Symplectic embeddings of concave toric domains into convex ones

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Plan for the talk

Section 1

Introduction

Symplectic embeddings

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We would like to better understand to what extent these obstructions are sharp.

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that are monotone under symplectic embeddings.

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On the other hand, there are also cases where ECH capacities are known to not be sharp, e.g. embeddings of one polydisc into another, embeddings of a polydisc into a ball.

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A *concave toric domain* is a toric domain X_Ω, where Ω is the closed region in the first quadrant underneath the graph of a convex function $f : [0, a] \longrightarrow [0, b]$, where a and b are positive real numbers, $f(0) = b$ and $f(a) = 0$.

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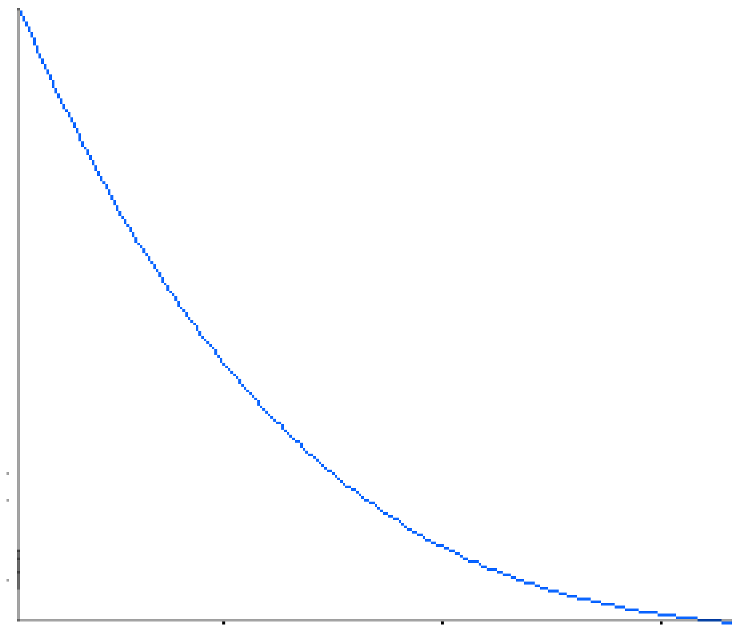
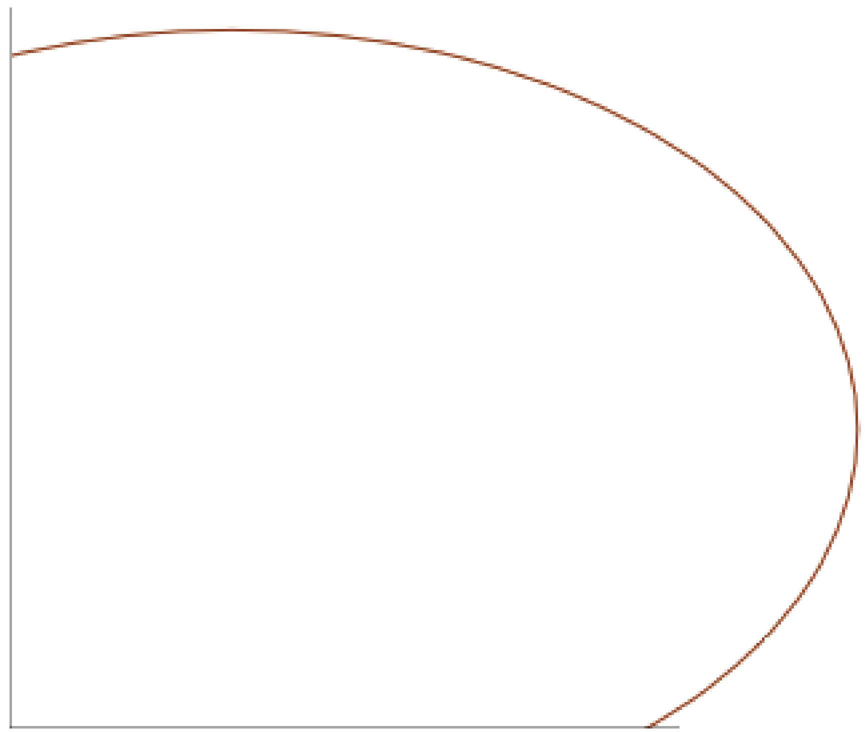
Examples



A concave domain



A convex domain

The main theorem

It turns out that for embeddings of concave domains into convex ones, the obstruction given by the ECH capacities is *sharp*:

Theorem (CG.)

