KPZ equation and directed polymers: exact results from the replica Bethe ansatz

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Alberto Rosso (LPTMS Orsay)
Thomas Gueudre (LPTENS)

P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)

P. Calabrese, P. Le Doussal, Phys. Rev. Letters 106 250603 (2011) and J. Stat. Mech. P06001 (2012) T. Gueudre, P. Le Doussal, EPL 100 26006 (2012).

- many models in "KPZ class" exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

- provide solution directly continuum KPZ eq./DP (at all times)

KPZ eq. is in KPZ class !

methods of integrable systems (Bethe Ansatz)+disordered systems (replica)

Outline:

- growth of 1D interfaces: KPZ equation, KPZ universality class

- random matrices largest eigenvalues: Tracy Widom universal distributions

- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition

- flat initial condition

- half-flat initial condition
  crossover flat/droplet

- KPZ in half space

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- not talk about:
  stationary initial condition

T. Inamura, T. Sasamoto
Phys. Rev. Lett. 108, 190603 (2012)

- other works/perspectives:

reviews KPZ: Corwin arXiv 1106.1596, H. Spohn..

also works by: V. Dotsenko, H. Spohn, Sasamoto

(math) Amir, Corwin, Quastel, Borodine,..

also G. Schehr, Reymenik, Ferrari, O'Connell,..

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

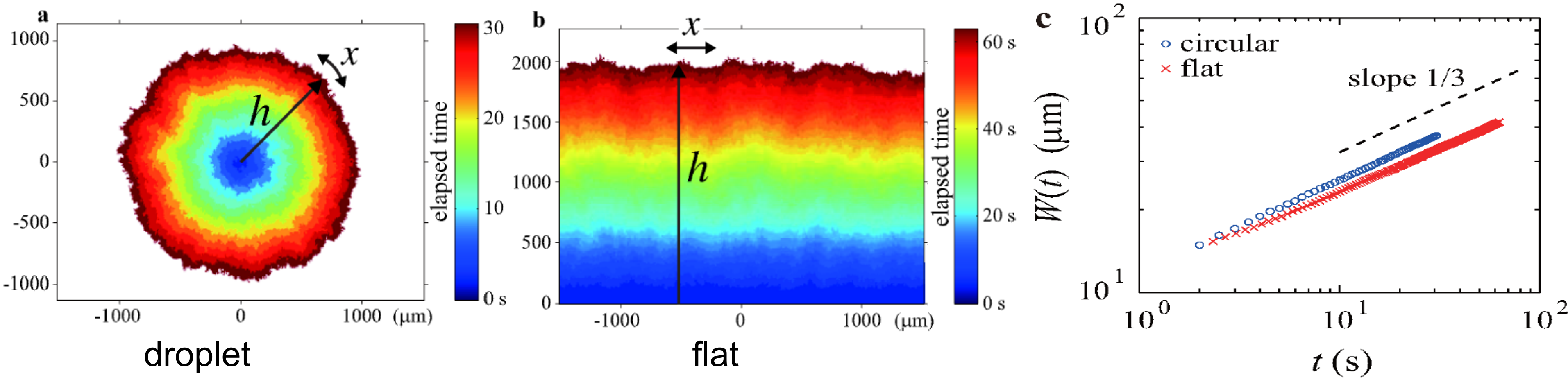
depends on some details of initial condition

flat	h(x,0) =0
wedge (droplet)	h(x,0) = - w |x|

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



droplet flat

$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t \to +\infty} v_\infty t + \chi\, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

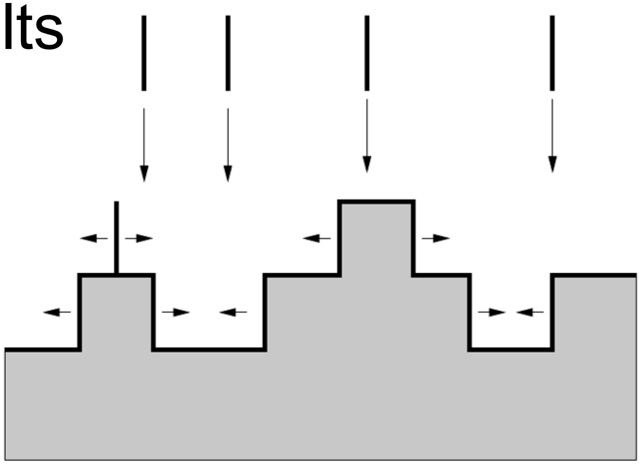
- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

