Bipartite Perfect Matching is in quasi-NC

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Joint work with Rohit Gurjar and Thomas Thierauf (University of Aalen, Germany).

Matching

$G = (V, E)$ is a graph with $n$ nodes and $m$ edges.

**Definition**

A matching in $G$ is a set $M \subseteq E$ such that each $v \in V$ is incident to at most one $e \in M$.

For a perfect matching (p.m.): substitute “exactly” for “at most” above. The perfect matching decision problem, PM, asks whether a given graph has a p.m.

The search problem, SEARCH-PM, asks for a p.m. in a graph if it exists. Matchings and perfect matchings have been widely studied in combinatorics and complexity theory.
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- A polynomial-time algorithm for PM due to Edmonds [Edm65].
- A fast randomized parallel (RNC) algorithm for PM due to Lovász [Lov79] (also Chari, Rohatgi, & Srinivasan [CRS95]).
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NC is the class of problems with uniform polynomial size circuits with polylogarithmic depth. For polylog-depth circuits solving PM, nothing better than exponential size was known.

Open

Is there a fast parallel nonrandomized (NC) algorithm for PM?
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Deterministic parallel algorithms
There are NC algorithms for certain types of graphs:

- $K_{3,3}$-free graphs (Vazirani [Vaz89]),
- graphs having polynomially many p.m.'s (Grigoriev & Karpinski [GK87], also Agrawal, Hoang, & Thierauf [AHT07])
- bipartite $d$-regular graphs (Lev, Pippenger, & Valiant [LPV81], also Sharan & Wigderson [SW96])
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- planar bipartite graphs (Datta, Kulkarni, & Roy [DKR10] and Tewari & Vinodchandran [TV12])

Our Work

Bipartite PM and SEARCH-PM are in quasi-NC.
That is, PM and SEARCH-PM on bipartite graphs have uniform circuits of depth $O(\log^2 n)$ and size $2^{O(\log^2 n)}$.
We also give an RNC$^2$ algorithm for bipartite PM using $O(\log^2 n)$ random bits.
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We also give an RNC² algorithm for bipartite PM using $O(\log^2 n)$ random bits.
Bipartite perfect matching in RNC

\( G \) bipartite with bipartition \( L = \{u_1, \ldots, u_{n/2}\} \) and \( R = \{v_1, \ldots, v_{n/2}\} \). Given a weight function \( w : E \to \mathbb{Z}^+ \), we extend \( w \) to sets of edges: for \( S \subseteq E \), define \( w(S) := \sum_{e \in S} w(e) \).

Define the \( n/2 \times n/2 \) matrix \( A_w = [a_{i,j}] \) as

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a_{ij} = \begin{cases} 
2^{w(e)} & \text{if } e = (u_i, v_j) \in E, \\
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Then

\[
\det(A_w) = \sum_{M \text{ a p.m. of } G} \text{sgn}(M) 2^{w(M)}.
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If \( G \) has no p.m., then \( \det(A_w) = 0 \) for any \( w \).
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Definition

A weight function $w$ is **isolating** if $G$ has a unique minimum weight p.m. with respect to $w$.

If $w$ is isolating, then $\det(A_w) \neq 0$, because the minimum weight term in $\det(A_w)$ does not cancel with other terms, which are strictly higher powers of 2.

Lemma (Isolation Lemma [MVV87])

Let $w(e)$ chosen uniformly at random from $\{1, \ldots, 2m\}$ for each edge $e$ independently. Then $w$ is isolating with probability $\geq 1/2$.

If $w$ is isolating, then computing $\det(A_w)$ gives the correct answer. This can be done in $\text{NC}^2$ (Berkowitz [Ber84]).
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We want to derandomize this lemma!

Let $E = \{ e_0, \ldots, e_{m-1} \}$, and define $w(e_i) = 2^i$ for all $i < m$. $w$ is clearly isolating, ...

but we cannot compute $\det(A_w)$ efficiently, because the matrix entries are too big.

Instead, we reduce the weights modulo small numbers $j$:  

**Definition**

Fix an integer $j > 1$. Define the weight function $w_{\mod j}$ as

$$w_{\mod j}(e) := w(e) \mod j$$

for all $e \in E$.

For some $t$ we choose later, define the set of weight functions

$$W_t := \{ w_{\mod j} \mid 2 \leq j \leq t \}.$$
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\( w \) is clearly isolating, 
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Instead, we reduce the weights modulo small numbers \( j \):

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Let $C = \langle e_1, \ldots, e_p \rangle$ be a cycle of $G$ with edges given in cyclic order. ($p$ is even because $G$ is bipartite.)

