Transport in RMT

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IAS workshop, nov. 6 2013

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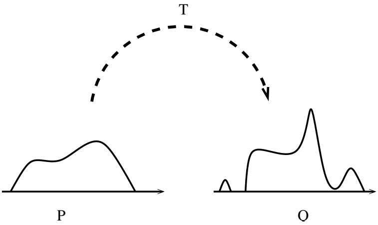
Outline

Transport in RMT

Transport

Let P, Q be two probability measures on $\mathbb{R}^d$ and $\mathbb{R}^{d'}$.
A transport map from P to Q is a measurable function $T : \mathbb{R}^d \to \mathbb{R}^{d'}$ so that for all bounded continuous function f

$$\int f(T(x))dP(x) = \int f(x)dQ(x).$$



That is $T(x)$ has law Q under P.

We denote $T\#P = Q$.

Fact (von Neumann [1932]) : If $P, Q \ll dx$, T exists.

Transport in the non-commutative setting

Non commutative laws are tracial states : $\tau : \mathbb{C}\langle X_1, \dots, X_d\rangle \to \mathbb{C}$

$$\tau(PP^*) \geq 0, \quad \tau(PQ) = \tau(QP), \quad \tau(I) = 1\,.$$

Here $(X_{i_1} \cdots X_{i_k})^* = X_{i_k} \cdots X_{i_1}$.

If τ, τ' are tracial states, can we build a transport map such that $F_1, \cdots, F_d$ so that $\tau = F\#\tau'$:

$$\tau(P(X_1, \dots, X_d)) = \tau'(P(F_1(X_1, \dots, X_d), \dots, F_d(X_1, \dots, X_d)))?$$

Examples of non-commutative laws

- Let $(X_1, \cdots, X_d)$ be d $N \times N$ Hermitian matrices,
$$\tau(P) := \frac{1}{N}\mathrm{Tr}\left(P(X_1, \cdots, X_d)\right).$$
Here $\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}$.
- Let $(X_1, \cdots, X_d)$ be d $N \times N$ Hermitian random matrices,
$$\tau(P) := \mathbb{E}[\frac{1}{N}\mathrm{Tr}\left(P(X_1, \cdots, X_d)\right)]$$
- Let $(X_1^N, \cdots, X_d^N)$ be d $N \times N$ Hermitian random matrices for $N \geq 0$ so that
$$\tau(P) := \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}\left(P(X_1^N, \cdots, X_d^N)\right)]$$
exists for all polynomial P.

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Results

Proofs

Perturbative non-commutative laws

Let $V \in \mathbb{C}\langle X_1, \ldots, X_d \rangle$ and set

$$\mathbb{P}_V^N(dX_1^N, \ldots, dX_d^N) = \frac{1}{Z_N} e^{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod 1_{\|X_i^N\| \le M} dX_i^N$$

Theorem (A-G –E. Maurel Segala (2006))

Let $M > 2$ be given and $W = W^*$. Let $V = \frac{1}{2}\sum X_i^2 + \epsilon W$. There exists $\epsilon(M, W) > 0$ so that for $|\epsilon| \le \epsilon(M, W)$ for any polynomial P

$$\tau_V(P) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}_V^N(X_1^N, \ldots, X_d^N)$$

Transport of non-commutative laws, perturbative case

$$\mathbb{P}_V^N(dX_1^N, \ldots, dX_d^N) = \frac{1}{Z_N} e^{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod 1_{\|X_i^N\| \le M} dX_i^N$$

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}^N_{\frac{1}{2}\sum X_i^2 + W}(X_1^N, \ldots, X_d^N)$$

Theorem (A-G –Shlyakhtenko (2012))

Let $M > 2$ and $W = W^*$. Let $\|P\| = \sum |\lambda_q(P)| 4^{deg(P)}$. There exists $\epsilon(M, W) > 0$ so that for $|\epsilon| \le \epsilon(M, W)$, there exists $(F, \tilde{F}) \in (\overline{\mathbb{C}\langle X_1, \ldots, X_d\rangle}^{\|.\|})^d$ so that

$$\tau_{\epsilon W} = F\#\tau_0 \qquad \tau_0 = \tilde{F}\#\tau_{\epsilon W}$$

Generalization to non perturbative setting

$$\mathbb{P}_V^N(dX_1^N, \ldots, dX_d^N) = \frac{1}{Z_N} e^{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod 1_{\|X_i^N\| \le M} dX_i^N$$

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}^N_{\frac{1}{2}\sum X_i^2 + W}(X_1^N, \ldots, X_d^N)$$

Theorem (WIP with Y-Dabrowski and D-Shlyakhtenko)

Assume that "$V = \frac{1}{2}\sum X_i^2 + W$ is strictly convex", then there exists $(F_i)_{1\le i\le d} \in (\overline{\mathbb{C}\langle X_1, \ldots, X_d\rangle})^d$ so that

$$\tau_W = F\#\tau_0$$

Application to transport of random matrices

Let $V=\sum X_i^2/2+\epsilon W$. Let $X^N=(X_1^N,\ldots,X_d^N)$ has law

$$\mathbb{P}_\epsilon^N(dX_1^N,\ldots,dX_d^N)=\frac{1}{Z_V^N}\exp\{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N)\}dX_1^N\cdots dX_d^N$$

