Algebraic cycles on holomorphic symplectic varieties

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Motivation

Let X be any complex projective manifold and $n \in \mathbb{N}$. Define the *symmetric product*

$$X^{(n)} := X^n / \mathfrak{S}_n .$$

Proposition

The cohomology ring

$$H^*(X^{(n)}, \mathbb{Q}) \simeq \mathrm{Sym}^n (H^*(X, \mathbb{Q})) := \left(H^*(X, \mathbb{Q})^{\otimes n}\right)^{\mathfrak{S}_n} .$$

However $X^{(n)}$ is a *singular* algebraic variety when $\dim X \geq 2$.

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What are the 'best' smooth models for $X^{(n)}$?
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The quotient stack/orbifold

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What is the reasonable cohomology theory for orbifolds ?

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Orbifold cohomology : global quotient case

M : a projective complex manifold with an action of a finite group G.

Define an auxiliary ring $H^*(M, G)$ as follows :

- As a G-graded vector space

$$H^*(M, G) := \oplus_{g \in G} H^{*-2\,\mathrm{age}(g)}(M^g).$$

- The stringy product $*$: for $u \in H(M^g)$ and $v \in H(M^h)$,

$$u * v := i_*\left(u|_{M^{g,h}} \cup v|_{M^{g,h}} \cup c_{top}(F_{g,h})\right) \in H(M^{gh}),$$

  where $F_{g,h}$ is some 'obstruction' vector bundle on $M^{g,h}$.
- Natural G-action : for $g, h \in G$ and $x \in M^g$, $h \cdot x := hx \in M^{hgh^{-1}}$. This action preserves the G-grading and the stringy product $*$.

Definition

Chen-Ruan's orbifold cohomology ring of $[M/G]$ is its invariant subring.

$$H^*_{orb}([M/G]) := H^*(M, G)^G.$$

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Second candidate

From now on, $X = S$ is a smooth projective surface. In this case, we have the Hilbert scheme of subschemes of length n on S :

$$S^{[n]} := \mathrm{Hilb}^n(S),$$

which is obviously birational to $S^{(n)}$.

Facts (miracle !)

- (Fogarty) $S^{[n]}$ is a smooth.
- The Hilbert-Chow morphism $\tau : S^{[n]} \to S^{(n)}$ is *crepant*.

Crepant=no discrepancy : $\tau^*(K_{S^{(n)}}) = K_{S^{[n]}}$.
Therefore $S^{[n]}$ is a minimal (best !) resolution of singularities of $S^{(n)}$.

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$[S^n/\mathfrak{S}_n]$ vs. $S^{[n]}$

Let S always be a smooth projective surface.

(Ruan) String theory : as 'best' smooth models of $S^{(n)}$, $[S^n/\mathfrak{S}_n]$ and $S^{[n]}$ are equally good !

Theorem

- *(Göttsche)* $H^*_{orb}([S^n/\mathfrak{S}_n]) \simeq H^*(S^{[n]})$ *as graded vector spaces.*
- *(Lehn-Sorger, Fantechi-Göttsche) When* $K_S = 0$,

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Remark : for S with $K_S \neq 0$, we have the more general result (Li-Qin) :

$$H^*_{orb}([S^n/\mathfrak{S}_n]) \simeq H^*_\tau(S^{[n]}),$$

where the RHS incorporates the *quantum corrections* coming from the Gromov-Witten invariants for curve classes contracted by the Hilbert-Chow morphism τ.

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Holomorphic symplectic varieties

Goal : generalize Fantechi-Göttsche-Lehn-Sorger theorem in various directions.

Note that for surface S with $K_S = 0$, $S^{[n]}$ is *holomorphic symplectic* in the following sense :

Definition

A smooth projective variety X is called *irreducible holomorphic symplectic* (or *hyperkähler*), if

- $\pi_1(X) = 1$;
- $H^{2,0}(X) = \mathbb{C} \cdot \eta$ with η a holomorphic *symplectic* 2-form.

Examples :

- $S^{[n]}$ with S a K3 surface ;
- $K_n(A)$: *generalized Kummer variety* associated to an abelian surface A ;
- Fano varieties of lines of cubic fourfolds.

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Hyperkähler crepant resolution conjecture (Ruan)

Conjecture (global quotient version)

Let M be a holomorphic symplectic variety with a *symplectic* action of a finite group G. Let X be a crepant (=symplectic) resolution of M/G. Then we have an isomorphism of graded *algebras* :

$$H^*_{orb}([M/G]) \simeq H^*(X).$$

My goal :

1. Formulate a motivic version of this conjecture ;
2. Try to prove it (maybe in some cases), hence Ruan's conjecture (in some cases).

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As before, let M be a holomorphic symplectic variety with a *symplectic* action of a finite group G. We can define its *orbifold motive* $\mathfrak{h}_{orb}([M/G])$ in the category of Chow motives CHM.

Conjecture

If X is a crepant resolution of M/G. We have an isomorphism of algebra objects in the category CHM :

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Why believe it ?

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Why believe it ?

As before, let M be a holomorphic symplectic variety with a *symplectic* action of a finite group G. We can define its *orbifold motive* $\mathfrak{h}_{orb}([M/G])$ in the category of Chow motives CHM.

Conjecture

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Supporting evidences

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- It fits good with Beauville's conjecture of multiplicative splitting of Bloch-Beilinson type of the Chow ring of holomorphic symplectic varieties.
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