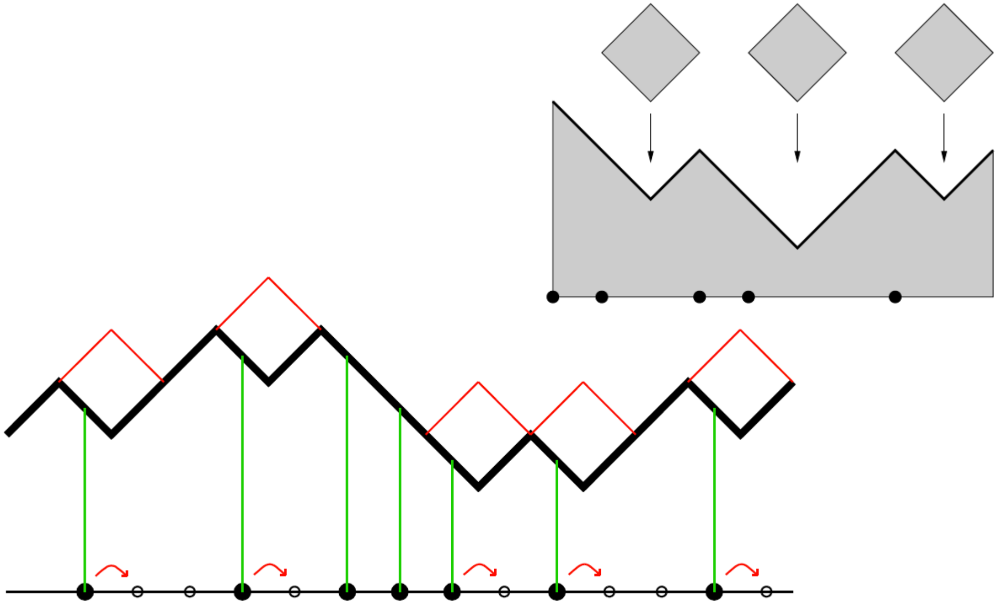
discrete models in KPZ class/exact results

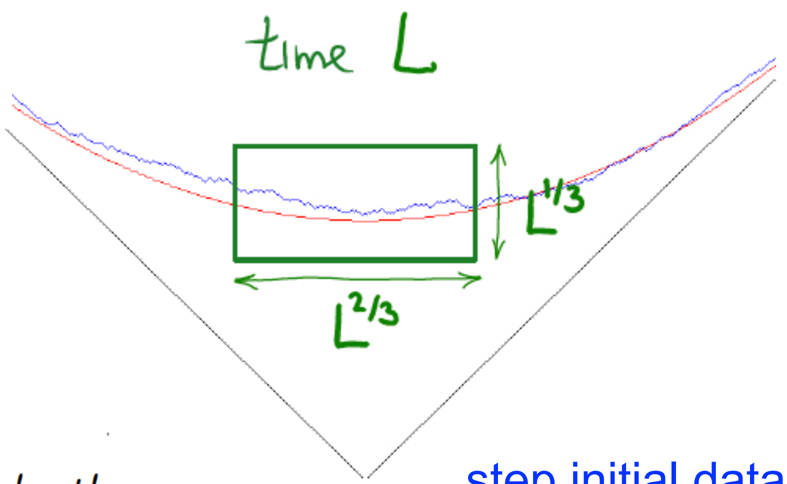
- polynuclear growth model (PNG)

Prahofer, Spohn, Baik, Rains (2000)

- totally asymmetric exclusion process (TASEP)







step initial data

Johansson (1999)

Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.

Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $\quad i = 1, ..N$

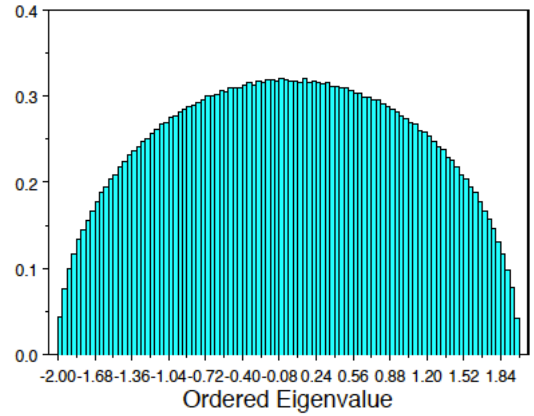
$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:

$\beta =$
- 1 (GOE) real symmetric
- 2 (GUE) hermitian
- 4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

