x \;\Big|\; (\alpha|_N - \alpha|_S)$

<u>GKM Theorem</u>: Suppose that

(a) M^T consists of isolated points; and

(b) M^S consists of isolated points and S^2s, for each $S \subset T$ of codimension 1.

Then $H_T^*(M;\mathbb{Q}) \cong \left\{ (f_v) \in \bigoplus_{v \in M^T} \mathbb{Q}[x_1, \ldots, x_d] \;\middle|\; \alpha_e | f_v - f_w \text{ for each } S_e^2 \right\}$.

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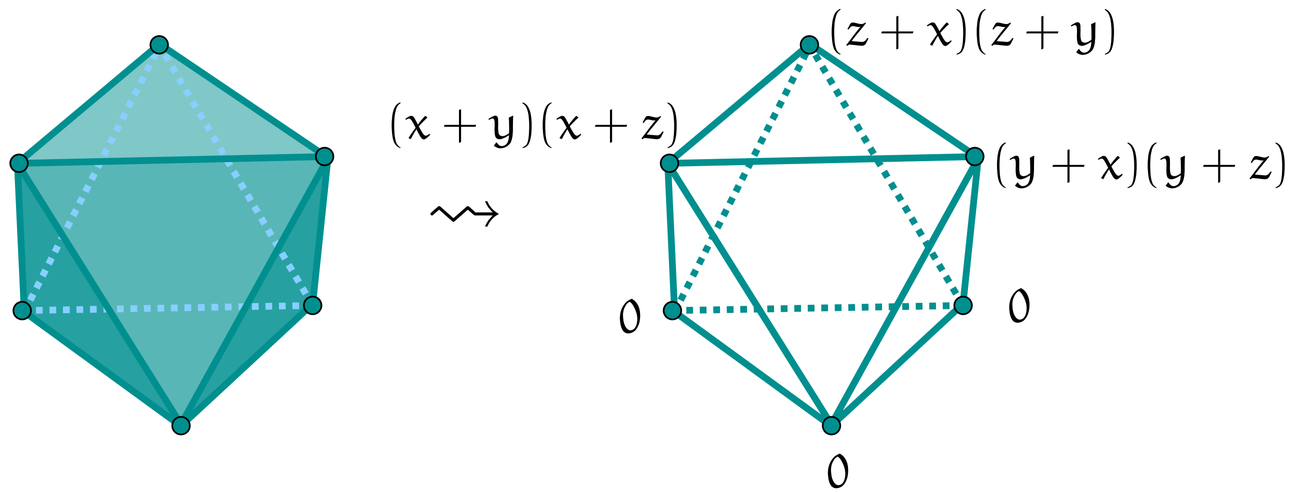