Let X_{Ω_1} be a concave toric domain, and let X_{Ω_2} be a convex one. Then there is a symplectic embedding

$$\mathrm{int}(X_{\Omega_1}) \longrightarrow \mathrm{int}(X_{\Omega_2})$$

if and only if $c_k(X_{\Omega_1}) \leq c_k(X_{\Omega_2})$ for all k.

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The ECH capacities of concave and convex domains are well-understood (and combinatorially interesting!).

Section 2

Idea of the proof

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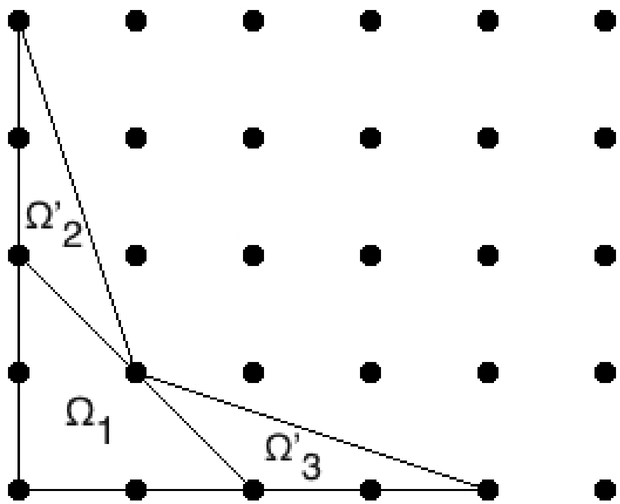

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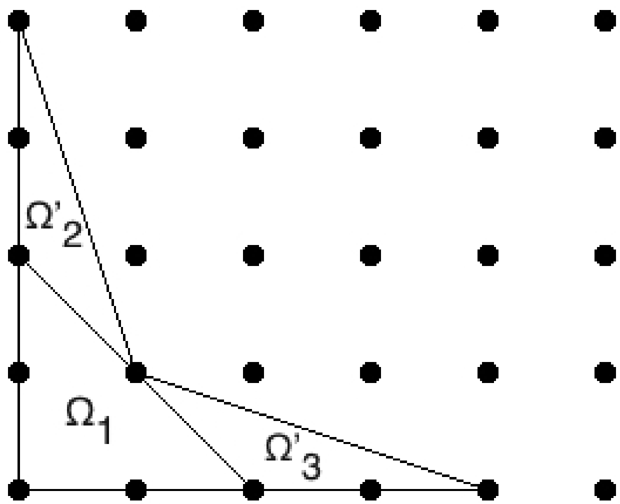

Decomposition of a concave domain

The key point is that X_{Ω_1} is a ball, while Ω_2' and Ω_3' are $SL_2(\mathbb{Z})$ equivalent to concave toric domains.

Weight sequences (cont.)

Assume that Ω is concave, and its upper boundary is piecewise linear, with rational nonsmooth points. We call such an Ω a *rational* concave domain.

