A Tight Bound for Hypergraph Regularity

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Joint work with Asaf Shapira

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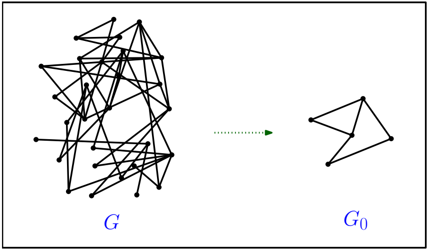

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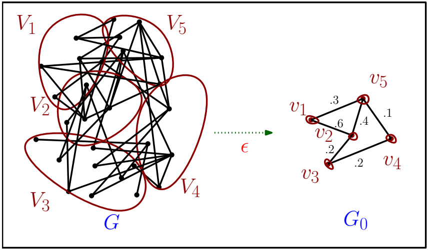

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- The number of H-free graphs [Erdős-Frankl-Rödl '86]

The Asymptotic Number of Graphs not Containing a Fixed Subgraph and a Problem for Hypergraphs Having No Exponent

P. Erdös[1], P. Frankl[2] and V. Rödl[3]

[1] Mathematical Institute of the Hungarian Academy of Science, P.O.B. 127, 1364 Budapest, Hungary
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Problem 6.1. *Suppose H is a $K_t(l,r)$-free r-uniform hypergraph on n vertices, $t > r$. Let ε be an arbitrarily small positive real $n > n_0(\varepsilon, r, t, l)$. Is it possible to remove εn^r edges from H so that the remaining hypergraph is $K_t(r)$-free?*

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Remark added in proof. Problem 6.1 has been recently positively answered by P. Frankl and V. Rödl. The proof uses an extension of Szemerédi's regularity lemma to hypergraphs.

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Theorem (Triangle Counting Lemma)

If G is an $n \times n \times n$ tripartite graph whose 3 bipartite graphs are ϵ-regular of densities $\alpha, \beta\,\gamma$ then the number of triangles in G is $(\alpha\beta\gamma \pm 7\epsilon)n^3$.

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There is a 4-partite 3-graph which is $K_4^{(3)}$-free even though each of the 4 triples of vertex classes is $o(1)$-regular:

- Let T be a balanced 4-partite random tournament (where the direction of each xy is chosen independently and uniformly).

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- Thus, each xyz forms an edge in H with probability $1/4$, and each of the 4 triples of vertex classes of H is $o(1)$-regular.
- It is easy to see that H is $K_4^{(3)}$-free.

Multiple Hypergraph Regularity Lemmas

Different versions of hypergraph regularity were proved by:

- Frankl-Rödl '02, Rödl-Skokan '04, Nagle-Rödl-Schacht '06
- Gowers '07
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Remark [Chung-Graham-Wilson '89]

In graphs, discrepancy, codegree, eigenvalues,... are poly-equivalent.

Upper Bounds for Hypergraph Regularity

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- $\mathrm{Ack}_1(n) = 2^n$
- $\mathrm{Ack}_2(n) = \mathrm{T}(n) = 2^{\cdot^{\cdot^{\cdot^{2}}}} \}$ n times
- $\mathrm{Ack}_3(n) = \mathrm{W}(n) = \mathrm{T}(\cdots(\mathrm{T}(1))\cdots)$ (n compositions)
- $\mathrm{Ack}_4(n) = \ldots$

Detour: Applications

Original motivation—a combinatorial proof of Szemerédi's Theorem:

Theorem (Szemerédi '74)

$\forall \delta > 0, k \in \mathbb{N} \ \exists N = N(\delta, k)$:
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- Obtaining such bounds for van der Waerden's and Szemerédi's Theorems (two special cases) were open problems for many decades (until solved by Shelah [JAMS '89] and Gowers [GAFA '01] respectively).

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- Weaker. (In fact, strictly weaker.)
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- The wowzer-type UB's come from constructing a regular partition in a sequence of steps, each applying the graph regularity lemma and thus increasing the partition size by a tower-type function.
- So the question is: Can we show that a sequence of applications of the graph regularity lemma is unavoidable?

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All known graph LB proofs fail to work vs. relaxed graph regularity.