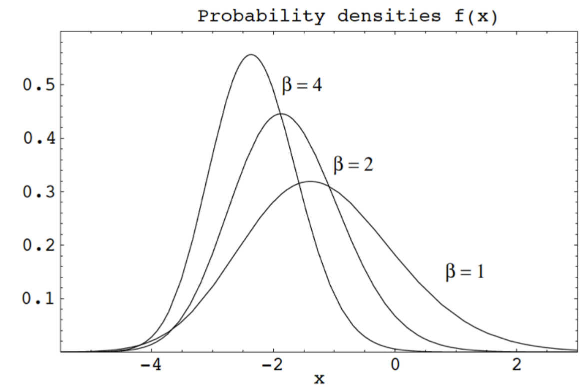
Tracy-Widom distributions (largest eigenvalue of RM)

Fredholm determinants

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$

GOE $$F_1(s) = Det[I - K_1]$$

$$K_1(x,y) = \theta(x)Ai(x+y+s)\theta(y)$$



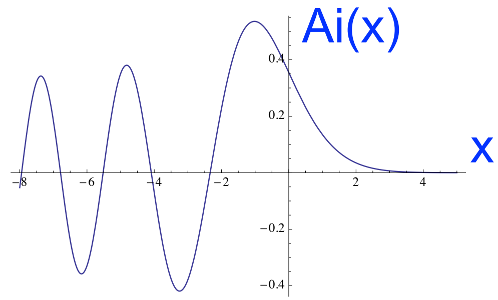
GUE $$F_2(s) = Det[I - K_2]$$

$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v)Ai(y+v)$$

Ai(x-E)

is eigenfunction E
particle linear potential



Exact results for height distributions for some discrete models in KPZ class

- PNG model

droplet IC

Baik, Deft, Johansson (1999)

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$

GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

$A_1(y)$ GOE

- similar results for TASEP

Johansson (1999), ...

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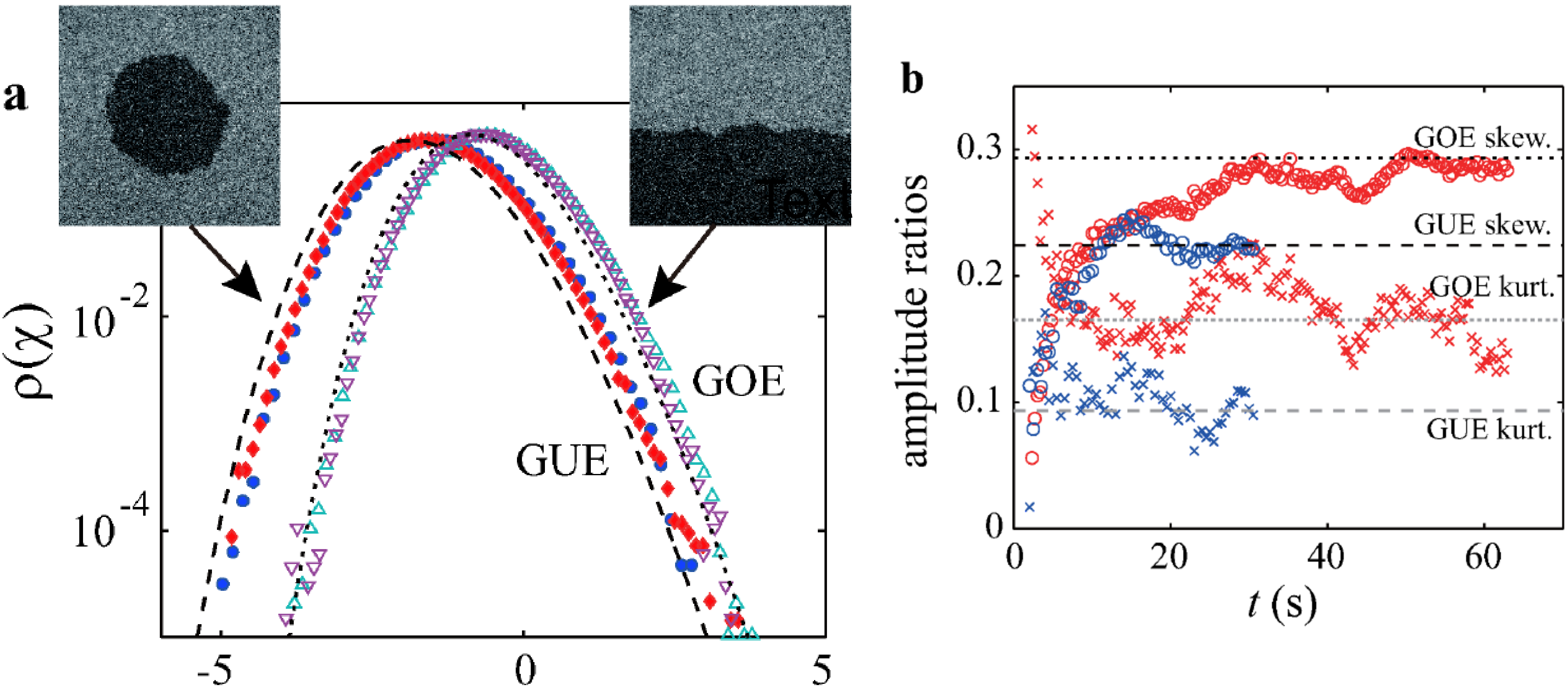
Question: is KPZ equation in KPZ class ?

