Dynamical generalizations of the Prime Number Theorem and disjointness of additive and multiplicative semigroup actions

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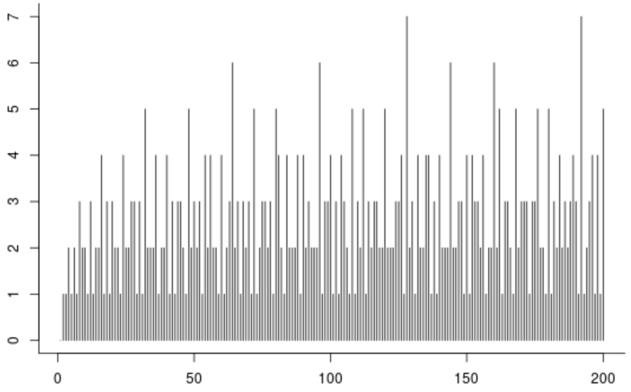
History and Background

Let $\Omega(n)$ denote the number of prime factors of n (when counted with multiplicities). For example, $\Omega(1) = 0$, $\Omega(p) = 1$, $\Omega(pq) = \Omega(p^2) = 2$, $\Omega(p_1^{e_1} \cdots p_k^{e_k}) = e_1 + \ldots + e_k$.

The following is a central question in multiplicative number theory:

Question

What is the distribution of the values of $\Omega(n)$.



Heuristics:

- The distribution of the values of $\Omega(n)$ follows no notable pattern. It appears to be random.
- Knowing $\Omega(n-1), \Omega(n-2), \ldots, \Omega(n-m)$ does not allow us to predict $\Omega(n)$.

Some Classical Results in Multiplicative Number Theory

The study of the distribution of the values of $\Omega(n)$ has a long and rich history and is closely related to fundamental questions about the prime numbers.

The natural density of a set $A \subset \mathbb{N}$ is defined as $d(A) = \lim_{N\to\infty} |\{1 \leqslant n \leqslant N : n \in A\}|/N$.

The following is a well-known equivalent form of the Prime Number Theorem.

Prime Number Theorem (von Mangoldt 1897, Landau 1911)

The sets $\{n \in \mathbb{N} : \Omega(n) \text{ is even}\}$ and $\{n \in \mathbb{N} : \Omega(n) \text{ is odd}\}$ have natural density $1/2$.

Pillai-Selberg Theorem (Pillai 1940, Selberg 1939)

For all $m \in \mathbb{N}$ and $r \in \{0, \ldots, m-1\}$ the set $\{n \in \mathbb{N} : \Omega(n) \equiv r \bmod m\}$ has natural density $1/m$.

A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is said to be uniformly distributed mod 1 if for any continuous $f\colon [0,1) \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) \;=\; \int_0^1 f(x)\,dx.$$

Erdős-Delange Theorem (Erdős 1946, Delange 1958)

For all irrational α the sequence $\Omega(n)\alpha$, $n \in \mathbb{N}$, is uniformly distributed mod 1.

A Dynamical Generalization of the Prime Number Theorem

1^{st} Main Result

Let X be a compact metric space and $T: X \to X$ a continuous map. Since

$$T^m \circ T^n = T^{m+n}, \qquad \forall m, n \in \mathbb{N},$$

the transformation T naturally induces an action of $(\mathbb{N}, +)$ on X. We call (X, T) an additive topological dynamical system. Every additive topological dynamical system (X, T) possesses at least one T-invariant Borel probability measure. If (X, T) admits only one such measure then the system is called uniquely ergodic.

Theorem A (Bergelson-R. 2020)

Let (X, μ, T) be uniquely ergodic. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\big(T^{\Omega(n)}x\big) \;=\; \int f \, d\mu$$

for every $x \in X$ and $f \in C(X)$.

One can interpret Theorem A as saying that for any uniquely ergodic system (X, T) and any point $x \in X$ the orbit $T^{\Omega(n)}x$ is uniformly distributed in the space X.

- Theorem A applied to rotation on two points recovers the Prime Number Theorem.