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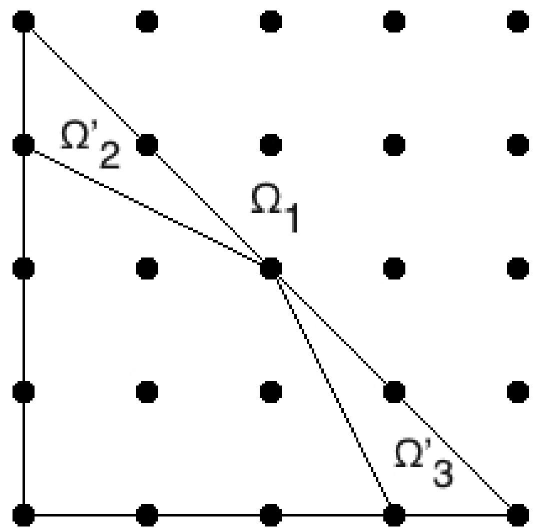
The numbers a_i are determined by Ω, and are called the *weight sequence* of Ω

More on weight sequences

If Ω is rational and convex, then there is a similar decomposition:



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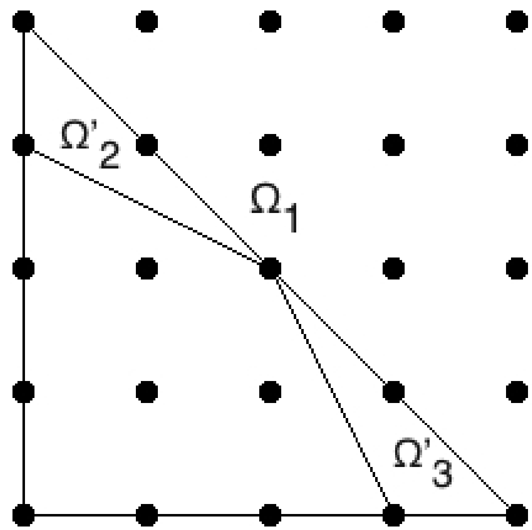

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Here the key point is that X_{Ω_1} is a ball, while Ω'_2 and Ω'_3 are affine equivalent to convex domains.

Weight sequences of convex domains

Thus, by using the weight sequence procedure for a concave domain, we find that if Ω is rational convex, there is a canonical packing

$$\coprod_{i=1}^{n} \operatorname{int}(B(b_i)) \coprod \operatorname{int}(X_\Omega) \longrightarrow B(b).$$

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The numbers $b, b_1 \ldots, b_n$ are called the *convex weight sequence* for Ω. We call b the *head*.

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If Ω_1 is concave, Ω_2 is convex, and we have a symplectic embedding

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In fact, the converse is also true:

Theorem (CG.)

Let X_{Ω_1} be a rational concave toric domain, and let X_{Ω_2} be a rational convex one. Let $(a_1, \ldots, a_n)$ be the weight sequence for Ω_1, and let $(b; b_1, \ldots, b_m)$ be the convex weight sequence for Ω_2.

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It is not hard to determine if such a ball packing exists, so this theorem is of potentially independent interest.

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This is because ECH capacities are sharp for all ball packings of a ball, so if $c_k(X_{\Omega_1}) \leq c_k(X_{\Omega_2})$, it is not hard to show that the required ball packing exists.

Section 3

The proof in more depth

Symplectic blowup

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Blowing up a concave domain

Let Ω be concave, and include X_Ω in some much larger open ball. Include this ball into a $(\mathbb{C}P^2, \omega)$ of the same volume. Now define a symplectic blow-up of $(\mathbb{C}P^2, \omega)$ as follows:

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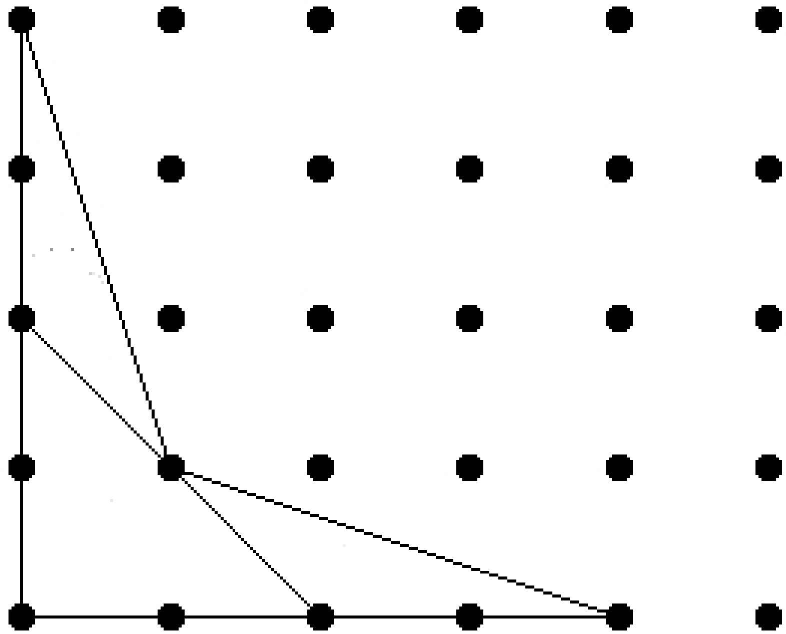
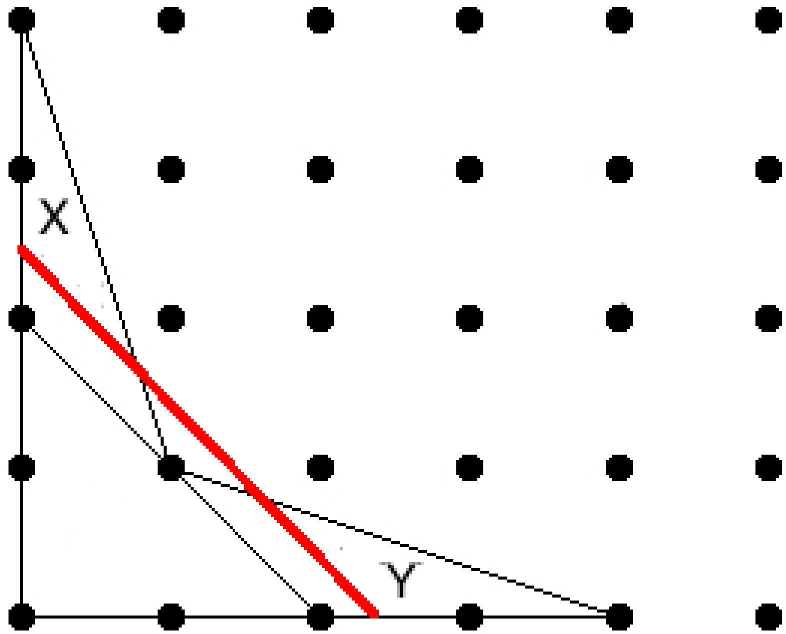
Let $a > 0$ be the largest real number such that Ω contains the triangle with vertices $(0,0), (a,0)$ and $(0,a)$, and let $\delta > 0$ be a small real number. Then there is a symplectic embedding

$$B(a+\delta) \longrightarrow (\mathbb{C}P^2, \omega).$$

Blow up along this embedding.



Decomposition of a concave domain



Blowing up a concave domain

Triangle with slant edge the red line gives a ball which we can blow up. This leaves an exceptional sphere.

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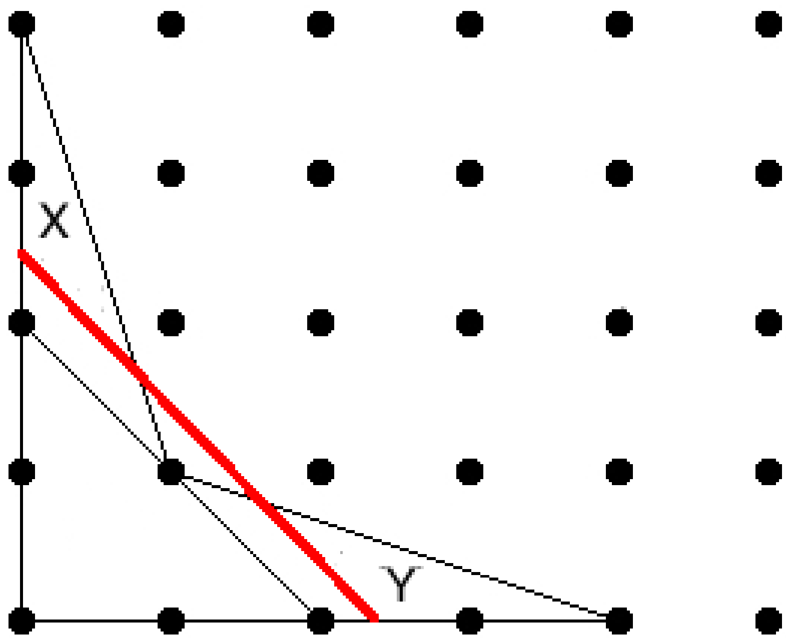
When we do this, we should choose the δ for the relevant blow up small enough that no previous spheres are completely removed.