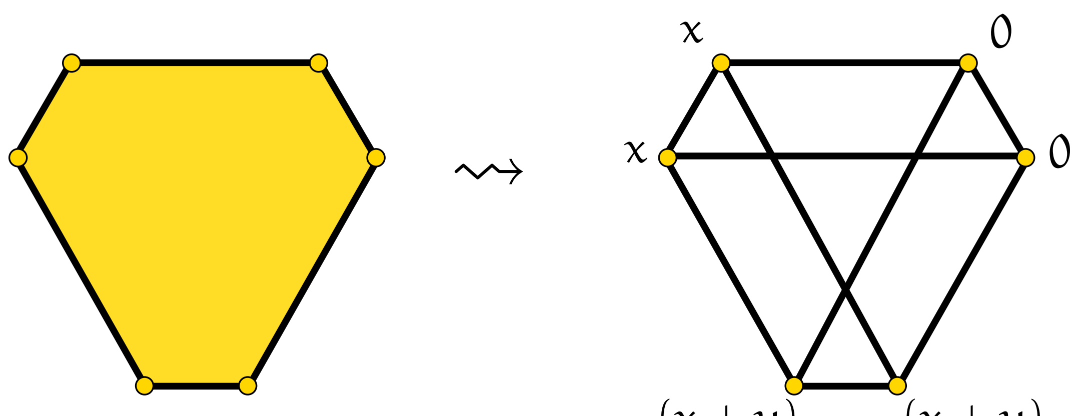

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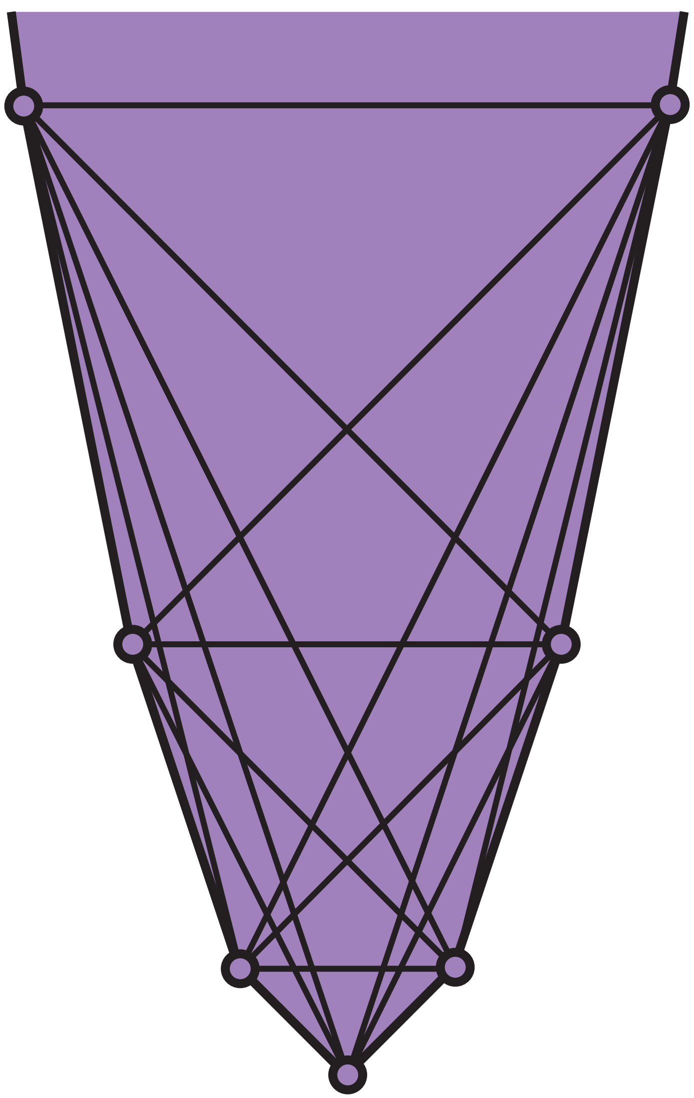
Equivariant invariants of ΩG



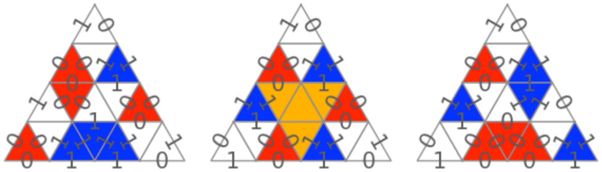
[66] Megumi Harada, André Henriques, and Tara S. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197(1): 198–221, 2005.

<u>Theorem</u> [Harada-Henriques-H.]: The GKM description works, even in infinite dimensional cases, for

- Equivariant cohomology $H_T^*(M;\mathbb{Q})$
  (Sometimes integrally!) $H_T^*(M;\mathbb{Z})$
- Equivariant K-theory $K_T^*(M)$
- Equivariant cobordism $MU_T^*(M)$

Further applications & generalizations

- Schubert calculus



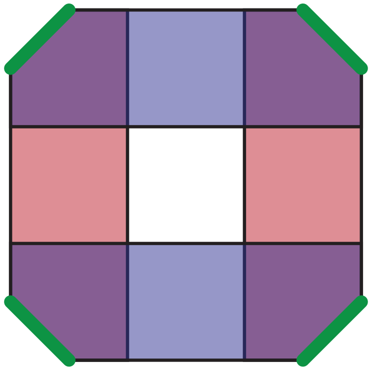
Questions:

- Does this lead to better / easier combinatorics?
- How do you program "subrings" rather than "quotients"?

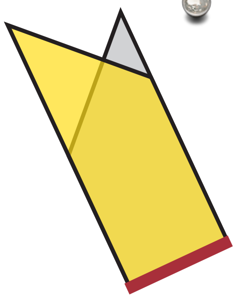
- Quantum invariants and Gromov-Witten theory

Question:

- How do you see quantum corrections in M^T?



