Incompressible Elasticity in 2D

Zhen(震) Lei(雷)

Fudan University

Joint with Thomas C. Sideris and Yi Zhou

Contents

- √ Incompressible Elasticity
- √ The Key Question and Its Difficulties
  - ⋆ some previous progress
- √ Almost Global Well-posedness of Small Solutions in 2D

Notations

The flow map $X(t, y)$:

It maps the material point $y \in \Omega_0$ at time $t = 0$ to the space position $x = X(t, y) \in \Omega_t$ at time t. (t, y) will be called Lagrangian coordinate, while (t, x) Euler

Notations

The flow map $X(t, y)$ generates a velocity field v, which, at time t and spatial position x, is given by:

$$v(t, x) = \frac{\partial X(t, y)}{\partial t}\bigg|_{y=X^{-1}(t,x)}.$$

Alternatively, one may also think that a given velocity field $v(t, x)$ generates the flow map by solving:

$$\frac{dX(t, y)}{dt} = v(t, x)\bigg|_{x=X(t,y)}, \quad X(0, y) = y.$$

Notations

For perfect fluid flows, the dynamics is determined by the following Lagrangian functional, which is related to the associated the kinetic energy:

$$\mathcal{L}(X;T,\Omega) = \frac{1}{2}\int_0^T \int_\Omega |v(t,X(t,y))|^2 dydt.$$

It is known that the first variation of $\mathcal{L}(X)$, under the incompressibility constraint, gives the well-known Euler equation.

Notations

Motion of elastic materials is also determined by their *elastic energies*. Define the deformation gradient $F(t,x)$ by:

$$F\big(t, X(t,y)\big) = \frac{\partial X(t,y)}{\partial y}. \tag{1}$$

Incompressibility means volume-preserving. In mathematics, that is

$$\det F \equiv 1 \tag{2}$$

since $\int_U dy \equiv \int_{X(t,U)} dX$ for any domain U.

Notations

Consider the most basic storage energy functionals

$$\widehat{W}(X(t,x)) = W\big(F(t,x)\big)$$

For isotropic materials, W depends on F only in terms of the invariants of $F^\top F$. In 2D, those are trace and determinant.

Perfect fluids: $W = W(\det F^\top F)$.

Hookean elastic case $W = \frac{1}{2}|F|^2$.

Incompressible Elasticity

The Lagrangian function in this case is

$$\mathcal{L}(X;T,\Omega) = \int_0^T \int_\Omega \frac{1}{2}|X_t(t,y)|^2 - \frac{1}{2}|F(t,X(t,y))|^2 + p(t,y)(\det F - 1)dydt.$$

Here $p(t,y)$ is a Lagrangian multiplier which is responsible for the incompressibility, which is equivalent to

$$\nabla \cdot v = 0.$$

Incompressible Elasticity

E-L equation:

$$X_{tt} - \Delta_y X + F^{-T} \nabla_y p = 0.$$

The incompressibility constraint:

$$\det \nabla X = 1.$$

Key Question

Key Question: To solve the flow map $X(t, \cdot)$, or equivalently, to solve the above incompressible elastic system.

We will formulate it in Euler coordinate: quasi-linear wave type equation. Current interests center around small-data global regularity.

Vector Fields

Suppose that X, p is a critical point of $\mathcal{L}$. If we define

$$\widetilde{X}(t,y) = Q(s)X(t, Q^\top(s)y), \quad \widetilde{p}(t,y) = p(\ldots),$$

where

$$Q(s) = e^{sA}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Then $\widetilde{X}, \widetilde{p}$ is also critical point of $\mathcal{L}$. This invariance group gives that

$$(\frac{\partial \Omega X}{\partial y})^\top (\partial_t^2 - \Delta_y) X + (\frac{\partial X}{\partial y})^\top (\partial_t^2 - \Delta_y) \Omega X + \nabla_y \Omega p = 0.$$

Vector Fields

Similarly, one can derive that

$$\Big(\frac{\partial \widetilde{S} X}{\partial y}\Big)^{\top}(\partial_t^2 - \Delta_y) X + \Big(\frac{\partial X}{\partial y}\Big)^{\top}(\partial_t^2 - \Delta_y)\widetilde{S} X + \nabla_y S p = 0.$$

where

$$S = t\partial_t + r\partial_r, \quad \widetilde{S} = S - 1.$$

and

$$\Omega X = \partial_\theta X + AX, \quad \Omega p = \partial_\theta p.$$

Unfortunately, there is no Lorentz invariance.

