Geometric Complexity Theory via Algebraic Combinatorics

Greta Panova

University of Pennsylvania
Institute for Advanced Study Princeton

IAS, CSDM Seminar

(Boolean) Complexity

Input: string of n bits, i.e. size($input$) $= n$.

Decision problems:

Is there an object, s.t.... ?

P $=$ solution can be found in time Poly(n)

NP $=$ solution can be *verified* in Poly(n) (polynomial witness)

NP **–Complete** $=$ in NP , and every NP problem can be reduced to it poly time; e.g.

Counting problems:

Compute $F(input) =?$

FP $=$ solution can be found in time Poly(n)

#P $=$ NP counting analogue; informally – $F(input)$ counts Exp-many objects, whose verification is in P .

(Boolean) Complexity

Input: string of n bits, i.e. size$(input) = n$.

Decision problems:

Is there an object, s.t.... ?

P = solution can be found in time Poly(n)

NP = solution can be *verified* in Poly(n) (polynomial witness)

NP **–Complete** = in NP , and every NP problem can be reduced to it poly time;

Counting problems:

Compute $F(input) = ?$

FP = solution can be found in time Poly(n)

#P = NP counting analogue; informally – $F(input)$ counts Exp-many objects, whose verification is in P .

The P vs NP Problem:

Is P = NP? Algebraic version: is VP = VNP?

(Boolean) Complexity

Input: string of n bits, i.e. $\text{size}(\mathit{input}) = n$.

Decision problems:

Is there an object, s.t.... ?

P = solution can be found in time Poly(n)

NP = solution can be *verified* in Poly(n) (polynomial witness)

NP **–Complete** = in NP , and every NP problem can be reduced to it poly time;

Counting problems:

Compute $F(\mathit{input}) =?$

FP = solution can be found in time Poly(n)

#P = NP counting analogue; informally – $F(\mathit{input})$ counts Exp-many objects, whose verification is in P .

The P vs NP Problem:

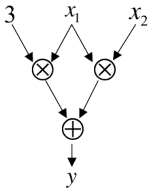
Is P = NP? Algebraic version: is VP = VNP?

An approach [Mulmuley, Sohoni]: **Geometric Complexity Theory**

VP vs VNP: determinant vs permanent

Arithmetic Circuits:

$y = 3x_1 + x_1x_2$



Polynomials $f_n \in \mathbb{F}[X_1, \ldots, X_n]$. Circuit – nodes are $+, \times$ gates, input – $X_1, \ldots, X_n$ and constants from $\mathbb{F}$.

Class VP (Valliant's P):
polynomials that can be computed with *poly*(n) large circuit (size of the associated graph).

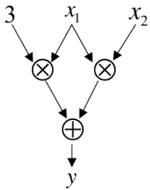
Class VNP:
the class of polynomials f_n, s.t. $\exists g_n \in$ VP with

$$f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).$$

VP vs VNP: determinant vs permanent

Arithmetic Circuits:

$$y = 3x_1 + x_1x_2$$



Polynomials $f_n \in \mathbb{F}[X_1, \ldots, X_n]$. Circuit – nodes are $+, \times$ gates, input – $X_1, \ldots, X_n$ and constants from $\mathbb{F}$.

Class VP (Valliant's P):
polynomials that can be computed with *poly*(n) large circuit (size of the associated graph).

Class VNP:
the class of polynomials f_n, s.t. $\exists g_n \in \mathrm{VP}$ with

$$f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).$$

Theorem[Bürgisser]: If VP = VNP, then P = NP if $\mathbb{F}$ - finite or the Generalized Riemann Hypothesis holds.

VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial p in any number of variables there exists some n such that

$$p = \det(A),$$

where A is an $n \times n$ matrix whose entries are affine linear polynomials.
The smallest n possible is called the *determinantal complexity* $\mathrm{dc}(p)$.
Example: $p = x_1^2 + x_1x_2 + x_2x_3 + 2x_1$, then

$$p = \det \begin{bmatrix} x_1 + 2 & x_2 \\ -x_3 + 2 & x_1 + x_2 \end{bmatrix}, \qquad \mathrm{dc}(p) = 2$$

VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial p in any number of variables there exists some n such that

$$p = \det(A),$$

where A is an $n \times n$ matrix whose entries are affine linear polynomials. The smallest n possible is called the *determinantal complexity* $\mathrm{dc}(p)$.