Let $F^N:\mathbb{R}^{N^d}\to\mathbb{R}^{N^2d}$ be the (optimal) transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_0^N$. Then, if ϵ is small enough, there exists a function $F\in\overline{\mathbb{C}\langle X_1,\ldots,X_d\rangle}^{\|.\|}$ so that

$$\int\sum_{i,j=1}^N\sum_{k=1}^d|F^N(X)_k(i,j)-F(X_1^N,\ldots,X_d^N)_k(i,j)|^2dP_0^N(X_1^N,\ldots,X_d^N)$$

vanishes as N goes to infinity.

Transport for β-models

$$d\mathbb{P}_N^V(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i$$

$$\lim_{N\to\infty}\frac{1}{N}\sum f(\lambda_i) = \int f(x)d\mu_V(x)$$

Theorem (Bekerman–Figalli–G 2013)

Assume V, W $C^{31}(\mathbb{R})$, with equilibrium measures μ_V, μ_W with connected support. Assume V, W are non critical. Then there exists $T_0 : \mathbb{R}\to\mathbb{R}$ C^{19}, $T_1 : \mathbb{R}^N \to \mathbb{R}^N$ C^1 so that

$$\|(T_0^{\otimes N} + \frac{T_1}{N})\#\mathbb{P}_N^V - \mathbb{P}_N^W\|_{TV} \le const.\sqrt{\frac{\log N}{N}}\,.$$

Universality for β-models

$$d\mathbb{P}_N^V(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i$$

Assume that there are T_0, T_1 smooth so that

$$\|(T_0^{\otimes N}+\frac{T_1}{N})\#\mathbb{P}_N^V - \mathbb{P}_N^W\|_{TV} \le \text{const.}\sqrt{\frac{\log N}{N}},$$

so that $$\sup_{1\le k\le N}\|T_1^{N,k}\|_{L^1(\mathbb{P}_N^V)} + \sup_{k,k'}\frac{|T_1^{N,k}-T_1^{N,k'}|}{\sqrt{N}|\lambda_k-\lambda_{k'}|} \le C\log N.$$

Corollary

There is universality at the edges and in the bulk.

C.f Bourgade, Erdös, Yau [1104.2272, 1306.5728] and M. Shcherbina [1310.7835].

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Proofs

Idea of the proof : Monge-Ampère equation

Consider probability measures P, Q on $\mathbb{R}^d$ and assume they have smooth densities

$$P(dx) = e^{-V(x)}dx \qquad Q(dx) = e^{-W(x)}dx\,.$$

Then $T\#P = Q$ is equivalent to

$$\begin{aligned} \int f(T(x))e^{-V(x)}dx &= \int f(x)e^{-W(x)}dx \\ &= \int f(T(y))e^{-W(T(y))}JT(y)dy \end{aligned}$$

with JT the Jacobian of T. Hence, it is equivalent to the Monge-Ampère equation

$$-V(x) = -W(T(x)) + \log JT(x)\,.$$

Non-commutative perturbative setting : commutative analogue

$$d\mathbb{P}_N^V(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum f(\lambda_i) = \int f d\mu_V$$

with $\mu_V = F\#\sigma$ iff

$$\frac{\beta}{2}\int \log\frac{F(x)-F(y)}{x-y}d\sigma(y) = V(F(x)) - \frac{1}{2}x^2 .$$

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Compare with Monge-Ampère equation with transport $F^{\otimes N}$: $\mathbb{P}_V^{1/2x^2}$ a.s

$$\beta\sum_{i<j}\log\frac{F(\lambda_i)-F(\lambda_j)}{\lambda_i-\lambda_j}+\sum\log F'(\lambda_i)=N\sum V(F(\lambda_i))-\frac{1}{2}\sum\lambda_i^2$$

Non-commutative perturbative setting

Let $V = \sum X_i^2/2 + W$ and put

$$\mathbb{P}_W^N(dX_1^N, \ldots, dX_d^N) = \frac{1}{Z_V^N} \exp\{-N\mathrm{Tr}(V(X_1^N, \ldots, X_d^N))\} dX_1^N \cdots dX_d^N \,.$$

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}_V^N(X_1^N, \ldots, X_d^N)$$

with $\tau_W = F\#\tau_0$ iff, with JF the Jacobian of F,

$$(1\otimes\tau_0 + \tau_0\otimes 1)\mathrm{Tr}\log JF = \left\{\frac{1}{2}\sum F(X)_j^2 + W(F(X))\right\} - \frac{1}{2}\sum X_j^2$$

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with $\tau_W = F\#\tau_0$ iff, with JF the Jacobian of F,

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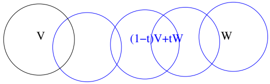
This equation has a unique solution $F_j = D_j G$ if W is small by a fixed point argument.