The effect of this is to remove the interior of a slightly larger concave domain containing Ω, and collapse the boundary of this domain to a chain of spheres.



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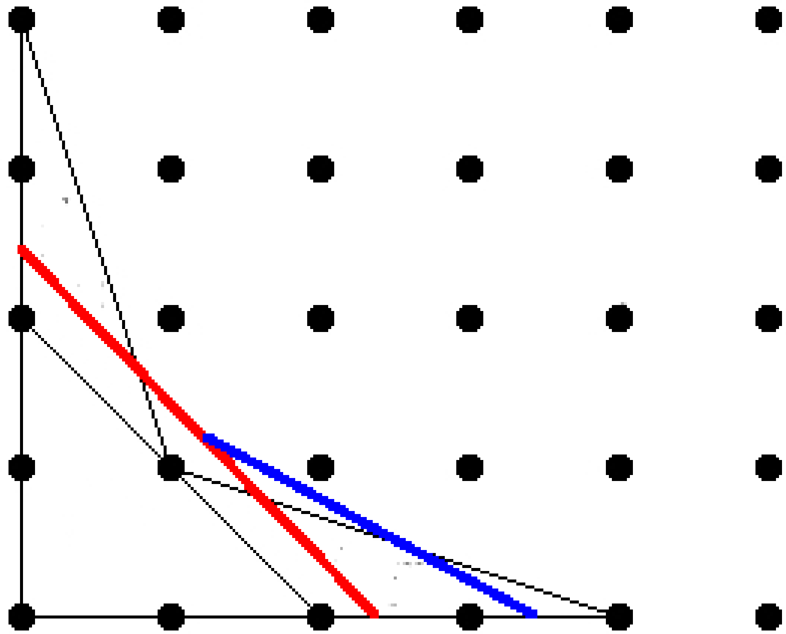

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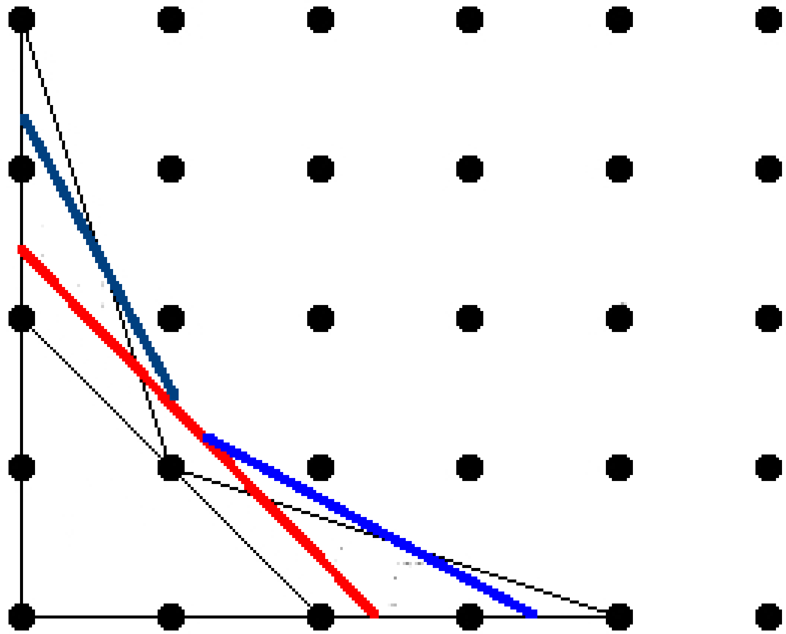