  Proof. Let $X = \{0,1\}$ and $T\colon x \mapsto x+1 \bmod 2$.This system is uniquely ergodic, with unique invariant measure μ given by $\mu(\{0\}) = \mu(\{1\}) = 1/2$.Let $f\colon \{0,1\} \to \mathbb{R}$ be defined as $f(0)=1$ and $f(1)=0$, and take $x=0$.Then

  $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\left(T^{\Omega(n)}x\right) = d(\{n : \Omega(n) \text{ is even}\}).$$

  Since $\int f\, d\mu = 1/2$, it follows from Theorem A that $d(\{n : \Omega(n) \text{ is even}\}) = 1/2$. □

- Theorem A applied to rotation on m points recovers the Pillai-Selberg Theorem.

  Proof. Let $X = \{0,1,\ldots,m-1\}$ and $T\colon x \mapsto x+1 \bmod m$. Take $f = 1_{\{r\}}$ and $x=0$. Then

  $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\left(T^{\Omega(n)}x\right) = d(\{n : \Omega(n) \equiv r \bmod m\}).$$

  It now follows from Theorem A that $d(\{n : \Omega(n) \equiv r \bmod m\}) = 1/m$. □

- Theorem A applied to irrational rotations on the circle recovers the Erdős-Delange Theorem.

  Proof. Let $X = \mathbb{R}/\mathbb{Z}$ and $T\colon x \mapsto x+\alpha \bmod 1$. If α is irrational then (X,T) is uniquely ergodic. Let $x=0$ and, for $h \in \mathbb{Z}\backslash\{0\}$, let $f(x) = e^{2\pi i h x}$. By Theorem A,

  $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h \Omega(n)\alpha} = 0, \qquad \forall h \in \mathbb{Z}\backslash\{0\}.$$

  In view of Weyl's equidistribution criterion, this is equivalent to the assertion that $\Omega(n)\alpha$, $n \in \mathbb{N}$, is uniformly distributed mod 1. □

Applications of Theorem A

Theorem (DeKoninck-Katai 2015)

If α is a non-Liouville number then the sequence $\Omega(n)^2\alpha$, $n \in \mathbb{N}$, is uniformly distributed mod 1?

- Theorem A applied to unipotent affine transformations on tori yields the following polynomial extensions of the Erdős-Delange Theorem, which includes an extension of the DeKoninck-Katai Theorem to all irrational α as a special case.

Corollary of Theorem A

Let $Q(n) = c_k n^k + \ldots + c_1 n + c_0$. Then $Q(\Omega(n))$, $n \in \mathbb{N}$, is uniformly distributed mod 1 if and only if at least one of the coefficients $c_1, \ldots, c_k$ is irrational.

- Theorem A applied to certain constant length substitution systems gives an analogue of a classical result of Gelfond, providing new insight into the digit expansion of $\Omega(n)$ in base q.

Corollary of Theorem A

Let $s_q(n)$ denote the sum of digits of n in base q. If m and $q-1$ are coprime then for all $r \in \{0, 1, \ldots, m-1\}$ the set of n for which $s_q(\Omega(n)) \equiv r \bmod m$ has asymptotic density $1/m$.

A word about the proof of Theorem A

Our proof of Theorem A is elementary and self-contained. In particular, we don't use any tools or results from analytic number theory.

Multiplicative Systems

Multiplicative systems

Recall, a additive topological dynamical system is a pair (X, T) where X is a compact metric space and T is continuous transformation on X, which we think of as an $(\mathbb{N}, +)$ action:

$$T^{n+m} = T^n \circ T^m, \qquad \forall n, m \in \mathbb{N}.$$

A multiplicative topological dynamical system is a pair (Y, S) where Y is a compact metric space and $S = (S_n)_{n\in\mathbb{N}}$ is an action of $(\mathbb{N}, \cdot)$ by continuous maps on Y, i.e.,

$$S_{nm} = S_n \circ S_m, \qquad \forall n, m \in \mathbb{N}.$$

Example

Since Ω has the property that $\Omega(nm) = \Omega(n) + \Omega(m)$ for all $n, m \in \mathbb{N}$, it turns any action of $(\mathbb{N}, +)$ into an action of $(\mathbb{N}, \cdot)$:

$$T^{\Omega(nm)} = T^{\Omega(n)+\Omega(m)} = T^{\Omega(n)} \circ T^{\Omega(m)}, \qquad \forall n, m \in \mathbb{N}.$$

Hence for any additive topological dynamical system (X, T), the pair (X, T^Ω) is a multiplicative topological dynamical system, where we use T^Ω to denote $(T^{\Omega(n)})_{n\in\mathbb{N}}$.

