Do NP-Hard Problems Require Exponential Time?

Andrew Drucker

IAS

April 8, 2014

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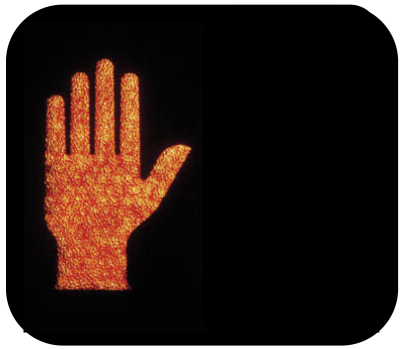

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- Want to solve NP-complete problems $\Longrightarrow$ must accept **compromise!**
- Popular approach: find **approximately-optimal** solutions.

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  - $(.5+\varepsilon)$-approx Max-LIN ($\mathbb{F}_2$): NP-Complete. [Håstad'97]

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- Sometimes we need an **exact** solution to an NP-C problem.
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(Strictly: $F(n)$'s above should be $O^*(F(n)) \triangleq F(n) \cdot |\mathit{instance}|^{O(1)}$.)

Example: Schöning's alg.

- **Given:** a k-CNF $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$.
  (each C_i an OR of $\leq k$ literals)
- **Goal:** find a satisfying solution to $\mathcal{F}$ if one exists.

Algorithm $A(\mathcal{F})$:

1. *Let $x \longleftarrow$ (random assignment).*
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- **Claim:** if $\mathcal{F} \in$ SAT, then

$$\Pr[A(\mathcal{F}) \text{finds a solution}] \geq 2^{-(1-c/k)n}.$$

$$\Longrightarrow \text{repeat for } 2^{(1-c/k)n} \text{ trials to find one w.h.p.!}$$

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- Suppose $\mathcal{F}(\mathbf{x}^*) = 1$. Let $x^t =$ state of x after t execs. of Step 2. Let

$$Y_t \quad \triangleq \quad ||x^t - \mathbf{x}^*||_1 \ .$$

- **Key fact:** if $Y_t > 0$, then $\Pr[Y_{t+1} = Y_t - 1] \ \geq \ 1/k$.
- Can lower-bound $\Pr[\min_t Y_t = 0]$ in terms of a biased random walk.

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Challenge *for complexity theory:* **explain** *the seeming differences in difficulty!*
Identify barriers *to further progress!*
Guide *search for faster algorithms!*

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- Could P $\neq$ NP conjecture imply that NP-C probs require exponential time?
- No idea. Seems hopeless!
- Influential approach: strengthen the conjecture!

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- Formally: for $k \geq 3$, define $s_k \in [0,1]$ by

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(allowing randomized algs with 1/3 error prob.)

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- $s_3 \leq s_4 \leq s_5 \leq \ldots$
- Best known: $s_3 \leq .388$, $s_4 \leq .555$, $s_k \leq 1 - \Theta(1/k)$.

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- Much stronger belief than $\mathsf{P} \neq \mathsf{NP}$.

- Payoff in explanatory power?

YES! *But, story is more complex than NP-completeness.*

- Issue: ETH studies dependence on key param.

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- Measures dimension of **search space**, not input size!

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- Consider **IND. SET** problem. Solvable in $O^*(1.23^N)$ time on N-vertex graphs.
- Can we hope for $2^{o(N)}$? Or, would that violate ETH?
- Given: $2^{o(N)}$-time alg for **IND. SET**; try to solve **3-SAT** instance $\mathcal{F}$. ($n \triangleq \#$ vars, $m \triangleq \#$clauses)
- Usual NP-C reduction: $\mathcal{F} \longrightarrow (G, k)$, where

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- $2^{o(N)}$ time alg for **IND. SET** $\Longrightarrow$ Solve **3-SAT** in time $2^{o(m)}$.

Theorem (IPZ)

Solve **k-SAT** *in time* $2^{o(m)}$ $\Longrightarrow$ *Solve* **k-SAT** *in time* $2^{o(n)}$ *!!*

- So, $2^{o(N)}$ time alg for **IND. SET** violates **ETH**.

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- Similar: **HAM PATH, DOMINATING SET, VERTEX COVER**.

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- **Planar IND. SET** problem: Solvable in $2^{O(\sqrt{N})}$ time on N-vertex planar graphs.
- Can we hope for $2^{o(\sqrt{N})}$?
- Usual NP-C reduction: 3-CNF $\mathcal{F} \longrightarrow$ (planar) (G, k), where
$$|V(G)| = \Theta(m^2) \ .$$
- Solve **Planar IND SET** in time $2^{o(\sqrt{N})} \Longrightarrow$ solve **3-SAT** in time $2^{o(m)}$.
Again, violates **ETH**!
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- Why focus on **3-SAT**? Is it WLOG?

  Could **3-SAT** be much easier than **4-SAT**??