- Torus manifolds (Masuda, Panov, Park)
- Toric origami manifolds (H-Pires)



Questions:

- What happens in the non-simply connected case?
- Can you determine manifolds up to cobordism?

Symplectic reduction

We have the moment map

$$\begin{array}{rcl} \Phi : M & \to & \mathfrak{t}^* \\ p & \mapsto & \left(\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^{\xi}(p) \end{array} \right). \end{array}$$

It can be used to prove localization results because ϕ^{ξ} behaves like a **Morse function**, with critical set M^T (for most ξ).

The moment map is also an **equivariant** map: $T \circlearrowright \Phi^{-1}(\mu)$ for every $\mu \in \mathfrak{t}$. If μ is a regular value, $\Phi^{-1}(\mu)$ is a manifold.

$$\begin{aligned} \dim(\Phi^{-1}(\mu)) &= \dim(M) - \dim(T) \\ &= 2n - d \end{aligned}$$

$$\begin{aligned} \dim(\Phi^{-1}(\mu)/T) &= \dim(M) - 2 \cdot \dim(T) \\ &= 2n - 2d \end{aligned}$$

Symplectic reduction

We have the moment map

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It can be used to prove localization results because ϕ^{ξ} behaves like a **Morse function**, with critical set M^T (for most ξ).

The moment map is also an **equivariant** map: $T \circlearrowright \Phi^{-1}(\mu)$ for every $\mu \in \mathfrak{t}$. If μ is a regular value, $\Phi^{-1}(\mu)$ is a manifold.

<u>Theorem</u> [Marsden-Weinstein]:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, and μ is a regular value of Φ, then

$$M//T(\mu) = \Phi^{-1}(\mu)/T$$

is symplectic, with at worst orbifold singularities.

