Recent developments of the mixed André-Oort conjecture

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Sep. 21, 2015

Universal elliptic curve

First we quickly review the theory of modular curve $Y(1)$.

1. Associate to an elliptic curve E there is an invariant $j(E)$ called the *j-invariant*. It satisfies $j(E) = j(E')$ iff $E \cong E'$.
2. Let $\mathbb{H}^+$ be the upper half plan. For any $\tau \in \mathbb{H}^+$, define $E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Then E_τ is an elliptic curve. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}^+$ by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$
3. Denote by $j\colon \mathbb{H}^+ \to \mathbb{C}$, sending $\tau \mapsto j(E_\tau)$. Then j is a complex analytic function. It satisfies $j(\tau) = j(\tau')$ iff $\tau \in \mathrm{SL}_2(\mathbb{Z})\tau'$. Therefore j induces a bijection
$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+ \cong \mathbb{C}.$$

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For any $N \geqslant 3$, we have a similar object $Y(N)$ satisfying:

Theorem

$Y(N)$ is an algebraic variety of dimension 1, and

$$Y(N) \leftrightarrow \{\text{elliptic curves with level-}N\text{-structure}\}/\sim .$$

Moreover, there is a uniformization $\mathbb{H}^+ \to Y(N)$.

The advantage of $Y(N)$ ($N \geqslant 3$) is that $Y(N)$ is then a fine moduli space, so that there is a universal family $\mathcal{E}(N)$ over it. This means $\pi\colon \mathcal{E}(N) \to Y(N)$ s.t. over each point $y \in Y(N)$, the fiber $\pi^{-1}(y)$ is isomorphic to the elliptic curve represented by y.

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Let us take a closer look at $\mathcal{E}(N)$. It has a uniformization $\mathbb{R}^2 \times \mathbb{H}^+ \to \mathcal{E}(N)$, which we view in two ways:

1. either fix $\mathbb{R}^2 \times \mathbb{H}^+ = \mathbb{C} \times \mathbb{H}^+$ and then let the lattices $\mathbb{Z}^2 \subset \mathbb{C}$ vary fiber by fiber;
2. or fix the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and let the complex structure of the $\mathbb{R}^2$'s vary fiber by fiber (over $\tau \in \mathbb{H}^+$, let $\mathbb{R}^2 \cong \mathbb{C}$ by $(a, b) \mapsto a + b\tau$).

We take the second point of view in this talk. In this way we can write $\mathcal{E}(N)$ as a quotient

$$\mathcal{E}(N) = (\mathbb{Z}^2 \rtimes \mathrm{SL}_2(1 + N\mathbb{Z}))\backslash(\mathbb{R}^2 \times \mathbb{H}^+).$$

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André-Oort for $\mathcal{E}(N)$

We say that a point of $\mathcal{E}(N)$ is *special* if it represents a torsion point on a CM elliptic curve.

Conjecture (André-Oort for $\mathcal{E}(N)$)

Let Z be an irreducible proper subvariety of $\mathcal{E}(N)$. If Z contains a Zariski dense subset of special points, then Z is either a special point, or a fiber of the projection $\mathcal{E}(N) \to Y(N)$, or a torsion section of the projection (means an irreducible component of a sub-group scheme of $\mathcal{E}(N)/Y(N)$).

This conjecture is a Theorem of André's.

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Universal abelian scheme

There is a similar theory for abelian varieties:

Theorem

1. *For any $N \in \mathbb{Z}_{>0}$, there is an algebraic variety $\mathcal{A}_g(N)$ s.t.*

$$\mathcal{A}_g(N) \leftrightarrow \{\textit{p.p.a.v. with level-N-structure of dim } g\}/\sim .$$

Moreover, there is a uniformization $\mathbb{H}_g^+ \to \mathcal{A}_g(N)$.

2. *When $N \geqslant 3$, $\mathcal{A}_g(N)$ is fine. So there is a universal abelian scheme $\mathfrak{A}_g(N)$ over $\mathcal{A}_g(N)$.*
3. *$\mathfrak{A}_g(N)$ has a uniformization $\mathcal{X}_{2g}^+ := \mathbb{R}^{2g} \times \mathbb{H}_g^+ \to \mathfrak{A}_g(N)$ and*

$$\mathfrak{A}_g(N) = (\mathbb{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(1 + N\mathbb{Z})) \backslash \mathcal{X}_{2g}^+.$$

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The conjecture is known for $\mathfrak{A}_g(N)$ after a series of work: Pila-Tsimerman/Klingler-Ullmo-Yafaev, Tsimerman, Gao.