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Given a weight function $w$, the circulation of $C$ with respect to $w$ is

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Given $w$, suppose $M_1 \neq M_2$ are min weight p.m.'s of $G$. Then $M_1$ and $M_2$ differ on disjoint cycles with zero circulation:
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We would like to choose a weight function from $W_t$ that gives nonzero circulation to as many cycles as possible. We cannot do this for all cycles at once, so we work in stages, starting with short cycles.

Lemma ([CRS95])

Let $s$ be a positive integer, and let $t = n^2 s$. Then for any set of $s$ many cycles $\{C_1, \ldots, C_s\}$ there exists a weight function $w \in W_t$ that gives nonzero circulation to all of $C_1, \ldots, C_s$.

We will apply this lemma with $s := n^4$. Each weight of $w$ is taken modulo some $j \leq t = n^2 s = n^6$, so needs only $6 \log n$ bits.
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The derived graph

Suppose $G$ has a p.m., and $w$ is a weight function on $G$.

**Definition**

The derived graph of $G$ with respect to $w$ is the subgraph $G^{(w)} := (V, E')$, where $E'$ is the union of all $w$-min weight p.m.'s of $G$.

**Key Lemma**

All cycles in $G^{(w)}$ have zero circulation with respect to $w$.

We proved this lemma using linear algebra. Later, an alternate combinatorial proof was found by Rao, Shpilka, & Wigderson (reported in Goldwasser & Grossman [GG15]).

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Removing short cycles from $G^{(w)}$

In steps, we remove cycles from derived graphs whose lengths are increasing powers of 2.
The next lemma says there are not too many of these.

**Lemma**

Let $G$ be a graph with no cycles of length $\leq r$ for some even $r$. Then $G$ has $\leq n^4$ cycles of length $\leq 2r$.

We can give all these cycles nonzero circulation by some weight function $w \in W_t$, where $t = n^6$.
So these cycles cannot exist in the derived graph with respect to $w$. 
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Stephen Fenner (Computer Science and Engineering Department, University of South Carolina, fenner@cse.sc.edu)
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Proof

Given a cycle $C$ of length $\leq 2r$, choose four vertices $u_0, u_1, u_2, u_3$ on the cycle such that the distance between adjacent vertices is $\leq r/2$. This is the only such cycle given $(u_0, \ldots, u_3)$. If there is another such cycle $C'$, then $C'$ forms a cycle with $C$ of length $\leq r$. Contradiction.
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Start with $G_0 := G$, a bipartite graph with a p.m.

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Proceed for $k := \lceil \log n \rceil - 1$ rounds to obtain $G_k$, which is a p.m.
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Proceed for $k := [\log n] – 1$ rounds to obtain $G_k$, which is a p.m.
The sequence of derived graphs

Start with \( G_0 := G \), a bipartite graph with a p.m.

- Choose \( w_1 \in W_t \) such that all cycles in \( G_0 \) of length \( \leq 4 \) have nonzero circulation.
- Let \( G_1 := G^{(w_1)} \). \( G_1 \) has a p.m. and no cycles of length \( \leq 4 \).
- Choose \( w_2 \in W_t \) such that all cycles in \( G_1 \) of length \( \leq 8 \) have nonzero circulation.
- Let \( G_2 := G_1^{(w_2)} \). \( G_2 \) has a p.m. and no cycles of length \( \leq 8 \).
- \( \ldots \)
- Choose \( w_i \in W_t \) such that all cycles in \( G_{i-1} \) of length \( \leq 2^{i+1} \) have nonzero circulation.
- Let \( G_i := G_{i-1}^{(w_i)} \). \( G_i \) has a p.m. and no cycles of length \( \leq 2^{i+1} \).
- \( \ldots \)

Proceed for \( k := \lceil \log n \rceil - 1 \) rounds to obtain \( G_k \), which is a p.m.
An isolating weight function for $G$

We must glue the weight functions $w_1, \ldots, w_k$ together into a single weight function.

Let $B$ be a strict bound on any edge weight from $w_1, \ldots, w_k$ (we may take $B := n^6$).

For every $e \in E$, define

$$w(e) = B^{k-1}w_1(e) + B^{k-2}w_2(e) + \cdots + B^0w_k(e).$$

Lemma

If $G$ has a p.m., then $w$ is isolating for $G$. 