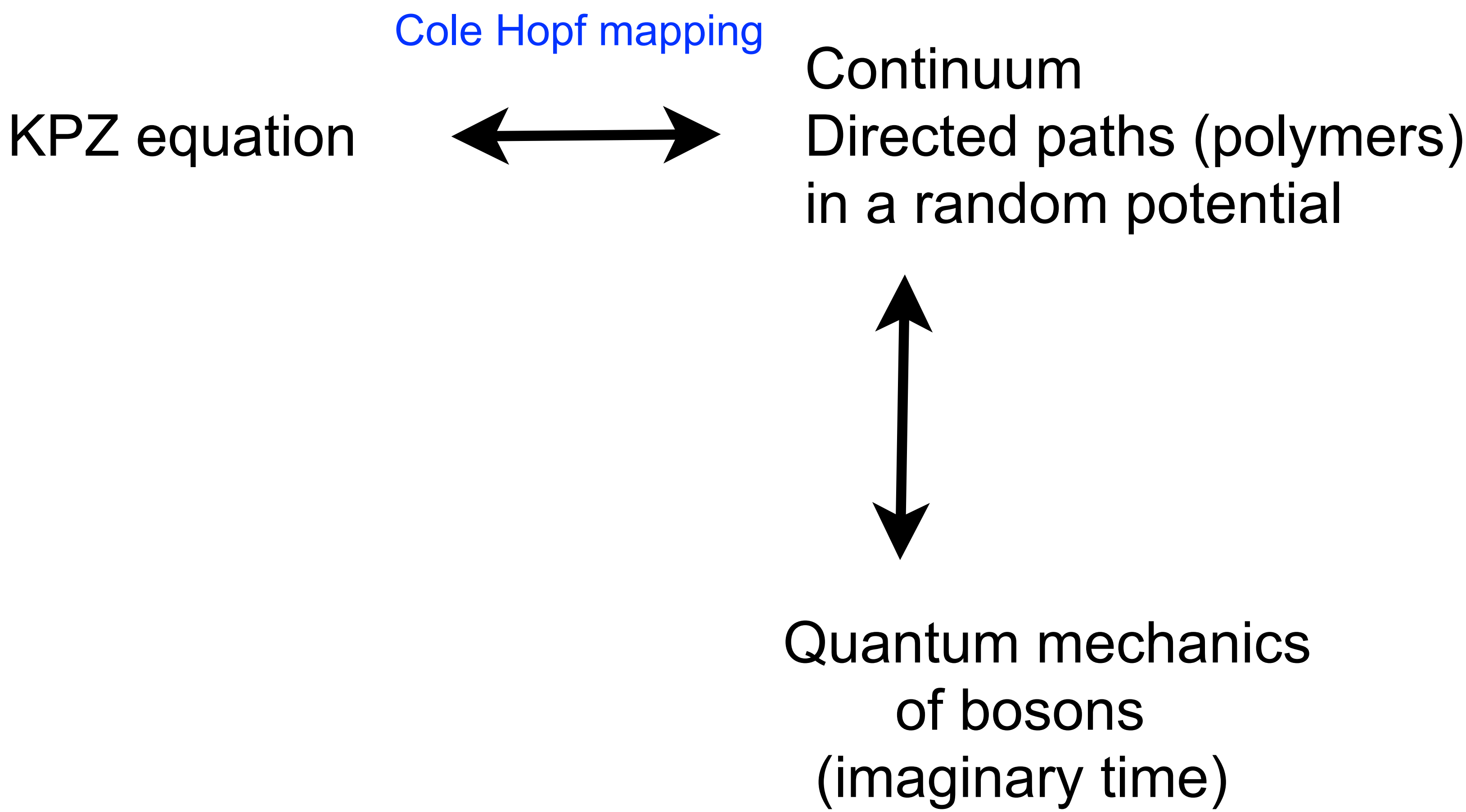
K. Takeuchi, M. Sano PRL (2010)+ H. Sasamoto, H. Spohn Nature (2011)

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$

skewness =

$$\frac{< (h- < h >)^3 >}{< (h- < h >)^2 >^{3/2}}$$




Cole Hopf mapping
KPZ equation
Continuum
Directed paths (polymers)
in a random potential
Quantum mechanics
of bosons
(imaginary time)

Continuum DP fixed endpoint/KPZ Narrow wedge (droplet)

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
  Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
  Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

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Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).

- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

Continuum DP one free endpoint/KPZ Flat (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Quastel, J. Ortmann and D. Remenik in preparation

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

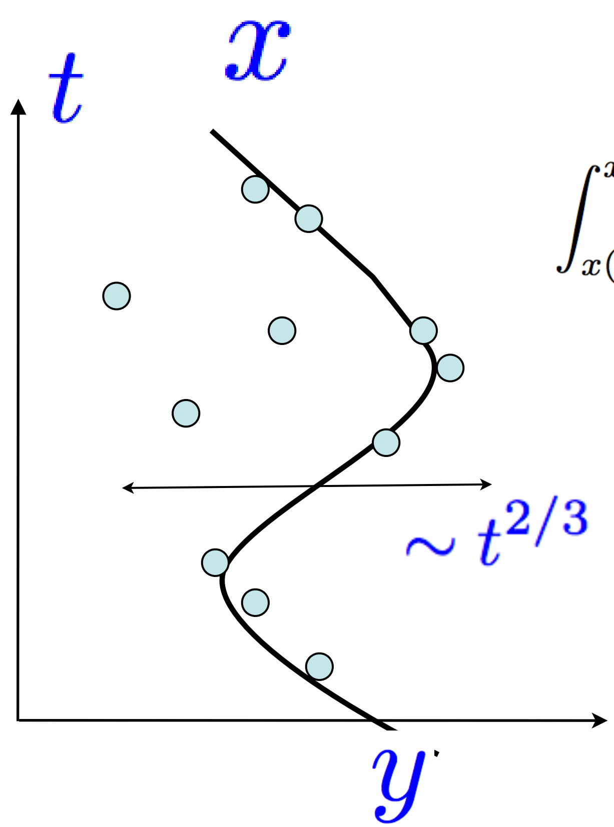
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c}\ \ \delta(t-t')\delta(x-x')$$

Feynman Kac

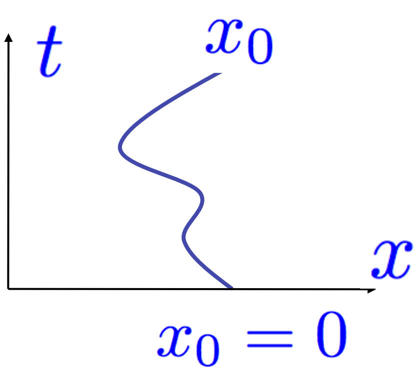
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu}h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu}h(y,t=0)}$$

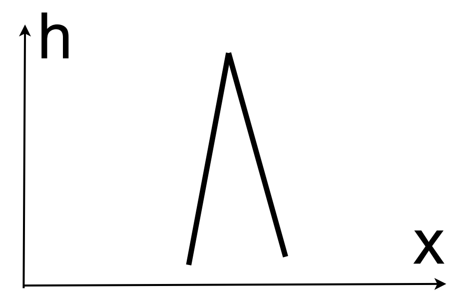
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



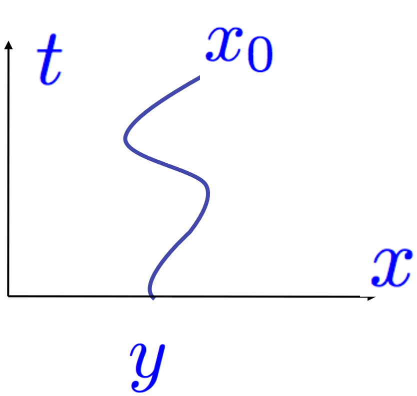
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $\overline{Z^n} = \int dZ\, Z^n P(Z) \qquad n \in \mathbb{N}$

"guess" the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

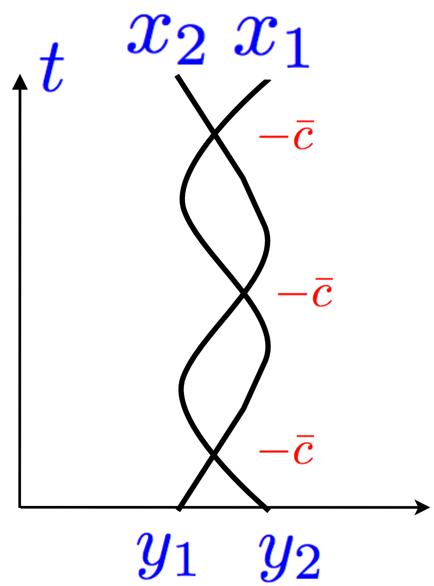
Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,t|y_1,0)..Z(x_n,t|y_n 0)} = \langle x_1,..x_n|e^{-tH_n}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

drop the tilde..