Sparse Regular Approximation Lemma (SRAL)

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Input: G with pn^2 edges.
Freedom: add/remove $1\% \cdot pn^2$ edges.
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Intuition

They were iterative, constructing the graph in "layers". However, if one is allowed to modify 1% of the edges, one can essentially stop the construction at a stage where the graph still has a regular partition of constant order.

Bounds for SRAL

Theorem (LB for SRAL, M.-Shapira '17)

Lower bound: $\mathrm{T}(\Omega(\log \frac{1}{p}))$.

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Remark

The same paper also proves a matching upper bound for SRAL, and deduces Fox's celebrated $\mathrm{T}(O(\log \frac{1}{\epsilon}))$ bound [Ann. of Math. '11] for the graph removal lemma.

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- A bipartite graph on (A, B) is $\langle\delta\rangle$-regular:
  $\forall A' \subseteq A, B' \subseteq B$, if $|A'| \geq \delta|A|, |B'| \geq \delta|B|$ then $d(A', B') \geq \frac{1}{2} d(A, B)$.

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Important difference from ϵ-regularity: Can prove LB for $\langle 2^{-30}\rangle$-regularity.

Our Lower Bounds, Formally

Theorem (LB for graph $\langle\delta\rangle$-regularity, M.-Shapira '18+)

$\forall p \in (0,1)$ $\exists$ *graph* G *of density* p :
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Theorem (Main result (for 3-graphs), M.-Shapira '18+)

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In fact, even trivial versions of these notions are stronger than our notion.

Detour: How Strong is Our Lower Bound?

Question

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Lemma

There are arbitrary large tripartite graphs of density $\approx \delta^5$ whose every pair of classes span a $\langle\delta\rangle$-regular graph and yet are triangle free.

How Strong is our Lower Bound - cont.

Reminder:

Definition ($\langle\delta\rangle$-regularity for graphs)

- A bipartite graph on (A, B) is $\langle\delta\rangle$-regular:
  $\forall A' \subseteq A, B' \subseteq B$, if $|A'| \geq \delta|A|, |B'| \geq \delta|B|$ then $d(A', B') \geq \frac{1}{2}d(A, B)$.

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- Remove each triangle and take a blow-up; $\langle\delta\rangle$-regularity is preserved.

□

Main Result: Proof Sketch

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Let $\mathcal{P}_1 \succ \cdots \succ \mathcal{P}_s$ be equipartitions with $|\mathcal{P}_{i+1}| = 2^{c|\mathcal{P}_i|}$.

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$\exists$ *graph* G *such that*

$\forall\epsilon$*-regular partition* $\mathcal{Z}$ *of* G *we have:*

$$\forall i : \quad \mathcal{Z} \prec_x \mathcal{P}_i \quad \Rightarrow \quad \mathcal{Z} \prec_{4x} \mathcal{P}_{i+1} \qquad \textit{for } x \geq \sqrt{\epsilon}\ .$$

The implication was improved to give a simpler proof of the bound $\mathrm{T}(1/\epsilon^c)$.

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Core Construction (special case)

Henceforth:

- **L** and **R** are vertex classes,
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- The partitions' orders can grow arbitrarily fast
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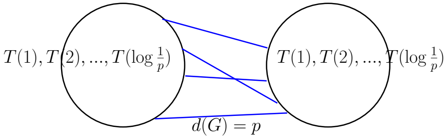
- The graph's property is one sided.

Core Construction ⇒ Graph Lower Bound

To prove our graph $\langle\delta\rangle$-regularity lower bound from Core Construction, put 4 copies along a 4-cycle.



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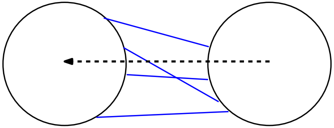

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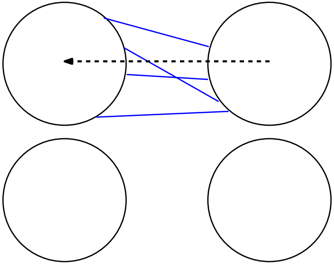