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This conjecture has a more general form:

Conjecture (André-Oort)

Let S be a connected mixed Shimura variety and let Z be an irreducible subvariety of S. If Z contains a Zariski dense subset of special points of S, then Z is a mixed Shimura subvariety.

Example

Examples of mixed Shimura varieties which are not pure: $\mathfrak{A}_g(N)$, universal Poincaré-biextension over $\mathcal{A}_g(N)$.

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André-Oort: Pila-Zannier method

Following the Pila-Zannier method, we have to do the following things to prove the André-Oort conjecture (let Σ be the set of special points of the mixed Shimura variety S):

1. Prove the Ax-Lindemann theorem; AL
2. Deduce from Ax-Lindemann a result about distribution (Pila-Tsimerman/Ullmo for pure, Gao for mixed);
3. Choose a "good" fundamental domain $\mathcal{F}$ for the uniformization $\mathrm{unif}\colon \mathcal{X}^+ \to S$ (Peterzil-Starchenko + Klingler-Ullmo-Yafaev + Gao) and associate to each special point a parameter (which we call the *complexity*);
4. Prove an upper bound for any point in $\mathrm{unif}^{-1}(\Sigma) \cap \mathcal{F}$ w.r.t. the complexity of its image in Σ (Pila-Tsimerman + Daw-Orr);
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6. Conclude. AO

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Ax-Lindemann in general case: weakly special subvarieties

Pink defined *weakly special subvarieties* for any mixed Shimura variety. We do not repeat the exact definition, but only look at the following example:

Example

Let Y be a subvariety of $\mathfrak{A}_g$. Then $\pi^{-1}(\pi(Y)) \to \pi(Y)$ is an abelian scheme. Let $\mathcal{C}$ be its isotrivial part, i.e. the largest isotrivial abelian subschemem of $\pi^{-1}(\pi(Y)) \to \pi(Y)$.

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Y is weakly special iff

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Theorem (Gao, pure case by Klingler-Ullmo-Yafaev)

Let S be a mixed Shimura variety and $\mathrm{unif}\colon \mathcal{X}^+ \to S$. Let $\widetilde{Z} \subset \mathcal{X}^+$ be a connected semi-algebraic subset. Then every irreducible component of $\mathrm{unif}(\widetilde{Z})^{\mathrm{Zar}}$ is weakly special.

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Gao reduced this lower bound for mixed Shimura varieties to the pure parts. Therefore for $\mathfrak{A}_g(N)$ (or more generally any mixed Shimura variety of abelian type), it suffices to prove the following result:

Conjecture (Edixhoven)

Let $x \in \mathcal{A}_g$ be a special point and let A_x denote the CM abelian variety parametrized by x. Let R_x be the center of $\mathrm{End}(A_x)$. Then there exists a constant $\delta(g) > 0$ s.t.

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Theorem (Masser-Wüstholz)

Let A, B be abelian varieties over a number field K which are isogenous over $\overline{K}$. Then there exists a constant $c_g > 0$ s.t.

$$\textit{minimum degree of isogenies } A \to B \ll_g \max(h_F(A), [K:\mathbb{Q}])^{c_g}$$

The term $[K:\mathbb{Q}]$ gives $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x|$ where $x \in \mathcal{A}_g$ parametrizes A. When A is CM, it is not hard to find B isogenous to A with the minimum degree of isogenies large enough. Hence to prove the desired lower bound, it suffices to bound $h_F(A)$ from above. The Colmez conjecture expresses $h_F(A)$ (A is CM) in an explicit formula whose terms can be bounded by classical results, and hence it suffices to prove the Colmez conjecture. But since $h_F(A)$ is bounded below by a constant depending only on $\dim A$, it is enough to prove an averaged version. This is known by recent work of Yuan-S.-W. Zhang/Andreatta-Goren-Howard-Madapusi Pera.

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Conclusion

Theorem

The André-Oort conjecture holds for any mixed Shimura variety of abelian type.

Thank you!