The permanent:

$$\mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} X_{i,\sigma(i)}.$$

Theorem:[Valiant] per_m is VNP-complete.

Conjecture (Valiant, VP $\neq$ VNP equivalent)

$\mathrm{dc}(\mathrm{per}_m)$ *grows superpolynomially.*

VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial p in any number of variables there exists some n such that

$$p = \det(A),$$

where A is an $n \times n$ matrix whose entries are affine linear polynomials. The smallest n possible is called the *determinantal complexity* $\mathrm{dc}(p)$.

The permanent:

$$\mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} X_{i,\sigma(i)}.$$

Theorem:[Valiant] per_m is VNP-complete.

Conjecture (Valiant, VP $\neq$ VNP equivalent)

$\mathrm{dc}(\mathrm{per}_m)$ *grows superpolynomially.*

Known: $\mathrm{dc}(\mathrm{per}_m) \leq 2^m - 1$ (Grenet 2011), $\mathrm{dc}(\mathrm{per}_m) \geq \frac{m^2}{2}$ (Mignon, Ressayre, 2004). Ryser's formula:

$$\mathrm{per}_m(X) = (-1)^m \sum_{S \subset [1..m]} (-1)^{|S|} \prod_{i=1}^{m} (\sum_{j \in S} X_{i,j})$$

Geometric Complexity Theory

GL_N action on polynomials: $A \in GL_N(\mathbb{C})$, $v := (X_1, \ldots, X_N)$,
$f \in \mathbb{C}[X_1, \ldots, X_N]$, then $A.f = f(A^{-1}v)$
(replaces variables with linear forms)

$GL_{n^2} \det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – **determinant orbit**.

$\Omega_n := \overline{GL_{n^2} \det_n}$ - **determinant orbit closure**.

$\mathrm{per}^n_m := (X_{1,1})^{n-m}\mathrm{per}_m$ – **the padded permanent**.

Geometric Complexity Theory

GL_N action on polynomials: $A \in GL_N(\mathbb{C})$, $v := (X_1, \ldots, X_N)$,
$f \in \mathbb{C}[X_1, \ldots, X_N]$, then $A.f = f(A^{-1}v)$
(replaces variables with linear forms)

$GL_{n^2} \det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – **determinant orbit**.

$\Omega_n := \overline{GL_{n^2} \det_n}$ - **determinant orbit closure**.

$\mathrm{per}_m^n := (X_{1,1})^{n-m}\mathrm{per}_m$ – **the padded permanent**.

Proposition (Lower bounds via geometry)

If $\mathrm{per}_m^n \notin \overline{GL_{n^2} \det_n}$, *then* $\mathrm{dc}(\mathrm{per}_m) > n$.

Geometric Complexity Theory

GL_N action on polynomials: $A \in GL_N(\mathbb{C})$, $v := (X_1, \ldots, X_N)$, $f \in \mathbb{C}[X_1, \ldots, X_N]$, then $A.f = f(A^{-1}v)$
(replaces variables with linear forms)

$GL_{n^2} \det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – **determinant orbit**.

$\Omega_n := \overline{GL_{n^2} \det_n}$ - **determinant orbit closure**.

$\mathrm{per}_m^n := (X_{1,1})^{n-m}\mathrm{per}_m$ – **the padded permanent**.

Proposition (Lower bounds via geometry)

If $\mathrm{per}_m^n \notin \overline{GL_{n^2} \det_n}$, *then* $\mathrm{dc}(\mathrm{per}_m) > n$.