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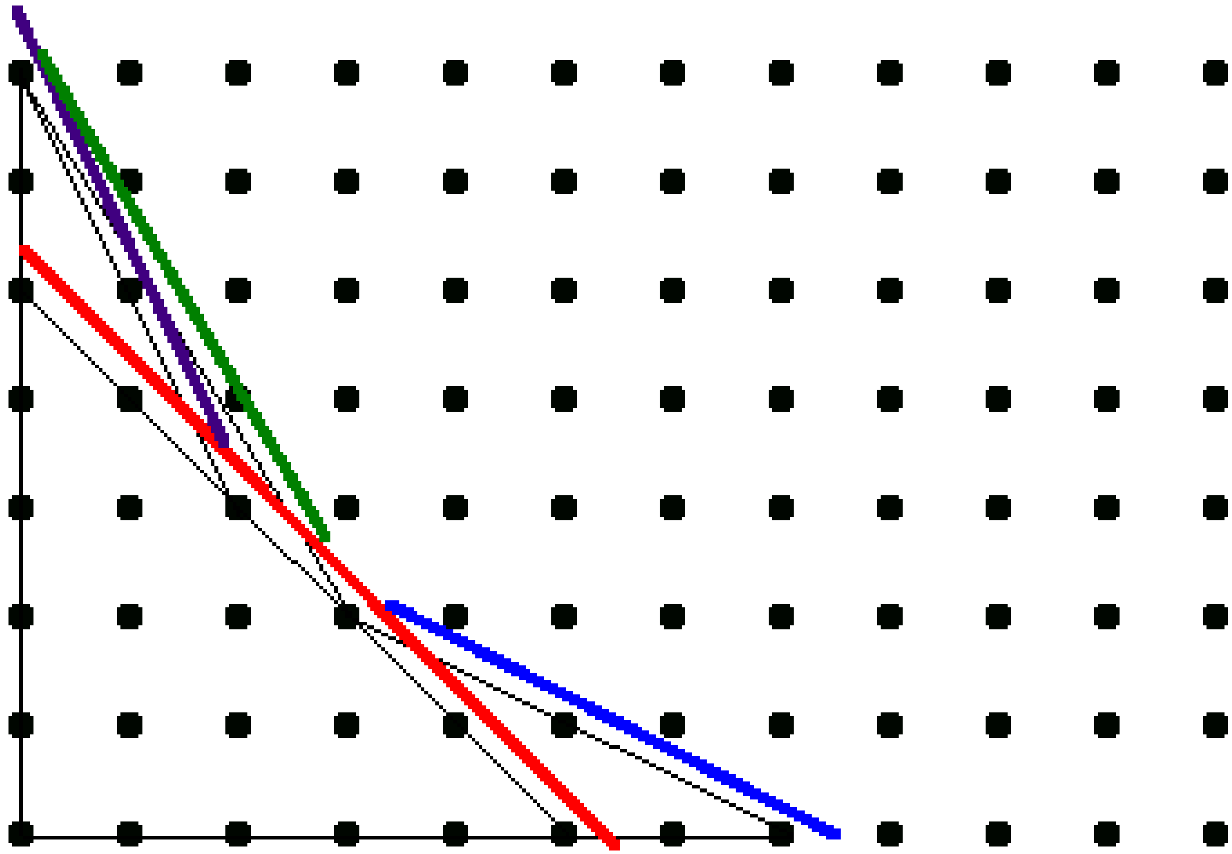

A completely worked example.

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We intersect Ω with the triangle with vertices $(0,0),(0,b-\delta)$ and $(b-\delta,0)$; this again gives two regions that are affine equivalent to concave domains, so we can apply the iterated blow up procedure from the previous slides after including $B(b-\delta)$ into a $(\mathbb{C}P^2,\omega)$ of the same volume.