Incom-Elasticity in Euler Chart

Incompressible Elasticity in Euler coordinate:

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \nabla \cdot \left(FF^T\right), \\ F_t + v \cdot \nabla F = \nabla v F, \\ \nabla \cdot v = 0. \end{cases}$$

Make use of the dispersive nature by studying small

$$(G, v) = (F - I, v).$$

Connection to Other System

- Add $\Delta v \Longrightarrow$ Viscoelasticity
- Ignore elastic force $\Longrightarrow$ Euler or Navier-Stokes
- By $\nabla \cdot F^\top = 0$, one may assume that $F = (\nabla^\perp \phi)^\top$. Then

$$\begin{cases} v_t + v \cdot \nabla v + \nabla \widetilde{p} = -\nabla \cdot (\nabla \phi \otimes \nabla \phi), \\ \phi_t + v \cdot \nabla \phi = 0, \\ \nabla \cdot v = 0. \end{cases}$$

MHD: ϕ is a scalar.

Main Difficulties

Linearization:

$$v_{tt} - \Delta v = 0, \quad G_{tt} - \Delta G = \nabla \times (\nabla \times G).$$

If $\nabla \times (\nabla \times G)$ can be treated as a forcing term, then the main part of the linearized system is of wave type. Fortunately, this is true because (thesis of L.)

$$\nabla \times G = Q(G, \nabla G).$$

Main Difficulties

So the key points for global or long time existence are

- √ dimension, which determines the time decay rate
- √ null structure of nonlinearies, which gives nonresonance along the light cone

Main Difficulties

In general, energy estimate gives (quadratic non)

$$\frac{dE_s(t)}{dt} \lesssim \|D^{s-2}v\|_{L^\infty} E_s(t).$$

Decay type estimate gives

$$\|D^{s-2}v(t,\cdot)\|_{L^\infty} \lesssim \frac{\sqrt{E_s}}{(1+t)^\alpha}.$$

- √ $\alpha > 1$: subcritical
- √ $\alpha = 1$: critical
- √ $\alpha < 1$: supercritical

Main Difficulties in 2D

Let $S = t\partial_t + r\partial_r$ be the scaling operator,

$\Omega_{ij} = x_i\partial_j - x_j\partial_i$ rotation and

$L_j = \Omega_{0j} = t\partial_j + x_j\partial_t$ Lorentz.

Theorem 1 (Klainerman). *Weighted inequality:*

$$|u(t,x)| \lesssim \frac{\sum_{|\alpha|\le[\frac{n}{2}]+1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2_x}}{(1+t+|x|)^{\frac{n-1}{2}}\big(1+\big|t-|x|\big|\big)^{\frac{1}{2}}}.$$

Main Difficulties in 2D

Hence,

- ✓ $n \geq 4$ subcritical: Global well-posedness (WP)
- ✓ $n = 3$ critical: Global WP under null condition, by Klainerman (86), Christodoulou (86).
- ✓ $n = 2$ supercritical: Global WP under double null conditions, Alinhac (01)

Main Difficulties in 2D

The elastic system is much more involved.

- √ Two different propagation speeds
- √ Null structure is hard to use.

Main Difficulties in 2D

Progress in 3D

- √ 3D compressible case: Sideris (97, 00), Agemi (00)
- √ 3D incompressible case: Sideris and Thomases (05, 06, 07)

Main Difficulties in 2D

Open questions:

- √ Wave systems with different speeds, 2D
- √ Compressible Elasticity, 2D
- √ Incompressible Elasticity, 2D

Main Result

Our main theorem is:

Theorem 2 (L., Sideris and Zhou, 12). *For sufficiently small initial data, the 2D incompressible elasticity is almost global well-posed.*

Here *almost global* means that if the norm of the initial data is of ϵ-order, then the lifespan of the solution is at least $\exp(C_0/\epsilon)$ for some $C_0 > 0$.

Viscoelasticity

With a viscous term Δu in momentum equation:

- √ 2D: Lin-Liu-Zhang (05), L. Zhou (05)
- √ 3D: L.-Liu-Zhou (08)
- √ 2D small strain: L.-Liu-Zhou (08), L. (10, 13)
- √ Survey: Lin (12)

Main Difficulties in 2D

In 2D elasticity, the difficulties

- ✓ dimension 2, the time decay rate $\frac{1}{\sqrt{1+t}}$ is supercritical
- ✓ structure of nonlinearies
- ✓ nonlocal nature

Vector Fields in Euler

Motivated by the invariance property of this equation in Lagrangian coordinate, we have

$$\begin{cases} \partial_t \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G \\ \quad = -\nabla \Gamma^\alpha p + \sum_{\beta+\gamma=\alpha} \Gamma^\beta v \cdot \nabla \Gamma^\gamma v + \nabla \cdot (\Gamma^\gamma G \Gamma^\beta G^T) \triangleq f_\alpha, \\ \partial_t G - \nabla \cdot \Gamma^\alpha G \\ \quad = \sum_{\beta+\gamma=\alpha} \nabla \Gamma^\beta v \Gamma^\gamma G - \Gamma^\gamma v \cdot \nabla \Gamma^\beta G \triangleq g_\alpha, \\ \nabla \cdot \Gamma^\alpha v = 0. \end{cases}$$