Non-perturbative setting : convex case

$\tau_0(dx) = e^{-V(x)}dx$



$\tau_1(dx) = e^{-W(x)}dx$

Non-perturbative setting : convex case

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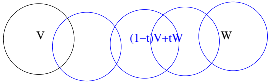

$\tau_1(dx) = e^{-W(x)}dx$

Define a flow $T_{s,t}$ so that $T_{s,t}\#\tau_{V_s} = \tau_{V_t}$, $V_t = (1-t)V + tW$,

$$T_{0,t} = T_{0,s} \circ T_{s,t} \,.$$

Non-perturbative setting : convex case

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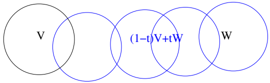

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$$T_{0,t} = T_{0,s} \circ T_{s,t} .$$

$\phi_t = \lim_{s\to t} T_{s,t} = \partial_t T_{0,t} \circ T_{0,t}^{-1}$ satisfies if $\phi_t = \nabla\psi_t$

$$L_t\psi_t = W - V$$

$L_t = \Delta - \nabla V_t.\nabla$ infinitesimal generator.

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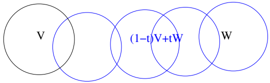

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$$L_t\psi_t = W - V$$

$L_t = \Delta - \nabla V_t.\nabla$ infinitesimal generator. $P_s^{V_t} = e^{sL_t}$,

$$\psi_t = \int_0^\infty [P_s^{V_t}(W-V) - \frac{1}{Z}\int (W-V)e^{-(1-t)V-tW}dx]ds \,.$$

One matrix case and approximate transport

$$d\mathbb{P}_N^V(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i$$

Find $T_t^N : \mathbb{R}^N \to \mathbb{R}^N$ "nice"

$$\sup_{t\in[0,1]} \| T_t^N \# \mathbb{P}_N^V - \mathbb{P}_N^{V_t} \|_{TV} \to 0 \qquad V_t = (1-t)V + tW$$

One matrix case and approximate transport

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Aim : Build ψ_t^N, $\partial_t T_t^N \circ (T_t^N)^{-1} = \nabla \psi_t^N$ so that

$$R_t^N(\psi) = L_t \psi_t^N - (V-W)$$

goes to zero in $L^1(\mathbb{P}_N^V)$. Then T_t^N solution of $\partial_t T_t^N = \nabla\psi_t^N(T_t^N)$ is an approximate transport. L_t the infinitesimal generator of Dyson BM in potential V_t.

One matrix case and approximate transport

Find

$$\psi_t^N(\lambda) = \sum_i [\psi_{0,t}(\lambda_i) + \frac{1}{N}\psi_{1,t}(\lambda_i)] + \frac{1}{N}\sum \psi_{2,t}(\lambda_i, \lambda_j)$$

so that

$$R_t^N(\psi) = L_t \psi_t^N - (V - W)$$

goes to zero in $L^1(\mathbb{P}_N^V)$.

One matrix case and approximate transport

Find

$$\psi_t^N(\lambda) = \sum_i [\psi_{0,t}(\lambda_i) + \frac{1}{N}\psi_{1,t}(\lambda_i)] + \frac{1}{N}\sum \psi_{2,t}(\lambda_i, \lambda_j)$$

so that

$$R_t^N(\psi) = L_t \psi_t^N - (V - W)$$

goes to zero in $L^1(\mathbb{P}_N^V)$. We find with $M_N = \sum(\delta_{\lambda_i} - \mu_{V_t})$

$$\begin{aligned} R_t^N &= N \int [\Xi \psi'_{0,t} + W - V](x) dM_N(x) + \cdots \\ & -\frac{\beta}{2N} \iint \frac{\psi'_{1,t}(x) - \psi'_{1,t}(y)}{x - y} dM_N(x) dM_N(y) + \ldots \end{aligned}$$

with $\quad \Xi f(x) = V_t'(x) f(x) - \beta \int \frac{f(x) - f(y)}{x - y} d\mu_{V_t}(y),$

One matrix case and approximate transport

Find

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$$\text{with} \quad \Xi f(x) = V'_t(x) f(x) - \beta \int \frac{f(x) - f(y)}{x - y} d\mu_{V_t}(y),$$

Ξ is invertible, with inverse $\Xi^{-1} f$ C^{r-1} if f C^r. Choose $\psi_{0,t}, \psi_{1,t}, \psi_{2,t}$ so that the first line vanishes, show the second is neglectable.

Open problems

- How far can we push this type of arguments to obtain isomorphisms classes for von Neumann algebras ?
- The local fluctuations for several matrix models should be the same as those of some $P(X_1, \ldots, X_n)$, X_i independent GUE. Can we prove local fluctuations for $P(X_1, \ldots, X_n)$, X_i independent GUE (e.g when P is the gradient of a convex function) ?
- The transport method is quite robust and should be “easily” adapted to other “one-matrix models”.