Example

Any completely multiplicative function $f\colon \mathbb{N} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ induces a natural action of $(\mathbb{N}, \cdot)$ on S^1 via $S_n(z) = f(n)z$ for all $n \in \mathbb{N}$ and $z \in S^1$.

2^{nd} Main Theorem

Theorem A (Bergelson-R. 2020)

Let (X, μ, T) be uniquely ergodic. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\big(T^{\Omega(n)}x\big) \;=\; \int f \, d\mu$$

for all $x \in X$ and $f \in C(X)$.

Question: Does Theorem A remain true if (X, T^{Ω}) is replaced by more general multiplicative systems (Y, S)?

Definition

We call a multiplicative topological dynamical system (Y, S) finitely generated if $\{S_p : p \text{ prime}\}$ is finite.

Theorem B (Bergelson-R. 2020)

Let (Y, ν, S) be finitely generated and strongly uniquely ergodic[1]. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g\big(S_n y\big) \;=\; \int g \, d\nu$$

for all $y \in Y$ and $g \in C(Y)$

[1] Slight strengthening of unique ergodicity for multiplicative systems

Disjointness of additive and multiplicative Systems

Sarnak's Liouville disjointness conjecture

Recall that the Liouville function is $\boldsymbol{\lambda}(n) = (-1)^{\Omega(n)}$.

Liouville disjointness conjecture

For any zero entropy additive topological dynamical system (X, T) we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)\boldsymbol{\lambda}(n) \;=\; 0$$

for all $x \in X$ and $f \in C(X)$,

We have learned from Theorems A and B that a natural generalization of $\boldsymbol{\lambda}(n)$ are sequences of the form $g(S_n y)$ coming from a multiplicative topological dynamical system (Y, S).

Question

If (X, T) is an additive topological dynamical system and (Y, S) is a multiplicative topological dynamical system, then what can be said about

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)g(S_n y),$$

where $x \in X$, $f \in C(X)$, $y \in Y$, and $g \in C(Y)$?

We call two bounded arithmetic functions $a, b\colon \mathbb{N} \to \mathbb{C}$ asymptotically independent if

$$\lim_{N\to\infty} \left[\frac{1}{N}\sum_{n=1}^{N} a(n)\overline{b(n)} - \left(\frac{1}{N}\sum_{n=1}^{N} a(n)\right)\cdot\left(\frac{1}{N}\sum_{n=1}^{N} \overline{b(n)}\right)\right] = 0. \tag{1}$$

Note that the Liouville disjointness conjecture says that $a(n) = f(T^n x)$ and $b(n) = \boldsymbol{\lambda}(n)$ are asymptotically independent.

Definition

Let (X, T) be an additive topological dynamical system and (Y, S) a multiplicative topological dynamical system. We call (X, T) and (Y, S) disjoint if for all $x \in X$, $f \in C(x)$, $y \in Y$, and $g \in C(Y)$ the sequences $a(n) = f(T^n x)$ and $b(n) = g(S_n y)$ are asymptotically independent.

Consider the multiplicative system (Y, S), where $Y = \{0, 1\}$ and $S_n(x) = x + \Omega(n) \bmod 2$ for all $n \in \mathbb{N}$. We refer to this system as multiplicative rotation on two points.

Liouville disjointness conjecture reformulated

Multiplicative rotation on two points is disjoint from every zero entropy additive topological dynamical system.

Our Conjecture

Heuristic

If (X, T) is a "low complexity" additive topological dynamical system and (Y, S) a "low complexity" multiplicative topological dynamical system and there are no "local obstructions", then (X, T) and (Y, S) are disjoint.