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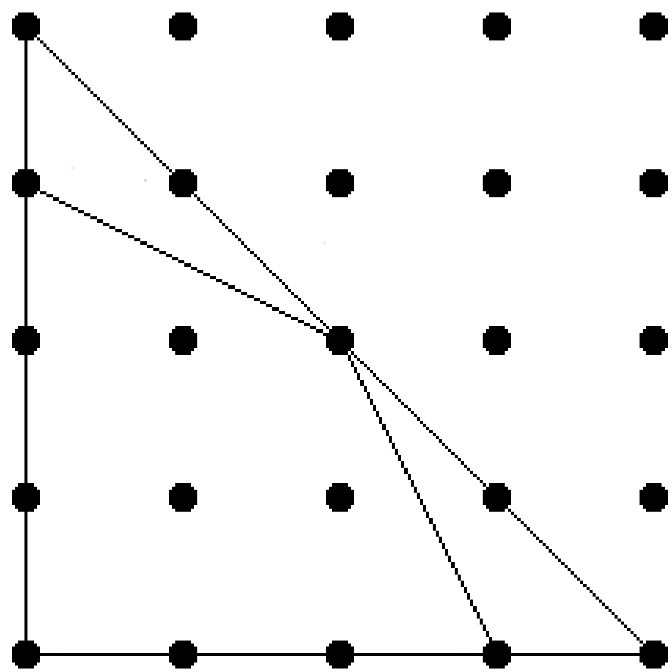
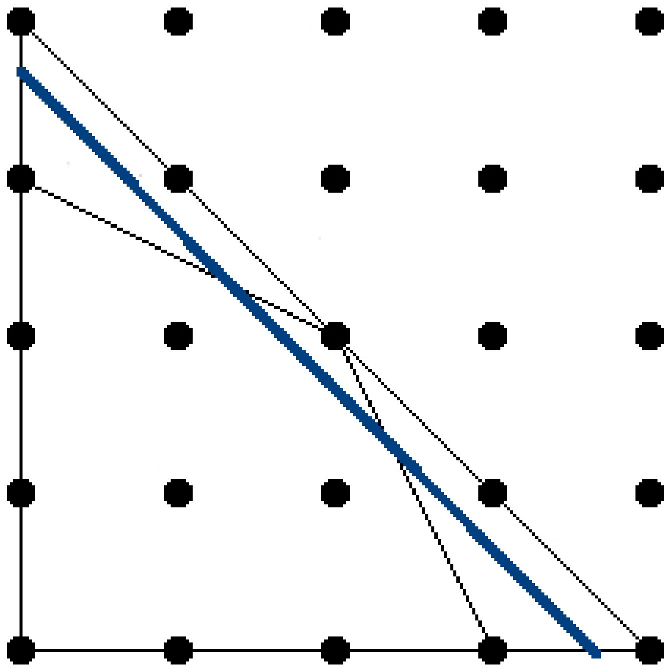
This removes the interior of the complement of a similar convex domainin in $B(b-\delta)$, and collapses the boundary to a chain of spheres.



Decomposition of a convex domain



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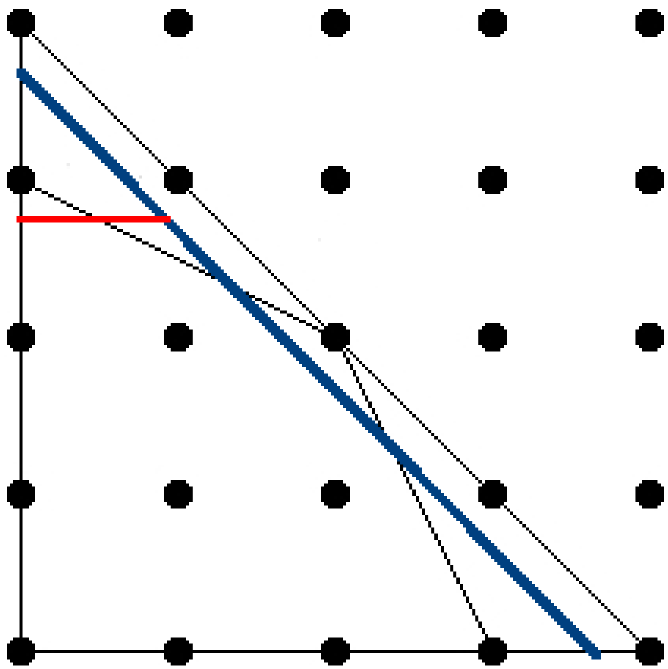