Conjecture (GCT: Mulmuley and Sohoni)

$\max\{n : \mathrm{per}_m^n \notin \overline{GL_{n^2} \det_n}\}(\leq \mathrm{dc}(\mathrm{per}_m))$ *grows superpolynomially.*

$$\mathrm{per}_m^n \in \overline{GL_{n^2} \det_n} \iff \underbrace{\overline{GL_{n^2}\mathrm{per}_m^n}}_{=:\Gamma_m^n} \subseteq \underbrace{\overline{GL_{n^2}\det_n}}_{\Omega_n}.$$

Geometric Complexity Theory

Proposition (Lower bounds via geometry)

If $\mathrm{per}_m^n \notin \overline{GL_{n^2}\,\mathrm{det}_n}$, *then* $\mathrm{dc}(\mathrm{per}_m) > n$.

Conjecture (GCT: Mulmuley and Sohoni)

$\max\{n : \mathrm{per}_m^n \notin \overline{GL_{n^2}\,\mathrm{det}_n}\}(\leq \mathrm{dc}(\mathrm{per}_m))$ *grows superpolynomially.*

$$\mathrm{per}_m^n \in \overline{GL_{n^2}\,\mathrm{det}_n} \iff \underbrace{\overline{GL_{n^2}\mathrm{per}_m^n}}_{=:\Gamma_m^n} \subseteq \underbrace{\overline{GL_{n^2}\mathrm{det}_n}}_{\Omega_n}.$$

Geometric Complexity Theory

Proposition (Lower bounds via geometry)

If $\mathrm{per}_m^n \notin \overline{GL_{n^2} \det_n}$, *then* $\mathrm{dc}(\mathrm{per}_m) > n$.

Conjecture (GCT: Mulmuley and Sohoni)

$\max\{n : \mathrm{per}_m^n \notin \overline{GL_{n^2} \det_n}\} (\leq \mathrm{dc}(\mathrm{per}_m))$ *grows superpolynomially.*

$$\mathrm{per}_m^n \in \overline{GL_{n^2} \det_n} \iff \underbrace{\overline{GL_{n^2} \mathrm{per}_m^n}}_{=:\Gamma_m^n} \subseteq \underbrace{\overline{GL_{n^2} \det_n}}_{\Omega_n}.$$

Exploit the symmetry! Coordinate rings as GL_{n^2} representations:

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

Definition (Representation theoretic obstruction)

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction**.
Its existence shows $\overline{GL_{n^2}\mathrm{per}_m^n} \not\subseteq \overline{GL_{n^2} \det_n}$ and so $\mathrm{dc}(\mathrm{per}_m) > n$!

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

Conjecture (GCT: Mulmuley-Sohoni)

There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

Conjecture (GCT: Mulmuley-Sohoni)

There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

If also $\delta_{\lambda,d,n} = 0$, then λ is an **occurrence obstruction**.

Conjecture (Mulmuley and Sohoni)

There exist occurrence obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda\vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

Conjecture (GCT: Mulmuley-Sohoni)

There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

If also $\delta_{\lambda,d,n} = 0$, then λ is an **occurrence obstruction**.

Conjecture (Mulmuley and Sohoni)

There exist occurrence obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

Theorem (Bürgisser-Ikenmeyer-P(FOCS 2016))

This Conjecture is false.

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.

Question: What are these $\delta_{\lambda,d,n}$ and $\gamma_{\lambda,d,n,m}$???

(Non)existence of obstructions

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathrm{VP} \neq \mathrm{VNP}$.
Question: What are these $\delta_{\lambda,d,n}$ and $\gamma_{\lambda,d,n,m}$???
Kronecker coefficients of the Symmetric Group:

$$\delta_{\lambda,d,n} \leq sk(\lambda, n^d) \leq g(\lambda, n^d, n^d)$$

(Symmetric Kronecker:
$sk(\lambda,\mu) := \dim \mathrm{Hom}_{S_{|\lambda|}}(\mathbb{S}^\lambda, S^2(\mathbb{S}^\mu)) = mult_\lambda \mathbb{C}[GL_{n^2}\det_n]_d)$
Plethysm coefficients: of GL.

$$a_\lambda(d[n]) := mult_\lambda Sym^d(Sym^n(V)) \geq \gamma_{\lambda,d,n,m}.$$

Problem (GCT program, "easy version")

Find λ, such that the $sk(\lambda,(n^d)) < a_\lambda(d[n])$?

Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2} \det_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}] = 0$.

Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$ *, for infinitely many* m*, there exists a partition* λ *occurring in* $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ *but not in* $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$*, where* $n = m^c$*.*

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$*. If* $g(\lambda, n^d, n^d) = 0$ *(so* $mult_\lambda \mathbb{C}[GL_{n^2} \det_n] = 0$*), then* $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}]) = 0$*.*

Theorem (Bürgisser-Ikenmeyer-P (FOCS'16))

Let n, d, m *be positive integers with* $n \geq m^{25}$ *and* $\lambda \vdash nd$*. If* λ *occurs in* $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$*, then* λ *also occurs in* $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$*. In particular, the Conjecture is false, there are no "occurrence obstructions".*

Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$

Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

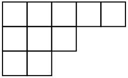
are the **Specht modules** $\mathbb{S}_\lambda$, indexed by

integer partitions $\lambda \vdash n$:

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,

$\lambda_1 + \lambda_2 + \cdots = n$, length $\ell(\lambda) = \ell$ ($=$ number of nonzero parts)

Young diagram of λ: 

($\lambda = (5, 3, 2)$, $\ell(\lambda) = 3$, $n = |\lambda| = 5 + 3 + 2 = 10$).

Basis for $\mathbb{S}_\lambda$: Standard **Y**oung **T**ableaux of shape λ:

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n}(\ldots\ldots\ldots)\mathbb{S}_\nu$$

Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} g(\lambda, \mu, \nu) \mathbb{S}_\nu$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} g(\lambda, \mu, \nu) \mathbb{S}_\nu$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$g(\lambda, \mu, \nu) = \dim \mathrm{Hom}_{S_n}(\mathbb{S}_\nu, \mathbb{S}_\lambda \otimes \mathbb{S}_\mu)$$

In terms of $GL(\mathbb{C}^m)$ modules V_λ, V_μ, V_ν

$$\mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \oplus_{\lambda,\mu,\nu} g(\lambda, \mu, \nu) V_\lambda \otimes V_\mu \otimes V_\nu$$

A bit of history

1873: Lie groups, *Lie, Klein....*

1896: Representations of finite groups, *Frobenius ...*

1923: Representations of Lie groups, *H. Weyl*. Quantum mechanics, *von Neumann*

1934: Tensor products of irreducible representations of Lie groups:
V_λ – irreducible representation of $GL_N(\mathbb{C})$.

$$V_\lambda \otimes V_\mu = \oplus_\nu c^\nu_{\lambda\mu} V_\nu$$

$c^\nu_{\lambda\mu}$ – Littlewood-Richardson coefficients.

A bit of history

1873: Lie groups, *Lie, Klein....*

1896: Representations of finite groups, *Frobenius ...*

1923: Representations of Lie groups, *H. Weyl*. Quantum mechanics, *von Neumann*

1934: Tensor products of irreducible representations of Lie groups: V_λ – irreducible representation of $GL_N(\mathbb{C})$.

$$V_\lambda \otimes V_\mu = \oplus_\nu c^\nu_{\lambda\mu} V_\nu$$

$c^\nu_{\lambda\mu}$ – Littlewood-Richardson coefficients.

Theorem (Littlewood-Richardson, 1934)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

A bit of history

1873: Lie groups, *Lie, Klein....*

1896: Representations of finite groups, *Frobenius ...*

1923: Representations of Lie groups, *H. Weyl*. Quantum mechanics, *von Neumann*

1934: Tensor products of irreducible representations of Lie groups:
V_λ – irreducible representation of $GL_N(\mathbb{C})$.

$$V_\lambda \otimes V_\mu = \oplus_\nu c^\nu_{\lambda\mu} V_\nu$$

$c^\nu_{\lambda\mu}$ – Littlewood-Richardson coefficients.