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**Lemma**

If $G$ has a p.m., then $w$ is isolating for $G$. 
Proof

- Notice that the edge sets of the $G_i$ form a descending chain, ending in a p.m. $M$ of $G$ (the edge set of $G_k$).
- Let $M' \neq M$ be some other p.m. of $G$.
- There must be some stage $i < k$ where $M$ and $M'$ are both in $G_i$ but $M'$ is not in $G_{i+1}$.
- Since $M$ and $M'$ are in $G_1, \ldots, G_i$, they both have the same minimum weight with respect to $w_1, \ldots, w_i$.
- But since $M'$ is not in $G_{i+1}$ (but $M$ is), it must be that $w_{i+1}(M') > w_{i+1}(M)$.
- This implies $w(M') > w(M)$, and so $w$ is isolating.
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The algorithm

We do not know which \( w_1, \ldots, w_k \) work, so we try them all in parallel. For all \( w_1, \ldots, w_k \in W_{n^6} \) in parallel:

- Compute \( w \) as above. (One of these choices of \( w \) must be isolating.)
- Compute \( \det(A_w) \) as in the RNC algorithm of [MVV87].
- If we ever find a nonzero determinant, answer “yes.”
- Else, answer “no.”

Each \( w_i \) takes \( 6 \log n \) bits to store, so \( w \) takes \( O(\log^2 n) \) bits. Processing them all in parallel can be done with circuits of size \( 2^{O(\log^2 n)} \) and depth \( O(\log^2 n) \).
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Then any set $S \subseteq E$ of edges naturally corresponds to its characteristic vector $(s_e)_{e \in E}$, where, for each edge $e \in E$,

$$s_e = \begin{cases} 
1 & \text{if } e \in S, \\
0 & \text{if } e \notin S.
\end{cases}$$

**Definition**

The perfect matching polytope $\text{PM}(G)$ is the convex hull of all the perfect matchings of $G$. 
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Lemma ([LP86])

If $G$ is bipartite, then a vector $\vec{x} = (x_e)_e$ is in $\text{PM}(G)$ if and only if

$$x_e \geq 0, \quad \sum_{e' \in \delta(v)} x_{e'} = 1,$$

for all $e \in E$ and $v \in V$, where $\delta(v)$ is the set of edges incident to $v$.

The $\Rightarrow$ direction is clear for any graph (not necessarily bipartite). The converse does not hold for general graphs.

We can extend any weight function $w$ to $\mathbb{R}^m$ by linearity:

$$w(\vec{x}) = \sum_{e \in E} w(e)x_e.$$
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Let $\vec{x}_1, \ldots, \vec{x}_t \in \text{PM}(G)$ be vectors corresponding to all the p.m.’s of $G$ with the same minimum weight $q$.

Set

$$\vec{x} = (x_e)_e = \frac{\vec{x}_1 + \cdots + \vec{x}_t}{t}.$$ 

Then $\vec{x} \in \text{PM}(M)$, and $w(\vec{x}) = q$.

Also, every entry of $\vec{x}$ in the derived graph $G'$ satisfies $x_e \geq \frac{1}{t}$. 
Let $\bar{x}_1, \ldots, \bar{x}_t \in \text{PM}(G)$ be vectors corresponding to all the p.m.'s of $G$ with the same minimum weight $q$.

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Then $\vec{x} \in \text{PM}(M)$, and $w(\vec{x}) = q$. Also, every entry of $\vec{x}$ in the derived graph $G'$ satisfies $x_e \geq \frac{1}{t}$. 
Suppose some cycle $C$ in the derived graph $G'$ has nonzero circulation. W.l.o.g., the blue edges outweigh the red edges.

Let $\tilde{y} = (y_e)_e$ be the vector obtained from $\tilde{x}$ by subtracting $\frac{1}{t}$ from the blue edges and adding $\frac{1}{t}$ to the red edges.

Then $\tilde{y} \in \text{PM}(G)$. Moreover,

$$w(\tilde{y}) = w(\tilde{x}) - \frac{c_w(C)}{t} < q.$$ 

But then there must be a p.m. of $G$ with weight $< q$. Contradiction.
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The RNC algorithm

Recall $w_{mod_j}(e_i) = 2^i \mod j$ for each edge $e_i \in E$ and $2 \leq j \leq t$. Instead of trying all of these weight functions, we let $j$ be a random prime.

Any set of $s$ many cycles has nonzero circulation with high probability. Doing this $k$ times gives random $w_1, \ldots, w_k$. 
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Other results

The following are all in quasi-NC:

- bipartite weighted PM with quasi-polynomially bounded integer weights
- maximum bipartite matching
- cycle cover with polynomially bounded integer weights
- subtree isomorphism
- max flow with polynomially bounded integer capacities
- constructing a depth-first search tree
Acknowledgments

I would like to thank Ran Raz and the rest of the IAS faculty for inviting me to give this talk.
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