Let us call an additive topological dynamical system (X, T) aperiodic if for all $f \in C(X)$ and $x \in X$ the sequence $a(n) = f(T^n x)$ is asymptotically independent from every periodic sequence.

Let us call an multiplicative topological dynamical system (Y, S) aperiodic if for all $g \in C(Y)$ and $y \in Y$ the sequence $b(n) = g(S_n x)$ is asymptotically independent from every periodic sequence.

Conjecture 1

If (X, T) is a zero entropy additive topological dynamical system and (Y, S) a finitely generated multiplicative topological dynamical system and either (X, T) or (Y, S) is aperiodic, then (X, T) and (Y, S) are disjoint.

Theorem C (Bergelson-R. 2020)

Conjecture 1 holds when (X, T) is a nilsystem.

Theorem D (Bergelson-R. 2020)

Conjecture 1 holds when (X, T) is a horocycle flow.

Proof of Theorem A

Theorem A (Bergelson-R. 2020)

Let (X, μ, T) be uniquely ergodic. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(T^{\Omega(n)}x\right) = \int f \, d\mu$$

for all $x \in X$ and $f \in C(X)$.

Main technical result from which Theorem A follows:

Theorem 1

For any bounded sequence $a\colon \mathbb{N} \to \mathbb{C}$ we have

$$\frac{1}{N} \sum_{n=1}^{N} a(\Omega(n)+1) = \frac{1}{N} \sum_{n=1}^{N} a(\Omega(n)) + \mathrm{o}_{N\to\infty}(1).$$

- Theorem 1 applied to $a(n) = (-1)^n \implies$ Prime Number Theorem
- Theorem 1 applied to $a(n) = \zeta^n$ where ζ is a root of unity $\implies$ Pillai-Selberg Theorem
- Theorem 1 applied to $a(n) = e^{2\pi n\alpha} \implies$ Erdős-Delange Theorem
- Theorem 1 applied to $a(n) = f(T^n x) \implies$ Theorem A.

For a finite and non-empty set $B \subset \mathbb{N}$ and a function $a\colon B \to \mathbb{C}$ we denote the Cesàro average of a over B and the logarithmic average of a over B respectively by

$$\mathop{\mathbb{E}}_{n\in B} a(n) := \frac{1}{|B|}\sum_{n\in B} a(n) \qquad \text{and} \qquad \mathop{\mathbb{E}^{\log}}_{n\in B} a(n) := \frac{\sum_{n\in B} a(n)/n}{\sum_{n\in B} 1/n}.$$

We will write $[N]$ for $\{1,\ldots,N\}$.

A well-known (and not hard to prove) fact from number theory asserts that for "large" s and for "almost all" $n \in \mathbb{N}$ the number of primes in the interval $[s]$ that divide n is approximately equal to $\sum_{p\leqslant s} 1/p$. This can be expressed more formally as

$$\lim_{s\to\infty} \lim_{N\to\infty} \mathop{\mathbb{E}}_{n\in[N]} \left| \mathop{\mathbb{E}^{\log}}_{p\in\mathbb{P}\cap[s]} \left(1 - p1_{p|n}\right)\right| = 0, \tag{2}$$

where $1_{p|n} = 1$ if p divides n and $1_{p|n} = 0$ otherwise. An equivalent form of (2), which will be particularly useful for our purposes, states that for all bounded arithmetic functions $a\colon \mathbb{N} \to \mathbb{C}$ one has

$$\lim_{s\to\infty} \limsup_{N\to\infty} \left| \mathop{\mathbb{E}}_{n\in[N]} a(n) - \mathop{\mathbb{E}^{\log}}_{p\in\mathbb{P}\cap[s]} \mathop{\mathbb{E}}_{n\in[N/p]} a(pn)\right| = 0. \tag{3}$$

An important role in our proof of Theorem A will be played by a variant of (3), asserting that

$$\limsup_{N\to\infty} \left| \mathop{\mathbb{E}}_{n\in[N]} a(n) - \mathop{\mathbb{E}^{\log}}_{m\in B} \mathop{\mathbb{E}}_{n\in[N/m]} a(mn)\right| \leqslant \varepsilon, \tag{4}$$

for some special types of finite and non-empty subsets $B \subset \mathbb{N}$.

