XI: Unexpected Spin Off: Moduli spaces:

This section is not about the calculus of variations; however it is intimately connected with it. It is also a lesson that the geometry behind the variational problem should not be ignored: it plays an essential role.

Associated with many variational problems, in particular harmonic maps and Yang-Mills equations, is a first order differential equation which implies the second order Euler-Lagrange equation. This is easy to illustrate: The basic second order linear equation associated with harmonic ructions and maps is Laplacian (u) = 0. The scale invariant space is n=2. If the image of u is a linear complex space, Laplacian(u)= 0 is implied by the Cauchy-Riemann equations d-bar u = 0, the first order equation associated to the second order one. It implies If the M is a region of the complex plane, and N is an almost complex manifold, the equivalent of this equation can be formulated, and the solution of d-bar u = 0 is called a pseudo-holomorphic curve (Gromov 1985). These form the basis of much of current symplectic topology.

Likewise, in Yang-Mills theory, we note that the curvature is a 2-form which canonically satisfies the Bianchi identity D_A^F_A = 0. The Euler-Lagrange equation for a critical point of the Yang-Mills functional J is

D_A*F_A = 0, so the first order equation. F_A = *F_A implies the Euler-Lagrange equations. These “self-dual” equations form the basis of many applications of gauge theory in topology.

First order equations can be formulated for integrals that are “above the critical dimension” or “unrenormalizeable” when the second order equations are otherwise out of reach. Also ate that by construction, the second order Euler-Lagrange equations are self-adjoint, with index 0, whereas first order equations have an index calculated via the Atiyah=Singer index theorem.   This index calculates the dimension of the “moduli spaces” of solutions to the first order equations. There are too many examples of these moduli spaces in topology, geometry and algebraic geometry to even begin a list. But in each case, the starting point is a variational problem.

XII. Present Day Mysteries.

There is, of course, much research going on in these subjects, whose framework came from the calculus of variations. Let me suggest that there are still uncharted waters, by mentioning two wildly different subjects:

A.Deep learning in computer science is the hottest topic going. It involves predicting from data some new information by minimizing a loss functional on a training set of data over a “deep” set of choices for the matching. Computer scientists tend to draw little pictures of minimizing the height of a graph. z= f(x,y), whereas they are thinking of minimizing a proper function over a very, very large dimensional manifold. Talk of minima, local minima, saddles and steepest descent gradient curves makes it clear that they are thinking in terms of calculus, but there is little geometry in the discussions I have heard.

B We have talked about minimizing the Dirichlet integral

$$J(u) = \int_M |du|^p \, dA.$$ For p = 2,

for functions u: M—>R, but we can do this in a standard way for any p. However, if we let p go to infinity, we are minimizing the Lipschitz constant of u. The equation for the limit is called the infinity Laplace equation. Craig Evans' comment to me was that "Everything we know about elliptic PDE is of no use". Another nice mystery.

I do have a bit of advice to give. Everything you can learn and know about the landscape of the problem turned out to be fruitful in the calculus of variations. The calculus of variations is not a black box. If you are doing differential geometry, everything you know about differential geometry is useful in setting up the variational problem. Same with topology, algebraic geometry. And it goes without saying in physics. This might be useful to keep in mind.

A. Deep Learning involves