Here Γ be any of

$$\{\partial_t, \partial_1, \partial_2, \Omega, S\}.$$

Proof

The modified rotation operator:

$$\Omega f = \begin{cases} \partial_\theta f, & \text{f scalar,} \\ \partial_\theta f + A f, & \text{f vector,} \\ \partial_\theta f + [A, f], & \text{f matrix.} \end{cases}$$

We often use the decomposition:

$$\nabla = \omega \partial_r + r^{-1} \omega^\perp \partial_\theta.$$

Proof

Based on the structures, we have:

$$\nabla \cdot \Gamma^\alpha G^\top = 0$$

and

$$\nabla^\perp \cdot \Gamma^\alpha G = h_\alpha,$$

where

$$(h_\alpha)_i = \sum_{\beta+\gamma=\alpha} \left[\Gamma^\beta G_{m1}\partial_m \Gamma^\gamma G_{i2} - \Gamma^\beta G_{m2}\partial_m \Gamma^\gamma G_{i1}\right].$$

Proof

Define the generalized energy by

$$E_k(t) = \sum_{|\alpha| \le k} \|\Gamma^\alpha(v, G)\|_{L^2}^2. \qquad (2)$$

We also define the weighted energy norm

$$X_k(t) = \sum_{|\alpha| \le k-1} \| < t - r > \nabla \Gamma^\alpha(v, G)\|_{L^2}. \qquad (3)$$

Proof

Structures:

✓ Due to the incompressibility $\nabla \cdot v = 0 = \nabla \cdot F^T$, the following are good unknowns near the light cone $r = t$:

$$v \cdot \omega, \quad G^T\omega \quad (\omega = x/r).$$

Proof

Structures:

✓ By identity (L.-Liu-Zhou, 08)

$$\partial_j G_{ik} - \partial_k G_{ij} = G_{mk}\partial_m G_{ij} - G_{mj}\partial_m G_{ik},$$

the following is a good unknown near the light cone $r = t$:

$$G\omega^{\perp}.$$

Proof

Structures: An extra intrinsic good unknown is

$$v + G\omega$$

This can be seen via Alinhac's ghost weight method, which was the first time to be applied for nonlocal problem.

Proof

The pressure satisfies null condition.

Lemma 3 (Estimate of pressure)**.** *We have*

$$\|\nabla\Gamma^\alpha p\|_{L^2} \lesssim \|f_\alpha\|_{L^2}$$

$$\|\nabla\Gamma^\alpha p\|_{L^2} \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\le|\gamma|}} \|\partial_j\Gamma^\beta v_i\Gamma^\gamma v_j - \partial_j\Gamma^\beta G_{ik}\Gamma^\gamma G_{jk}\|_{L^2},$$

for all $|\alpha| \le k-1.$

Proof

Lemma 4 (Structures). *Define*

$$\begin{cases} L_k = \sum_{|\alpha|\le k}\left[|\Gamma^\alpha v| + |\Gamma^\alpha G|\right], \\ N_k = \sum_{|\alpha|\le k-1}\left[t\big(|f_\alpha| + |g_\alpha| \right. \\ \qquad\qquad \left. +|\nabla\Gamma^\alpha p|\big) + (t+r)|h_\alpha|\right]. \end{cases}$$

Then for all $|\alpha| \le k-1$ *(First two B-T-L),*

$$r|\partial_r\Gamma^\alpha v\cdot\omega| \lesssim L_k$$

$$r|\partial_r\Gamma^\alpha G^\top\omega| \lesssim L_k$$

$$r|\partial_r\Gamma^\alpha G\omega^\perp| \lesssim L_k + N_k.$$

Proof

Lemma 5 (Better Decay of Good Unknowns near Light Cone)**.** *For* $|\alpha| \le k-2$, *we have*

$$\|r\Gamma^\alpha v \cdot \omega\|_{L^\infty} + \|r\Gamma^\alpha G^\top \omega\|_{L^\infty} \lesssim E_{|\alpha|+2}^{1/2}.$$

Proof. Sobolev imbedding on sphere + incompressibility. □

Proof

Lemma 6 (Structures). *Recall that*

$$\begin{cases} L_k = \sum_{|\alpha|\le k} \big[|\Gamma^\alpha v| + |\Gamma^\alpha G|\big], \\ N_k = \sum_{|\alpha|\le k-1} \Big[t\big(|f_\alpha| + |g_\alpha| \\ \qquad\qquad + |\nabla\Gamma^\alpha p|\big) + (t+r)|h_\alpha|\Big]. \end{cases}$$