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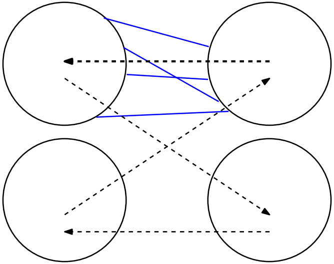

$$|\mathcal{L}_i| = |\mathcal{R}_{i+1}|$$

Core Construction (general case)

Theorem (Core Construction)

$\exists$ *equipartitions* $\mathcal{G}_1 \succ \cdots \succ \mathcal{G}_s$ *of* $\mathbf{L} \times \mathbf{R}$ *with* $|\mathcal{G}_j| = 2^j$ *such that* $\forall G \in \mathcal{G}_j$ $\forall \langle 2^{-28} \rangle$*-regular partition* $(\mathcal{L}, \mathcal{R})$ *of* G *we have:*

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- Had Core Construction held without the one-sided assumption then one would have been able to prove wowzer-type LB for graph $\langle\delta\rangle$-regularity and thus also for Szemerédi's regularity lemma.
- In other words, if one wishes to have a construction that holds with arbitrarily fast growing orders, then one has to introduce one-sidedness.

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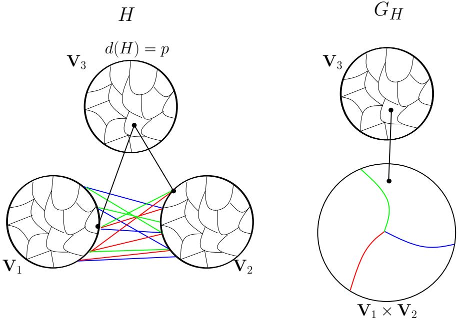
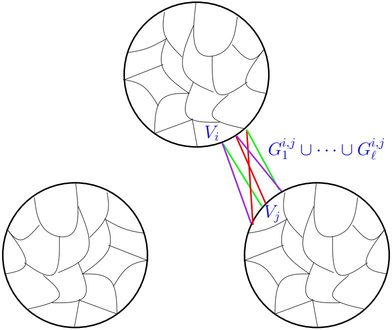


The Definition of 3-Graph $\langle\delta\rangle$-Regularity

A 2-partition $\mathcal{P}$ consists of a vertex equipartition $V_1, \dots, V_t$, and an edge equipartition $K[V_i, V_j] = G_1^{i,j} \cup \dots \cup G_\ell^{i,j}$ $(\forall i \neq j)$.



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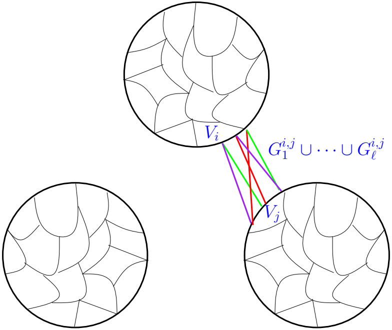

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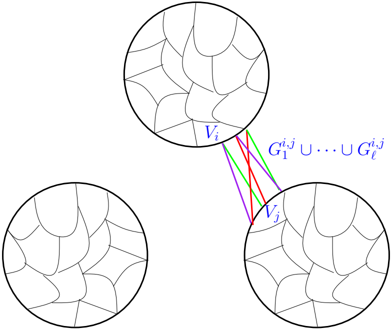

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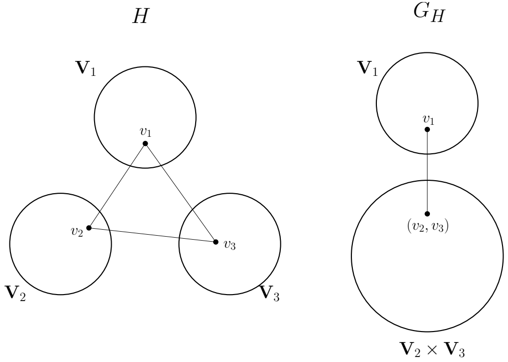
Definition ($\langle\delta\rangle$-good partition)

A 2-partition is $\langle\delta\rangle$-good if every bipartite graph $G_\ell^{i,j}$ is $\langle\delta\rangle$-regular.