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\le i<j\le n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0 | e^{-tH_n} | x_0, .. x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

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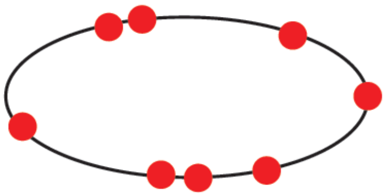
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0) \Psi_\mu(x_0..x_0) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

- flat initial condition

$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi_\mu^*(x_0, .x_0) \int_{y_1,.y_n} \Psi_\mu(y_1, .y_n) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



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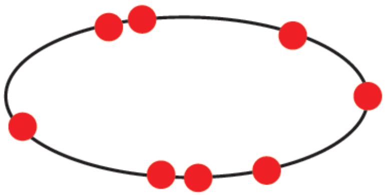

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n\ge\ell>k\ge1} \left(1 - \frac{ic\ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

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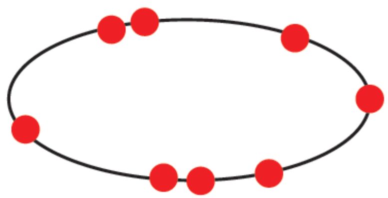

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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell\neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

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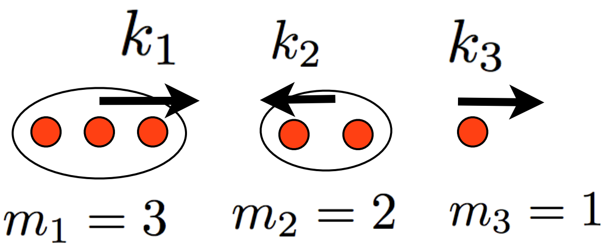
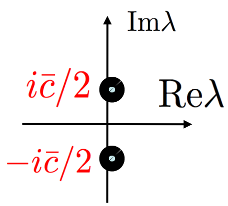
- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$E_\mu = \sum_{j=1}^{n_s}(m_j k_j^2 - \frac{\bar{c}^2}{12}m_j(m_j^2-1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_\mu \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

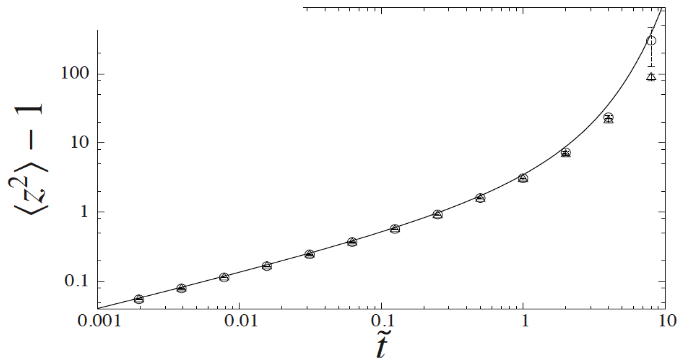
$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\ldots m_{n_s})_n} \qquad n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2 c^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda\to\infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

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what we aim to calculate= Laplace transform of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\leq i<j\leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

↑

↓

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \dots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s, x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \; det[K(v_j, v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1, v_2) = \; - \int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1 - v_2)} \; \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x) \;\; = Det[I + K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1, v_2) \equiv \theta(v_1) K(v_1, v_2) \theta(v_2)$$

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2) large time limit $\lambda = +\infty$ $\quad \frac{e^{\lambda y}}{1 + e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v - v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i-m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)}) + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\,(1; 2 - 2ik, 2 + 2ik; -z)}{\sinh\,(2\pi k)\,\Gamma\,(2 - 2ik)\,\Gamma\,(2 + 2ik)}\,, \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1 - u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)]\,.$$

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large time limit

$$\lim_{\lambda \to +\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda \to +\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) = \theta(y_1 + y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