Proposition

Let $B \subset \mathbb{N}$ be finite and non-empty. For any arithmetic function $a\colon \mathbb{N} \to \mathbb{C}$ bounded in modulus by 1 we have

$$\limsup_{N\to\infty} \left| \mathop{\mathbb{E}}_{n\in[N]} a(n) - \mathop{\mathbb{E}^{\log}}_{m\in B} \mathop{\mathbb{E}}_{n\in[N/m]} a(mn) \right| \leqslant \left(\mathop{\mathbb{E}^{\log}}_{m\in B} \mathop{\mathbb{E}^{\log}}_{n\in B} \Phi(n,m) \right)^{1/2}, \tag{5}$$

where $\Phi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}\cup\{0\}$ is the function $\Phi(m,n) := \gcd(m,n) - 1$.

Proof.

We will show that

$$\lim_{N\to\infty} \mathop{\mathbb{E}}_{n\in[N]} \left| \mathop{\mathbb{E}^{\log}}_{m\in B} \left(1 - m1_{m|n}\right) \right|^2 = \mathop{\mathbb{E}^{\log}}_{l\in B} \mathop{\mathbb{E}^{\log}}_{m\in B} \Phi(l,m). \tag{6}$$

From this (5) follows by Cauchy-Schwarz. By expanding the square on the left hand side of (6) we get

$$\mathop{\mathbb{E}}_{n\in[N]} \left| \mathop{\mathbb{E}^{\log}}_{m\in B} \left(1 - m1_{m|n}\right) \right|^2 = 1 - 2\Sigma_1 + \Sigma_2, \tag{7}$$

where $\Sigma_1 := \mathbb{E}_{n\in[N]} \mathbb{E}^{\log}_{m\in B} m1_{m|n}$ and $\Sigma_2 := \mathbb{E}_{n\in[N]} \mathbb{E}^{\log}_{l,m\in B} (l1_{l|n})(m1_{m|n})$. Note that $\mathbb{E}_{n\in[N]} m1_{m|n} = 1 + \mathrm{O}(1/N)$ and therefore

$$\Sigma_1 = 1 + \mathrm{O}\left(\tfrac{1}{N}\right). \tag{8}$$

Similarly, since $\mathbb{E}_{n\in[N]} lm1_{l|n}1_{m|n} = \gcd(l,m) + \mathrm{O}(1/N)$, we have

$$\Sigma_2 = \mathop{\mathbb{E}^{\log}}_{l\in B} \mathop{\mathbb{E}^{\log}}_{m\in B} \gcd(l,m) + \mathrm{O}\left(\tfrac{1}{N}\right) = 1 + \mathop{\mathbb{E}^{\log}}_{l\in B} \mathop{\mathbb{E}^{\log}}_{m\in B} \Phi(l,m) + \mathrm{O}\left(\tfrac{1}{N}\right). \tag{9}$$

□

We denote by $\mathbb{P}_2$ the set of 2-almost primes, i.e., $\mathbb{P}_2 = \{n \in \mathbb{N} : \Omega(n) = 2\}$.

Lemma

For all $\varepsilon \in (0,1)$ and $\rho \in (1, 1+\varepsilon]$ there exist finite and non-empty sets $B_1, B_2 \subset \mathbb{N}$ with the following properties:

1. $B_1 \subset \mathbb{P}$ and $B_2 \subset \mathbb{P}_2$;
2. $|B_1 \cap [\rho^j, \rho^{j+1})| = |B_2 \cap [\rho^j, \rho^{j+1})|$ for all $j \in \mathbb{N} \cup \{0\}$;
3. $\mathbb{E}^{\log}_{m \in B_1} \mathbb{E}^{\log}_{n \in B_1} \Phi(m,n) \leqslant \varepsilon$ as well as $\mathbb{E}^{\log}_{m \in B_2} \mathbb{E}^{\log}_{n \in B_2} \Phi(m,n) \leqslant \varepsilon$, where $\Phi(m,n) := \gcd(m,n) - 1$.