Definition (The auxiliary graph G_H)

Let H be a 3-partite 3-graph H on $(\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3)$.
Define a bipartite graph $G_H = G_H(\mathbf{V}^1,\ \mathbf{V}^2 \times \mathbf{V}^3)$ on $(\mathbf{V}^1, \mathbf{V}^2 \times \mathbf{V}^3)$ by

$$E(G_H) = \left\{\, (v_1, (v_2, v_3)) \,\middle|\, (v_1, v_2, v_3) \in E(H) \,\right\} .$$



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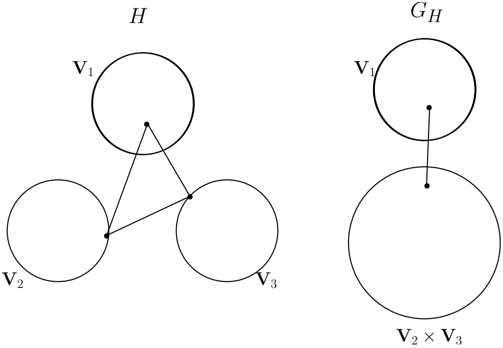

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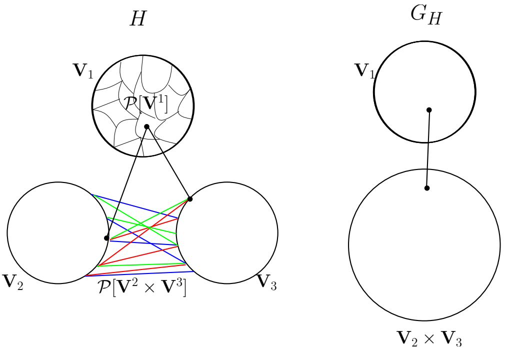

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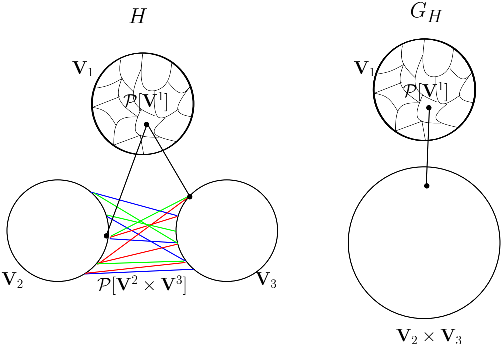

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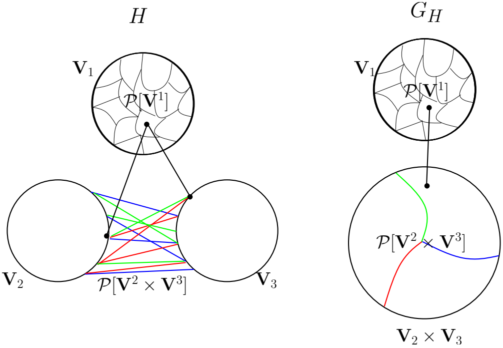

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3-graph $\langle\delta\rangle$-regularity - cont.

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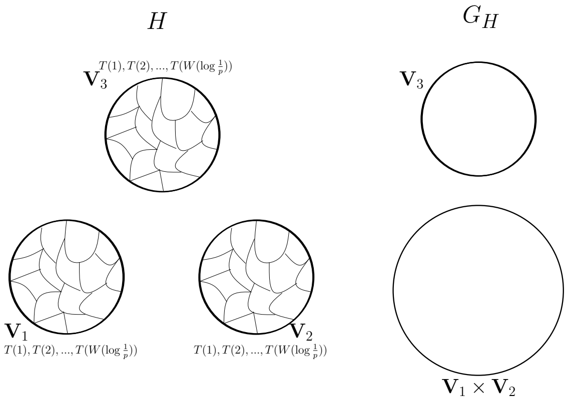
Our 3-graph Construction



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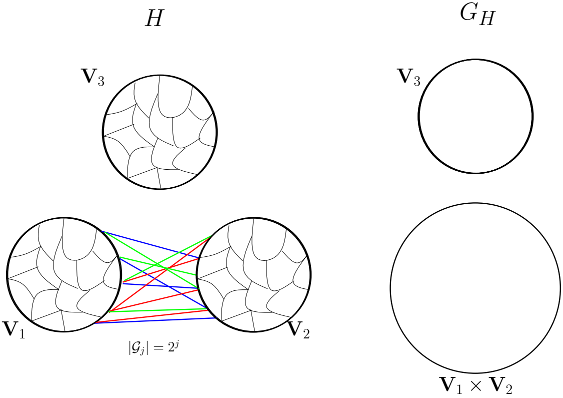