For all $|\alpha| \le k-1$,

$$(t \pm r)|\nabla\Gamma^\alpha v \pm \nabla\cdot\Gamma^\alpha G \otimes \omega| \lesssim L_k + N_k.$$

Proof

Take a look at the proof which seems not transparent:
Using $S = t\partial_t + r\partial_r$ and the equation:

$$t\nabla\Gamma^\alpha v + r\partial_r\Gamma^\alpha G = S\Gamma^\alpha G - tg_\alpha$$
$$t\nabla\cdot\Gamma^\alpha G + r\partial_r\Gamma^\alpha v = S\Gamma^\alpha v - tf_\alpha + t\nabla\Gamma^\alpha p.$$

This is rearranged as follows:

$$\begin{aligned} t\nabla\Gamma^\alpha v + r\nabla\cdot\Gamma^\alpha G\otimes\omega &= r[\nabla\cdot\Gamma^\alpha G\otimes\omega - \partial_r\Gamma^\alpha G] \\ &\quad + S\Gamma^\alpha G - tg_\alpha \\ t\nabla\cdot\Gamma^\alpha G\otimes\omega + r\nabla\Gamma^\alpha v &= r[\nabla\Gamma^\alpha v - \partial_r\Gamma^\alpha v\otimes\omega] \\ &\quad + [S\Gamma^\alpha v - tf_\alpha + t\nabla\Gamma^\alpha p]\otimes\omega. \end{aligned}$$

Proof

Lemma 7 (Estimate of Nonlinearities Using Weighted Energy). *We have*

$$\|N_k(t)\|_{L^2} \lesssim E_k(t) + E_k(t)^{1/2} X_k(t)^{1/2}.$$

Proof. Away from the light cone, using weighted energy. Near the light cone, using the better estimate for good unknowns. □

Proof

Lemma 8 (Estimate of Weighted Energy). *If* $E_k(t) \ll 1$, *then* $X_k(t) \lesssim E_k(t)^{1/2}$.

Proof: By structures, the main contribution of $X_k(t)^2$ is E_k and

$$\sum_{|\alpha| \le k-1} [\|(t-r)\nabla\Gamma^\alpha v\|_{L^2}^2 + \|(t-r)\nabla \cdot \Gamma^\alpha G\|_{L^2}^2.$$

Proof

Then use structures and the decomposition:

$$\nabla\Gamma^\alpha v = \quad \tfrac{1}{2}[\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega] + \tfrac{1}{2}[\nabla\Gamma^\alpha v - \nabla\cdot\Gamma^\alpha G\otimes\omega] \tag{-4}$$

and

$$\nabla\cdot\Gamma^\alpha G = \tfrac{1}{2}[\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega]\omega - \tfrac{1}{2}[\nabla\Gamma^\alpha v - \nabla\cdot\Gamma^\alpha G\otimes\omega]\omega,$$

Proof

Lemma 9 (Further Better Decay of Good Unknowns near Light Cone)**.** *Let* $k \geq 4$, $E_k \ll 1$, $\omega = x/|x|$. *Then we have*

$$\|r(\partial_r\Gamma^\alpha v + \partial_r\Gamma^\alpha G\omega)\|_{L^2} + \|r\partial_r\Gamma^\alpha G\omega^\perp\|_{L^2} \lesssim E^{1/2}_{|\alpha|+1},$$

$$\|r(\Gamma^\alpha v + \Gamma^\alpha G\omega)\|_{L^\infty} + \|r\Gamma^\alpha G\omega^\perp\|_{L^\infty} \lesssim E^{1/2}_{|\alpha|+1}.$$

Proof. Using the equations and structures. □

Remark 10. This is as good as linear wave equations.

Proof

Now we are ready to derive a critical energy estimate near the light cone, based on a very delicate estimate on pressure and nonlinearities, and the application of a ghost weight method by Alinhac.
The critical energy estimate away from the light cone is due to the Klainerman-Sideris's weighted L^2 energy estimate.

Proof

The final estimate:

$$\widetilde{E}_k'(t) \leq C_0(1+t)^{-1}\widetilde{E}_k(t)^{3/2}, \quad 0 \leq t < T.$$

Here $E_k \sim \widetilde{E}_k$. This implies that $E_k(t)$ remains bounded by $2\epsilon^2$ on a time interval of order $T \sim \exp(C_0/\epsilon)$.

致谢

Thank you very much!!

谢 谢 关 注 !!