Theorem (Littlewood-Richardson, 1934)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

(LR tableaux of shape $(7,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(7,4,3)}_{(3,1)(4,3,2)} = 2$)

A bit of history

1873: Lie groups, *Lie, Klein....*

1896: Representations of finite groups, *Frobenius ...*

1923: Representations of Lie groups, *H. Weyl*. Quantum mechanics, *von Neumann*

1934: Tensor products of irreducible representations of Lie groups:
V_λ – irreducible representation of $GL_N(\mathbb{C})$.

$$V_\lambda \otimes V_\mu = \oplus_\nu c^\nu_{\lambda\mu} V_\nu$$

$c^\nu_{\lambda\mu}$ – Littlewood-Richardson coefficients.

Theorem (Littlewood-Richardson, 1934)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

1938: Tensor product of irreducible representations of S_n, Kronecker coefficients, *Murnaghan*:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} g(\lambda, \mu, \nu) \mathbb{S}_\nu$$

The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$*, i.e. a family of combinatorial objects* $\mathcal{O}_{\lambda,\mu,\nu}$*, s.t.* $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$*.*

The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in #P .

Classical motivation: (Littlewood–Richardson: for $c^{\nu}_{\lambda,\mu}$, $\mathcal{O}_{\lambda,\mu,\nu} = \{$ LR tableaux of shape ν/μ, type $\lambda\}$)

Theorem (Murnaghan)

If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^{\nu}_{\lambda\mu}.$$

Modern motivation:

1. A positive combinatorial formula " $\iff$ " Computing Kronecker coefficients is in #P .
2. **Geometric Complexity Theory.**
3. Invariant Theory, moment polytopes [see Bürgisser, Christandl,Mulmuley, Walter, Oliveira, Garg, Wigerson etc]

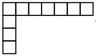
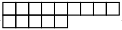
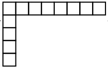
The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda,\mu,\nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda,\mu,\nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in $\#$P .

Results since then:

Combinatorial formulas for $g(\lambda,\mu,\nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n-k,k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k,k)$, $\lambda = (n-r,r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n-k,1^k)$ (), [*Blasiak, 2012*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].

The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in #P .

Bounds and positivity:

[Pak-P]: $g(\lambda, \mu, \mu) \geq |\chi^\lambda(2\mu_1 - 1, 2\mu_2 - 3, \ldots)$ when $\mu = \mu^T$.
Corollaries: $g(\lambda, \mu, \mu) > c\frac{2^{\sqrt{2k}}}{k^{9/4}}$ for $\lambda = (|\mu| - k, k)$, and $diag(\mu) \geq \sqrt{k}$.

Complexity results:

[Bürgisser-Ikenmeyer]: KRON is in GapP.
(Littlewood-Richardson, i.e. KRON's special case, is #P -complete)

[Pak-P]: If ν is a hook, then KronPositivity is in P. If λ, μ, ν have fixed length there exists a linear time algorithm for deciding $g(\lambda, \mu, \nu) > 0$.

[Ikenmeyer-Mulmuley-Walter]: KronPositivity is NP -hard.

[Bürgisser-Christandl-Mulmuley-Walter]: membership in the moment polytope is NP and coNP .

Back to GCT: Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Back to GCT: Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[\overline{GL_{n^2} \det_n}] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}]) = 0$.

Back to GCT: Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2} \det_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}]) = 0$.

Theorem (Bürgisser-Ikenmeyer-P (FOCS'16))

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$, then λ also occurs in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$. In particular, the Conjecture is false, there are no "occurrence obstructions".

No occurrence obstructions I: positive Kroneckers

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n \times d, n \times d) = 0$, then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m)}] = 0$.

Proof:

$\bar{\lambda} := (\lambda_2, \lambda_3, \ldots) \vdash |\lambda| - \lambda_1$

Theorem (Kadish-Landsberg)

If $mult_\lambda \mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}] > 0$, then $|\bar{\lambda}| \leq md$ and $\ell(\lambda) \leq m^2$.

Theorem (Degree lower bound, [IP])

If $|\bar{\lambda}| \leq md$ with $a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.

No occurrence obstructions I: positive Kroneckers

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n \times d, n \times d) = 0$, then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}]) = 0$.

Proof:
$\bar{\lambda} := (\lambda_2, \lambda_3, \ldots) \vdash |\lambda| - \lambda_1$

Theorem (Kadish-Landsberg)

If $mult_\lambda \mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}] > 0$, then $|\bar{\lambda}| \leq md$ and $\ell(\lambda) \leq m^2$.