Proof of Theorem 1.

Let $a \colon \mathbb{N} \to \mathbb{C}$ be bounded. Our goal is to show that

$$\lim_{N \to \infty} \left| \mathop{\mathbb{E}}_{n \in [N]} a(\Omega(n)+1) \; - \; \mathop{\mathbb{E}}_{n \in [N]} a(\Omega(n)) \right| \; = \; 0. \tag{10}$$

Let $\varepsilon \in (0,1)$ and $\rho \in (1, 1+\varepsilon]$ be arbitrary and find two finite sets $B_1, B_2 \subset \mathbb{N}$ satisfying conditions 1, 2, and 3 of the above Lemma. Combining the Proposition with part 3 of the Lemma gives

$$\mathop{\mathbb{E}}_{n \in [N]} a(\Omega(n)+1) \; = \; \mathop{\mathbb{E}^{\log}}_{p \in B_1} \; \mathop{\mathbb{E}}_{n \in [N/p]} a(\Omega(pn)+1) \; + \; \mathrm{O}(\varepsilon^{1/2}) \; + \; \mathrm{o}_{N \to \infty}(1) \tag{11}$$

as well as

$$\mathop{\mathbb{E}}_{n \in [N]} a(\Omega(n)) \; = \; \mathop{\mathbb{E}^{\log}}_{q \in B_2} \; \mathop{\mathbb{E}}_{n \in [N/q]} a(\Omega(qn)) \; + \; \mathrm{O}(\varepsilon^{1/2}) \; + \; \mathrm{o}_{N \to \infty}(1). \tag{12}$$

Proof of Theorem 1 (cont.)

Since B_1 is comprised only of primes, we have $a\big(\Omega(pn)+1\big) = a\big(\Omega(n)+2\big)$ for all $p \in B_1$. Similarly we have $a\big(\Omega(qn)\big) = a\big(\Omega(n)+2\big)$ for all $q \in B_2$, because B_2 is comprised only of 2-almost primes. So (13) and (14) become

$$\mathop{\mathbb{E}}_{n\in[N]} a\big(\Omega(n)+1\big) \;=\; \mathop{\mathbb{E}^{\log}}_{p\in B_1} \mathop{\mathbb{E}}_{n\in[N/p]} a\big(\Omega(n)+2\big) \;+\; \mathrm{O}(\varepsilon^{1/2}) \;+\; \mathrm{o}_{N\to\infty}(1) \tag{13}$$

as well as

$$\mathop{\mathbb{E}}_{n\in[N]} a\big(\Omega(n)\big) \;=\; \mathop{\mathbb{E}^{\log}}_{q\in B_2} \mathop{\mathbb{E}}_{n\in[N/q]} a\big(\Omega(n)+2\big) \;+\; \mathrm{O}(\varepsilon^{1/2}) \;+\; \mathrm{o}_{N\to\infty}(1). \tag{14}$$

Finally, note that if p and q belong to the same ρ-adic interval $[\rho^j, \rho^{j+1})$ then

$$\mathop{\mathbb{E}}_{n\in[N/p]} a\big(\Omega(n)+2\big) \;=\; \mathop{\mathbb{E}}_{n\in[N/q]} a\big(\Omega(n)+2\big) \;+\; \mathrm{O}(\rho-1).$$

Since B_1 and B_2 have the same cardinality when restricted to $[\rho^j, \rho^{j+1})$ for every $j \in \mathbb{N}$, we obtain

$$\left|\mathop{\mathbb{E}}_{n\in[N]} a\big(\Omega(n)+1\big) \;-\; \mathop{\mathbb{E}}_{n\in[N]} a\big(\Omega(n)\big)\right| \;=\; \mathrm{O}(\eta-1) \;+\; \mathrm{O}(\varepsilon^{1/2}) \;+\; \mathrm{o}_{N\to\infty}(1).$$

Using $\eta \in (1, 1+\varepsilon]$ and letting ε tend to 0 finishes the proof of (10). □

Thank you