Example of symplectic reduction

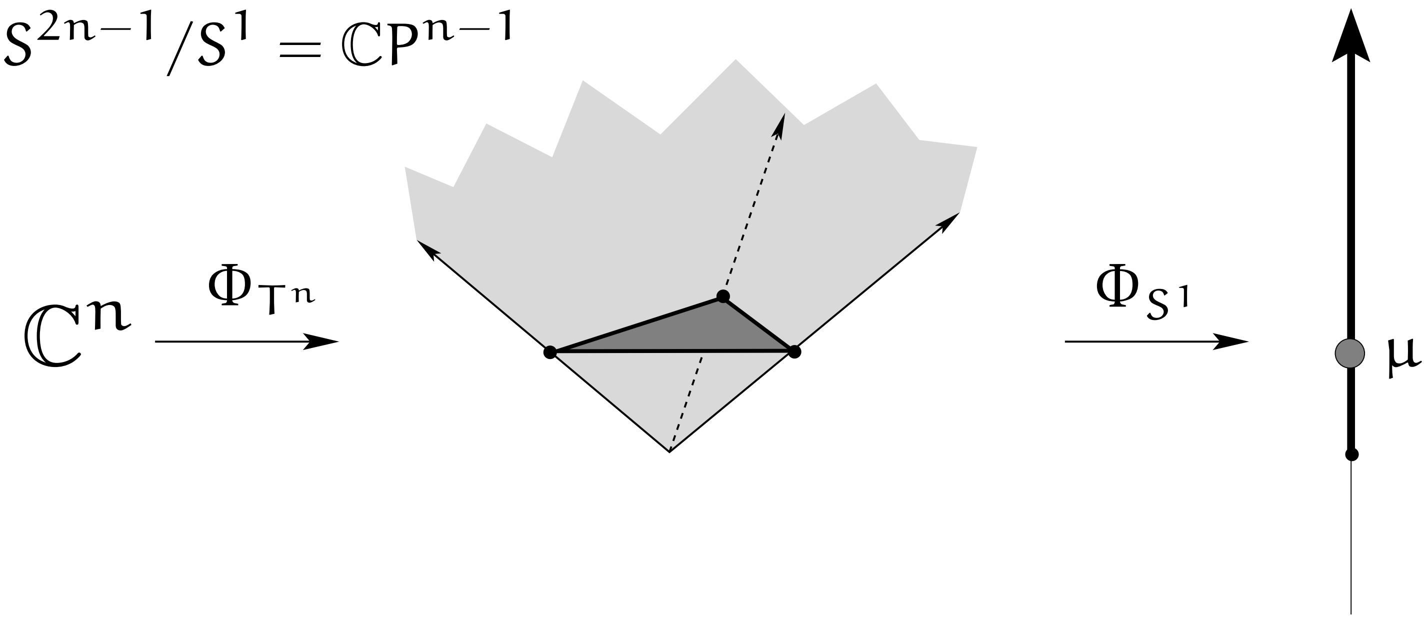
$S^1 \circlearrowright \mathbb{C}^n$

$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n)$

$$\begin{aligned} \Phi : \mathbb{C}^n &\to \mathbb{R} \\ (z_1, \ldots, z_n) &\mapsto \sum |z_i|^2 \end{aligned}$$

$$\Phi^{-1}(\mu) = \left\{ (z_1, \ldots, z_n) \;\middle|\; \sum |z_i|^2 = \mu \right\} = S^{2n-1}$$

$$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$$

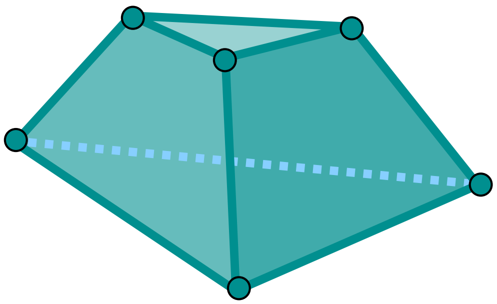


More examples

Delzant's Theorem:

$$\left\{ \begin{array}{c} \text{compact toric} \\ \text{symplectic manifolds} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{simple rational} \\ \text{smooth convex polytopes} \end{array} \right\}$$

$$\mathbb{C}^n//T^d \quad \leftrightsquigarrow$$



$$\left. \begin{array}{l} T^n \circlearrowright \mathcal{F}\ell ags(\mathbb{C}^n) \\ T^n \circlearrowright \mathcal{G}r(k, \mathbb{C}^n) \\ \;T \circlearrowright \mathcal{O}_\lambda \end{array} \right\} \mathcal{O}_\lambda//T \text{ is a weight variety.}$$

Yet another example

$$\mathcal{P}ol_3(a_1,\dots,a_n) = \frac{\left\{(\overrightarrow{v_1},\dots,\overrightarrow{v_n}) \;\middle|\; \overrightarrow{v_i} \in \mathbb{R}^3, |v_i| = a_i, \sum \overrightarrow{v_i} = \overrightarrow{0}\right\}}{SO(3)}$$

$$= S^2_{a_1} \times \cdots \times S^2_{a_n} /\!\!/ SO(3)$$

$$\begin{tikzcd}
 & (\mathbb{C}^2)^n \arrow[dl, "/\!\!/ U(2)"'] \arrow[dr, "/\!\!/ T^n"] & \\
\mathcal{G}r(2,\mathbb{C}^n) \arrow[dr, "/\!\!/ T^{n-1}"'] & & S^2 \times \cdots \times S^2 \arrow[dl, "/\!\!/ SO(3)"] \\
 & \mathcal{P}ol_3(a_1,\dots,a_n) &
\end{tikzcd}$$

Cohomology of symplectic reductions

$$\Phi^{-1}(\mu) \hookrightarrow M \rightsquigarrow H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(\Phi^{-1}(\mu);\mathbb{Q})$$

Kirwan's Theorem:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, then

$$\kappa_\mu : H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(\Phi^{-1}(\mu);\mathbb{Q}) \cong H^*(\Phi^{-1}(\mu)/T;\mathbb{Q})$$

is a surjection (with isomorphism when μ is a regular value).

Theorem [H-Tolman]:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold with connected stabilizer subgroups and if M^T is torsion free, then κ_μ is surjective over $\mathbb{Z}$.

Technique: The map $||\Phi||^2$ is minimally degenerate.

Theorem [H-Karshon]:

Minimal degeneracy is a local condition.