Theorem (Degree lower bound, [IP])

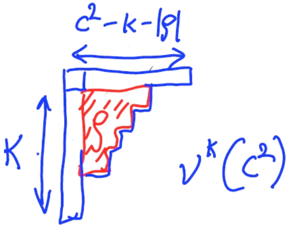
If $|\bar{\lambda}| \leq md$ with $a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.

Theorem (Kronecker positivity, [IP])

Let $\lambda \vdash dn$. Let $\mathcal{X} := \{(1), (2 \times 1), (4 \times 1), (6 \times 1), (2, 1), (3, 1)\}$.
(a) If $\bar{\lambda} \in \mathcal{X}$, then $a_\lambda(d[n]) = 0$.
(b) If $\bar{\lambda} \notin \mathcal{X}$ and $m \geq 3$ such that $\ell(\lambda) \leq m^2$, $|\bar{\lambda}| \leq md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$.

Kronecker positivity I: hook-like λs

Proposition (Ikenmeyer-P)

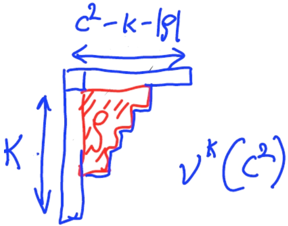


If there is an a, such that $g(\nu^k(a^2), a \times a, a \times a) > 0$ *for all* k*, s.t.* $k \notin H^1(\rho)$ *and* $a^2 - k \notin H^2(\rho)$ *for some sets* $H^1(\rho), H^2(\rho) \subset [\ell, 2a+1]$*,*
then $g(\nu^k(b^2), b \times b, b \times b) > 0$ *for all* k*, s.t.* $k \notin H^1(\rho)$ *and* $b^2 - k \notin H^2(\rho)$ *for all* $b \geq a$*.*

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{k : g(\nu^k(c^2), c \times c, c \times c) > 0\}$, we have
Claim: Suppose that $k \in P_c$, then $k, k + 2c + 1 \in P_{c+1}$.

Kronecker positivity I: hook-like λs

Proposition (Ikenmeyer-P)



If there is an a, such that $g(\nu^k(a^2), a \times a, a \times a) > 0$ *for all* k, *s.t.* $k \notin H^1(\rho)$ *and* $a^2 - k \notin H^2(\rho)$ *for some sets* $H^1(\rho), H^2(\rho) \subset [\ell, 2a+1]$, *then* $g(\nu^k(b^2), b \times b, b \times b) > 0$ *for all* k, *s.t.* $k \notin H^1(\rho)$ *and* $b^2 - k \notin H^2(\rho)$ *for all* $b \geq a$.

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{k : g(\nu^k(c^2), c \times c, c \times c) > 0\}$, we have
Claim: Suppose that $k \in P_c$, then $k, k + 2c + 1 \in P_{c+1}$.

Corollary

We have that $g(\lambda, h \times w, h \times w) > 0$ *for* $\lambda = (hw - j - |\rho|, 1^j + \rho)$ *for most "small" partitions* ρ *and all but finitely many values of* j.

Kronecker positivity II: squares, and decompositions

Theorem (Ikenmeyer-P)

Let $\nu \notin \mathcal{X}$ and $\ell = max(\ell(\nu)+1, 9)$, $a > 3\ell^{3/2}$, $b \geq 3\ell^2$ and $|\nu| \leq ab/6$. Then $g(\nu(ab), a \times b, a \times b) > 0$.

Proof sketch: decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k-1) \times k) + \sum_{k=2}^{\ell} y_k((k-1) \times 2).$$

Kronecker positivity II: squares, and decompositions

Theorem (Ikenmeyer-P)

Let $\nu \notin \mathcal{X}$ and $\ell = max(\ell(\nu)+1, 9)$, $a > 3\ell^{3/2}$, $b \geq 3\ell^2$ and $|\nu| \leq ab/6$. Then $g(\nu(ab), a \times b, a \times b) > 0$.