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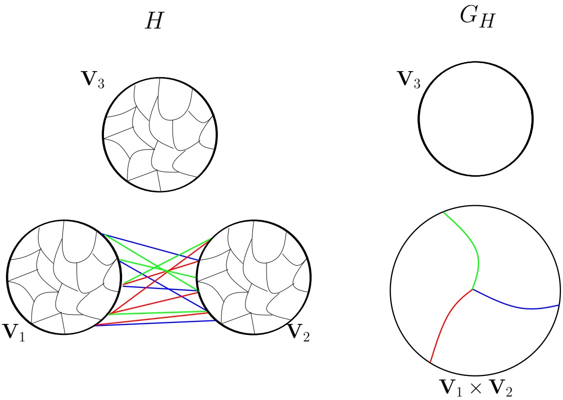

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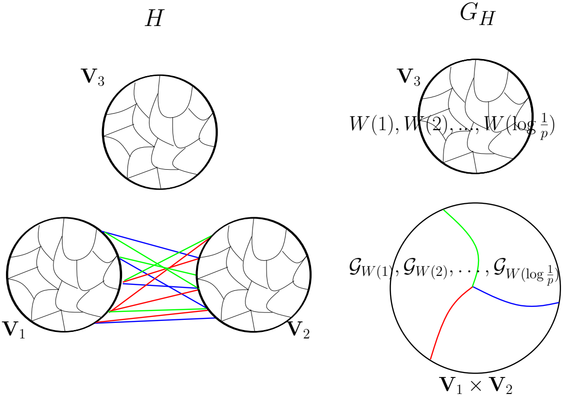

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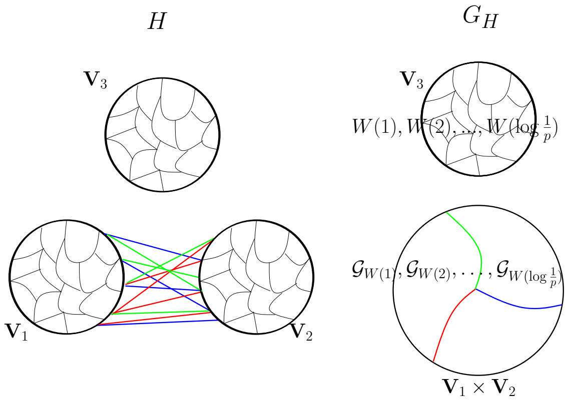

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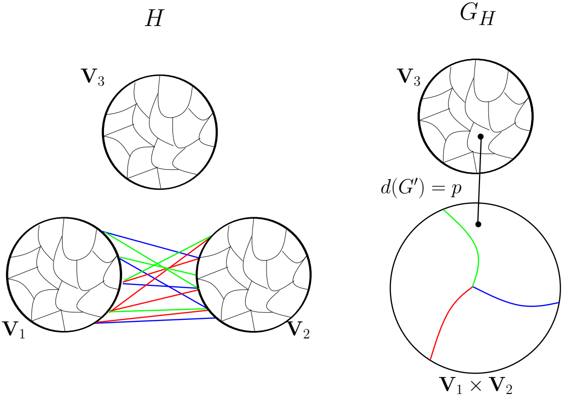

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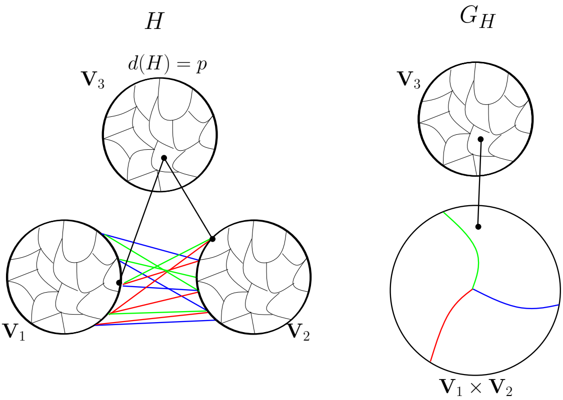