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A stronger hypothesis

$$s_k \triangleq \inf\{\varepsilon : k\text{-SAT decidable in time } O^*(2^{\varepsilon n})\} .$$

Exponential Time Hypothesis (ETH) (IPZ'97)

$$s_3 > 0 .$$

Best known: $s_k \leq 1 - \Theta(1/k)$. Why not "go for broke?"

Strong Exp. Time Hypothesis (**SETH**) (IP'98)

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More consequences of ETH, SETH

- Many more runtime LBs shown under ETH, SETH.
- Strong power to explain dependence on natural input parameters.
- Major implications for **parametrized complexity theory**

[Downey, Fellows]; [Lokshtanov, Marx, Saurabh survey]

Parametrized problems

- Many problem instances have associated **integer parameter** — gives some indication of difficulty.

- E.g., VERTEX COVER:

  Given: (G, k)

  Decide: does G have a vertex cover of size k?

- Goal of "parametrized algorithm" design: design algs that are "fast when k is small."

Parametrized problems

- VERTEX COVER:

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  [Eisenbrand, Grandoni'04; Pătraşcu, Williams'10]

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- **Treewidth** of a graph G: $tw(G) =$ measure of "fatness" of G.

- Known: many NP-C graph problems solvable fast on low-treewidth graphs (by dynamic prog.), in time*

$$c^{tw(G)} \cdot n^{O(1)} \quad \text{for some } c.$$

*(Given a tree decomposition.)

- [Lokshtanov, Marx, Saurabh '11]: **Strong ETH** $\Longrightarrow$ some of these algorithms are **optimal!**

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How well can we approximate IND SET on n-vertex graphs in subexponential time?

Consider obtaining an r-approximation to max ind. set size, $r = r(n) = \omega(1)$.

Theorem (Chitniz, Hajiaghayi, Kortsarz'13)

Can get r-approximation in time $O^*(2^{n/r})$.

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Theorem (Aingworth, Chekuri, Indyk, Motwani'96; Roddity, Vassilevska Williams'13)

For a simple graph G on n verts, m edges, can compute 3/2-approximation to diameter(G) in (expected) time $\widetilde{O}(m\sqrt{n})$.

Theorem (Roddity, Vassilevska Williams'13)

If we can estimate diameter(G) to approx. factor $(3/2-\varepsilon)$ in time $O(m^{2-\delta})$, then **SETH** *fails.*

- **SETH** $\Longrightarrow$ Detailed info about complexity of a **poly-time** computation!

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Further afield

- [Abboud, Vassilevska Williams'14]: Improvements in certain **dynamic algorithms** for graph problems $\Rightarrow$ $\neg$SETH.
- [Bringmann, this morning]: Compute Fréchet distance in $n^{2-\varepsilon}$ time $\Rightarrow$ $\neg$SETH.
- Seems likely to see more results of this kind...

The key theorem

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k-SAT *in time* $O^*(2^{\varepsilon m})\ \forall \varepsilon > 0 \quad \Longrightarrow \quad$ **k-SAT** *in time* $O^*(2^{\varepsilon n})\ \forall \varepsilon > 0.$

$m = \#$ clauses$(\mathcal{F})$, $\qquad n = \#$ variables$(\mathcal{F})$.

- Let's see the proof ideas.
  Main challenge: for general "dense" $\mathcal{F}$, may have $m \gg n$.
- Ideal approach: give a "sparsification" reduction:

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$$m' \ , \ n' \ \leq \ O(n) \ .$$

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k-SAT *in time* $O^*(2^{\varepsilon m})\ \forall \varepsilon > 0 \quad \Longrightarrow \quad$ **k-SAT** *in time* $O^*(2^{\varepsilon n})\ \forall \varepsilon > 0.$

$m = \#$ clauses$(\mathcal{F})$, $\qquad n = \#$ variables$(\mathcal{F})$.

- Let's see the proof ideas.
  Main challenge: for general "dense" $\mathcal{F}$, may have $m \gg n$.
- Ideal approach: give a "sparsification" reduction:

$$\mathcal{F} \longrightarrow^{ptime} \mathcal{F}' \qquad SAT(\mathcal{F}) = SAT(\mathcal{F}')$$

$$m',\ n' \leq O(n)\ .$$

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$$\mathcal{F} \quad \rightarrow^{2^{o(n)}\ time} \quad \mathcal{G}^1, \mathcal{G}^2, \ldots, \mathcal{G}^s \qquad s \;=\; 2^{o(n)}$$

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Sparsification Lemma (IPZ'97)

Fix $k \geq 3, \varepsilon > 0$.
There exists a reduction $\mathcal{F} \rightarrow \mathcal{G}^1, \ldots, \mathcal{G}^s$, computable in time $O^*(2^{\varepsilon n})$, such that

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Proof of sparsification lemma (debt to D. Scheder's notes!)

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Thanks!