Proof sketch: decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k-1)\times k) + \sum_{k=2}^{\ell} y_k((k-1)\times 2).$$

Crucial facts:

- $g(k \times k, k \times k, k \times k) > 0$ [Bessenrodt-Behns].
- Transpositions: $g(\alpha, \beta, \gamma) = g(\alpha, \beta^T, \gamma^T)$ (with $\beta = \gamma = wxh$)
- Hooks and exceptional cases: $g(\lambda, h \times w, h \times w) > 0$ for all $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for $|\rho| \leq 6$ and almost all js.
- Semigroup property for positive triples: $g(\alpha^1+\alpha^2, \beta^1+\beta^2, \gamma^1+\gamma^2) \geq \max(g(\alpha^1,\beta^1,\gamma^1), g(\alpha^2,\beta^2,\gamma^2))$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d)$, as $n \to \infty$.
$\mathrm{St}^1(\rho) := \{(n, d) \mid g((nd - |\rho|, \rho), n \times d, n \times d)\} = a_\rho(d)$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd-|\rho|,\rho), n\times d, n\times d) = a_\rho(d)$, as $n\to\infty$.
$\mathrm{St}^1(\rho) := \{(n,d) \mid g((nd-|\rho|,\rho), n\times d, n\times d)\} = a_\rho(d)$.

Proposition (Ikenmeyer-P)

Fix ρ, and let $(n,d)\in St^1(\rho)$, which is true in particular if $n\geq|\rho|$. Let $\lambda=(nd-|\rho|,\rho)$. Then $g(\lambda, n\times d, n\times d)\geq a_\lambda(d[n])$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd-|\rho|,\rho), n\times d, n\times d) = a_\rho(d)$, as $n\to\infty$.
$\mathrm{St}^1(\rho) := \{(n,d) \mid g((nd-|\rho|,\rho), n\times d, n\times d)\} = a_\rho(d)$.

Proposition (Ikenmeyer-P)

Fix ρ, and let $(n,d)\in St^1(\rho)$, which is true in particular if $n\geq|\rho|$. Let $\lambda=(nd-|\rho|,\rho)$. Then $g(\lambda, n\times d, n\times d)\geq a_\lambda(d[n])$.

Proof: $\lambda=\mu+d(n-m)$. Suppose $g(\lambda,n\times d,n\times d)<a_\lambda(d[n])$:

KL'14: If $\mu\vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m}])\geq a_\mu(d[m])$, where $V_m := Sym^m\mathbb{C}^{m^2}$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd-|\rho|,\rho), n\times d, n\times d) = a_\rho(d)$, as $n\to\infty$.
$\mathrm{St}^1(\rho) := \{(n,d) \mid g((nd-|\rho|,\rho), n\times d, n\times d)\} = a_\rho(d)$.

Proposition (Ikenmeyer-P)

Fix ρ, and let $(n,d)\in St^1(\rho)$, which is true in particular if $n\geq|\rho|$. Let $\lambda=(nd-|\rho|,\rho)$. Then $g(\lambda, n\times d, n\times d)\geq a_\lambda(d[n])$.

Proof: $\lambda=\mu+d(n-m)$. Suppose $g(\lambda,n\times d,n\times d)<a_\lambda(d[n])$:

KL'14: If $\mu\vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m)}] \geq a_\mu(d[m])$, where $V_m := Sym^m\mathbb{C}^{m^2}$.

Stability: $g(\lambda,n\times d,n\times d) = g(\mu,m\times d,m\times d)$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd-|\rho|,\rho), n\times d, n\times d) = a_\rho(d)$, as $n\to\infty$.
$\mathrm{St}^1(\rho) := \{(n,d) \mid g((nd-|\rho|,\rho), n\times d, n\times d)\} = a_\rho(d)$.

Proposition (Ikenmeyer-P)

Fix ρ, and let $(n,d)\in St^1(\rho)$, which is true in particular if $n\geq|\rho|$. Let $\lambda=(nd-|\rho|,\rho)$. Then $g(\lambda, n\times d, n\times d)\geq a_\lambda(d[n])$.