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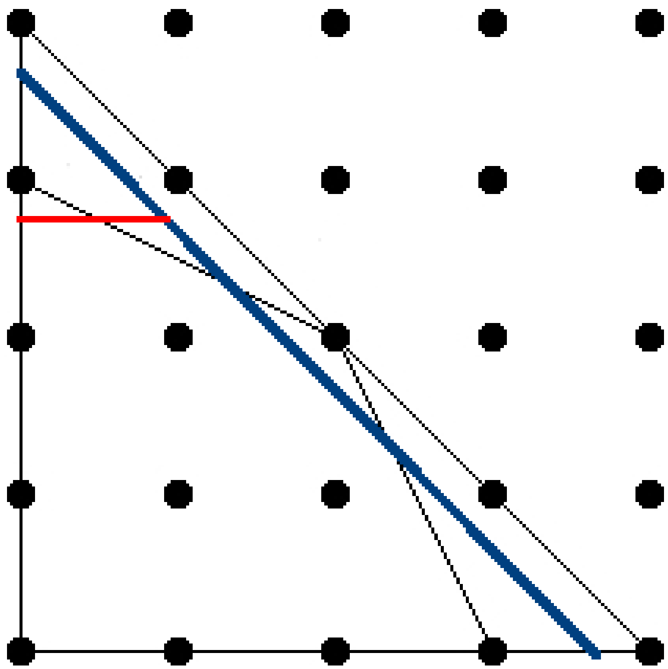

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Proposition

Let Ω_1 be a rational concave domain and let Ω_2 be a rational convex domain. Let m be the length of the weight expansion for Ω_1, and let $n+1$ be the length of the convex weight expansion for Ω_2. If there is a symplectic form ω on $\mathbb{C}P^2\#(m+n)\overline{\mathbb{C}P^2}$ such that there is a symplectic embedding

$$\mathcal{C}_{\Omega_1,\delta_1} \sqcup \hat{\mathcal{C}}_{\Omega_2,\delta_2} \longrightarrow \mathbb{C}P^2\#(m+n)\overline{\mathbb{C}P^2},$$

then there is a symplectic embedding $X_{\Omega_1} \longrightarrow \operatorname{int}(X_{\Omega_2})$.

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- *Step 2*: Blow up along $\epsilon \cdot \Omega_1$ and Ω_2. This gives two chains of spheres with the right intersection pattern, but the spheres in $\mathcal{C}_{\epsilon \cdot \Omega_1, \delta_1}$ are too small.

Sketch of rest of proof

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- *Step 3*: Correct the area of the spheres by using the inflation procedure of Lalonde and McDuff.

A few remarks

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To prove the theorem for domains that are not rational, we use an approximation argument.

Section 4

The geometric meaning of ECH capacities

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Theorem (Choi, CG., Frenkel, Hutchings, Ramos)

The ECH capacities of a a concave toric domain X_Ω with weight expansion $(a_1, a_2, \ldots,)$ are given by

$$c_k(X_\Omega) = c_k(\coprod_i B(a_i)).$$

A similar result holds for convex domains.

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Theorem (Choi, CG.)

The ECH capacities of a convex toric domain X_Ω with convex weight expansion $(b; b_1, b_2, \ldots)$ are given by

$$c_k(X_\Omega) = c_{ECH}(B(b)) - c_{ECH}(\coprod B(b_i)).$$

Here, $-$ denotes the "sequence subtraction" operation defined by Hutchings.

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Luckily, Hutchings has recently found new obstructions coming from ECH that are stronger than ECH capacities in many situations.