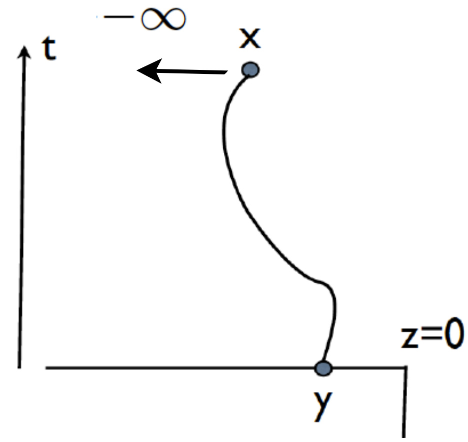
how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\rm hs}(x,t) \equiv Z_{\rm flat}(x,t)$$

$$Z_{\rm hs,w}(x,t) = \int_{-\infty}^{0} dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x) e^{wx}$$



$$\Big(\prod_{\alpha=1}^{n} \int_{-\infty}^{0} dy_\alpha e^{wy_\alpha}\Big) \Psi_\mu(y_1 \dots y_n) = \sum_P A_P G_{P\lambda}$$

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

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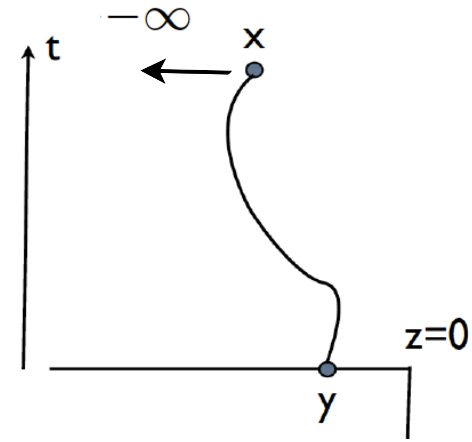

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miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w+i\lambda_\alpha)} \prod_{1\le\alpha<\beta\le n} \frac{2w+i\lambda_\alpha+i\lambda_\beta-1}{2w+i\lambda_\alpha+i\lambda_\beta}$$

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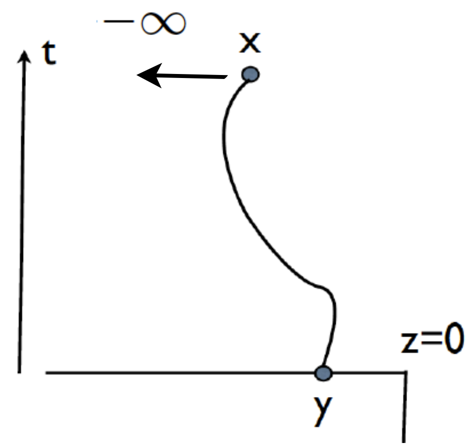

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strings:

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j+1-2a)$$

$$a = 1, ..., m_j$$

$$\int^w \Psi_\mu = n!(-2)^n \prod_{i=1}^{n_s} S^w_{m_i,k_i} \prod_{1\le i<j\le n_s} D^w_{m_i,k_i,m_j,k_j}$$

$$D^w_{m_1,k_1,m_2,k_2} = (-1)^{m_2} \frac{\Gamma(1-z+\frac{m_1+m_2}{2})\Gamma(z+\frac{m_1-m_2}{2})}{\Gamma(1-z+\frac{m_1-m_2}{2})\Gamma(z+\frac{m_1+m_2}{2})} \qquad z = ik_1 + ik_2 + 2w$$

$$S^w_{m,k} = \frac{(-1)^m \Gamma(z)}{\Gamma(z+m)} \qquad z = 2ik + 2w.$$

in double limit $\lim_{x\to-\infty, w\to 0}$

$$S^w_{m_i,k_i} \to \frac{(-1)^{m_i}}{2\Gamma(m_i)} 2\pi\delta(k_i) + s^0_{m_i,k_i}$$

expand the product $\prod_i S_i \prod_{i<j} D_{ij}$

each momentum k_ℓ appears only in exactly one pole

$$D^w_{m_i,k_i,m_j,k_j} \to (-1)^{m_i} m_i \delta_{m_i,m_j}\ 2\pi\delta(k_i+k_j) + d^w_{m_i,k_i,m_j,k_j}$$

pairing of string momenta and pfaffian structure emerges second method:

calculate: $$\prod_{\alpha=1}^{n}\int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n)$$

use Bethe equations: $$e^{i\lambda_j L} = \prod_{\ell\neq j}\frac{\lambda_j-\lambda_\ell-i\bar{c}}{\lambda_j-\lambda_\ell+i\bar{c}}$$