Cohomology of symplectic reductions

$$\Phi^{-1}(\mu) \hookrightarrow M \quad \rightsquigarrow \quad H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(\Phi^{-1}(\mu);\mathbb{Q})$$

<u>Kirwan's Theorem</u>:

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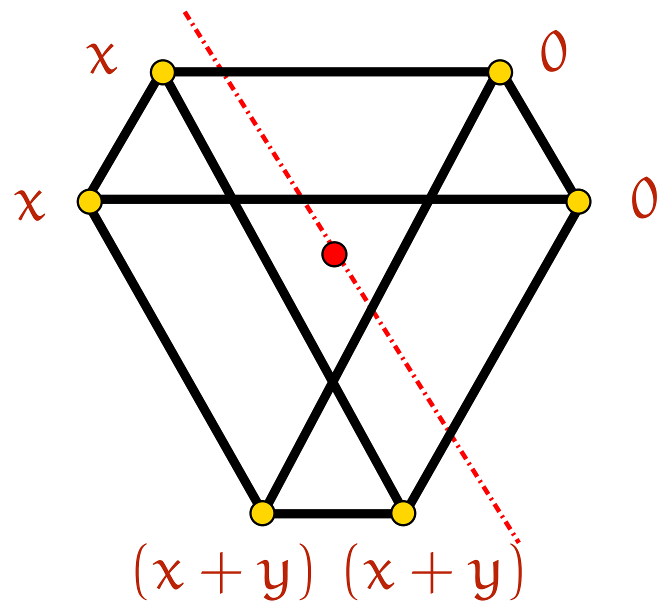
is a surjection (with isomorphism when μ is a regular value).

<u>Theorem</u> [Tolman-Weitsman, Goldin]:

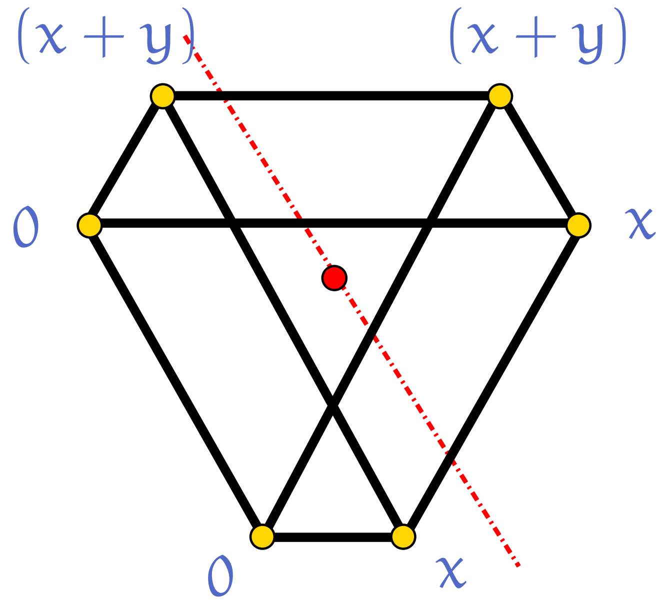
The ideal $\ker(\kappa_\mu)$ is computable in terms of localization.

$$\begin{tikzcd}
\ker(\kappa_\mu) \arrow[r, hook] \arrow[dr, hook] & H_T^*(M) \arrow[r, two heads] \arrow[d, hook] & H^*(M/\!/T(\mu)) \\
& H_T^*(M^T) &
\end{tikzcd}$$

Computing $\ker(\kappa_\mu)$



Computing $\ker(\kappa_\mu)$



Invariants of symplectic reductions that are orbifolds

Theorem [Goldin-H-Knutson]:

When M//T is an orbifold,

$$\bigoplus_{t \in T} H_T^*(M^t) \longrightarrow H_{CR}^*(M//T)$$

is surjective, with computable kernel.

degree 0 Gromov-Witten invariants

Invariants of symplectic reductions that are orbifolds

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is surjective, with computable kernel.

Examples

- Symplectic toric orbifolds
- Weight varieties

Invariants of symplectic reductions that are orbifolds

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When $M//T$ is an orbifold,

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Generalizations

- Equivariant K-theory (with Goldin, Harada, Kimura)
- Explicit computations in K-theory (with Goldin, Harada)
- Preliminary computations of $QH^*(\mathcal{P}ol_3(\vec{\alpha}))$ (with Chen, Taipale; building on Gonzalez, Woodward, Ziltener, et al)

The End

Thank you!!