Proof: $\lambda = \mu + d(n-m)$. Suppose $g(\lambda, n\times d, n\times d) < a_\lambda(d[n])$:

KL'14: If $\mu \vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m)}] \geq a_\mu(d[m])$, where $V_m := Sym^m\mathbb{C}^{m^2}$.

Stability: $g(\lambda, n\times d, n\times d) = g(\mu, m\times d, m\times d)$.

GCT: If $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m)}] \geq g(\lambda, n\times d, n\times d)$ then $dc(f_m) > n$ for some $f_m\in V_m$.

Kronecker vs plethysm: inequality of multiplicities

Stability[Manivel]: $g((nd-|\rho|,\rho), n\times d, n\times d) = a_\rho(d)$, as $n\to\infty$.
$\mathrm{St}^1(\rho) := \{(n,d) \mid g((nd-|\rho|,\rho), n\times d, n\times d)\} = a_\rho(d)$.

Proposition (Ikenmeyer-P)

Fix ρ, and let $(n,d)\in St^1(\rho)$, which is true in particular if $n\geq|\rho|$. Let $\lambda=(nd-|\rho|,\rho)$. Then $g(\lambda, n\times d, n\times d)\geq a_\lambda(d[n])$.

Proof: $\lambda = \mu + d(n-m)$. Suppose $g(\lambda, n\times d, n\times d) < a_\lambda(d[n])$:

KL'14: If $\mu\vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m}]\geq a_\mu(d[m])$, where $V_m := Sym^m\mathbb{C}^{m^2}$.

Stability: $g(\lambda, n\times d, n\times d) = g(\mu, m\times d, m\times d)$.

GCT: If $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m})]\geq g(\lambda, n\times d, n\times d)$ then $dc(f_m)>n$ for some $f_m\in V_m$.

$$\Longrightarrow mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m}V_m})]\geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n\times d, n\times d)$$

$$\Longrightarrow \max_{f\in V_m} dc(f_m) > n \to \infty$$

Plethysm positivity

Theorem (Bürgisser-Ikenmeyer-P (FOSC'16))

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$, then λ also occurs in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$. In particular, the Obstruction Existence Conjecture is false, there are no "occurrence obstructions".

Proof ideas:

- For $mult_\lambda \mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}] > 0$ we must have $\lambda_1 > d(n-m)$.
- (Valiant): $dc(X_1^s + \cdots + X_k^s) \leq ks$, hence...
  $\ell^{n-s}(v_1^s + \cdots + v_k^s) \in \Omega_n$ for $n \geq ks$.
- If a highest weight vector of weight $-\lambda$ does not vanish on Ω_n (or in particular, on the power sums), then $\delta_{\lambda,n} = mult_\lambda \mathbb{C}[\Omega_n] > 0$.
- Then $\delta_{\lambda,n} > 0$, because of the existence of λ-highest weight vectors in $Sym^d Sym^n V$, i.e. $a_\lambda(d[n]) > 0$ via explicit tableaux constructions: tableaux T of shape λ, content $d \times n$....

1	1	1	1	2	2	2	3	3	4	4	4	4	5
2	2	3	3	3									
4	5	5	5	5									

· decomposition into building blocks + regrouping

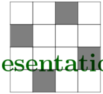
Next time:

- Matrix Powering vs permanent and the symmetric Kronecker coefficients.
- Iterated Matrix Multiplication vs permanent model.
- Matrix Multiplication lower bounds via GCT.
- Some combinatorics and bounds on the Kronecker coefficients.

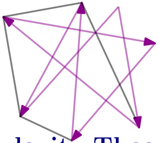
Thank you!

Algebraic Geometry

$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

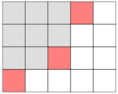


Representation Theory



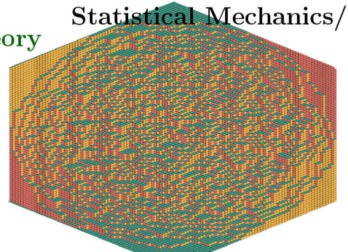
Complexity Theory

P vs NP



Algebraic Combinatorics

Statistical Mechanics/



Probability

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}$$