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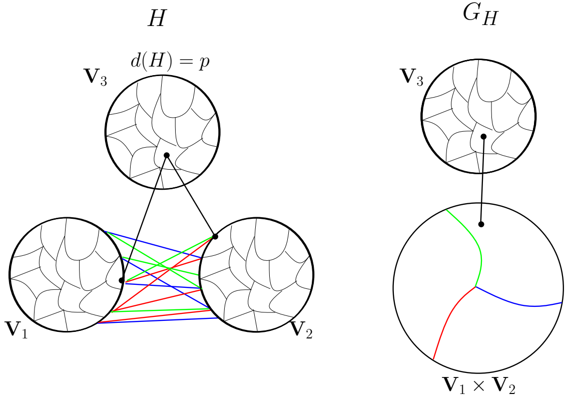

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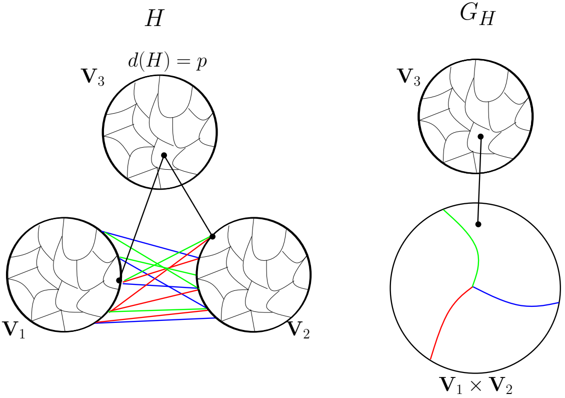

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Finally, take several copies of H along a (tight) 6-cycle.

Open Questions

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We now know that "k-graph SRAL" has an $\mathrm{Ack}_k(\Omega(\log \frac{1}{p}))$ lower bound.

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Open Question

Come up with a weaker notion than hypergraph regularity that has primitive recursive bounds and yet is useful.

Thank you!

The construction of our 3-graph

For $1 \leq j \leq 3$ fix canonical partitions $\mathcal{V}_1^j \succ \mathcal{V}_2^j \succ \cdots$ with $|\mathcal{V}_i^j| \approx \mathsf{T}(i)$.

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Main claim

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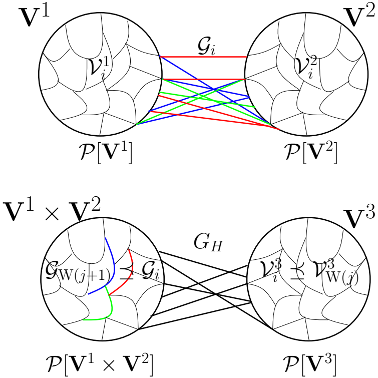

Lower Bounds Proof for Graph $\langle\delta\rangle$-Regularity

The construction uses the following graph operation:

Modified blow-up of a bipartite graph G:

- replace each vertex x of G by a set of $2^{\Omega(|V(G)|)}$ new vertices X
- replace each edge (u, v) with a bipartite graph on (U, V) as follows: letting $U' \subseteq U$ be a random half, replace (u, v) by $K(U', V)$.

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Intuition

If G has a "unique" regular partition then so does its modified blow-up.

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Application:

Theorem (Roth's Theorem, '53)

For every subset $A \subseteq [n] = \{1, 2, \ldots, n\}$,

$$|A| \geq \epsilon n \textit{ and } n \geq n_0(\epsilon) \quad \Rightarrow \quad A \textit{ contains a 3-AP.}$$

TRL $\Rightarrow$ Roth's Theorem

Theorem (Roth's Theorem)

$\forall A \subseteq [n]: \quad |A| \geq 0.01n \;\Rightarrow\;$ *A contains a* 3-*AP.*

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□

Best known bounds:

$$\epsilon^{\ln(1/\epsilon)} \leq \mathrm{Rem}(\epsilon) \leq \mathrm{T}(1/\epsilon)$$

$$n^{-1/\sqrt{\log n}} \leq r_3(n) \leq\approx (\log n)^{-1}$$