=> integral vanishes for generic state

oberve: requires pairs opposite rapidities second method:

calculate: $$\prod_{\alpha=1}^{n}\int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n) = \langle\Phi_0|\mu\rangle$$

is the overlap with uniform state

$$\Phi_0(x_1,..x_n) = 1$$

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Can be seen as interaction quench in Lieb-Liniger model with initial state BEC (c=0)

de Nardis et al., arXiv 1308.4310

overlap is non zero only for parity invariant states $\{\lambda_1,-\lambda_1,..,\lambda_{n/2},-\lambda_{n/2}\}$

infinity of conserved charges $$Q_p = \sum_{\alpha=1}^{n}\lambda_\alpha^p$$

second method: calculate: $$\prod_{\alpha=1}^{n}\int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n) = \langle \Phi_0|\mu\rangle$$

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$$\langle\Phi_0|\mu\rangle = n!c^{n/2}\prod_{\alpha=1}^{n/2}\frac{1}{\lambda_\alpha^2}\prod_{1\leq\alpha<\beta\leq n/2}\frac{(\lambda_\alpha-\lambda_\beta)^2+c^2}{(\lambda_\alpha-\lambda_\beta)^2}\frac{(\lambda_\alpha+\lambda_\beta)^2+c^2}{(\lambda_\alpha+\lambda_\beta)^2}\times\det G^Q.$$

$$G^Q_{\alpha\beta} = \delta_{\alpha\beta}(L+\sum_{\gamma=1}^{n/2}K^Q(\lambda_\alpha,\lambda_\gamma)) - K^Q(\lambda_\alpha,\lambda_\beta)$$

$$K^Q(x,y) = K(x-y)+K(x+y),$$

$$K(x) = \frac{2c}{x^2+c^2}.$$

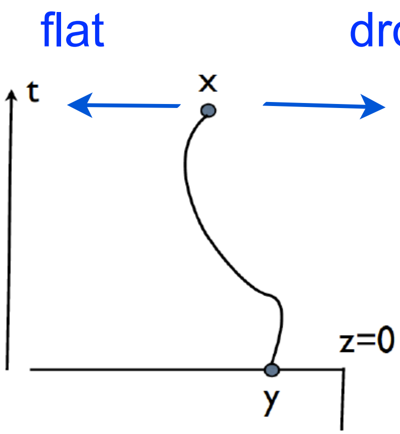
Brockmann, arXiv1402.1471.

P. Calabrese, P. Le Doussal, arXiv 1402.1278

large L limit, overlap for strings

partially recovers the moments Z^n for flat

Back to half-flat: crossover from droplet to flat



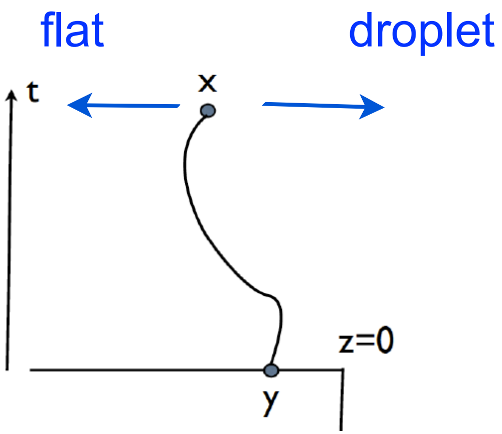
$$x = \lambda^2 \tilde{x} = (t/4)^{2/3} \tilde{x}$$

$$w = \frac{\tilde{w}}{\lambda} = \frac{\tilde{w}}{(t/4)^{1/3}}$$

$$g_\lambda(s) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, s)$$

$$Z(n_s, s) = \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} 2^{m_j+1} \int \frac{dk_j}{2\pi} dy_j \frac{(-1)^{m_j} \Gamma(\frac{2ik_j+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik_j+2\tilde{w}}{\lambda} + m_j)} Ai(y_j + 4k_j^2 + s) e^{\lambda m_j (y_j - ik_j \tilde{x})}$$

$$\times \prod_{1 \le i < j \le n_s} D^w_{m_i, k_i/\lambda, m_j, k_j/\lambda} \times det[\frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j}]_{n_s \times n_s}$$



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$$\times \prod_{1 \le i < j \le n_s} D^w_{m_i, k_i/\lambda, m_j, k_j/\lambda} \times det[\frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j}]_{n_s \times n_s}$$

simplification in large time limit $\lambda \to +\infty$ assume $D^w_{m_i, k_i/\lambda, m_j, k_j/\lambda} \to 1$

$$Z(n_s, s) = \prod_{j=1}^{n_s} \int_{v_j > 0} \det M(v_i, v_j)|_{n_s \times n_s}$$

$$M(v_1, v_2) = \int \frac{dk}{2\pi} dy Ai(y + 4k^2 + ik\tilde{x} + v_1 + v_2 + s) e^{-2ik(v_1 - v_2)} \phi_\lambda(k, y)$$

$$\phi_\lambda(k, y) = \sum_{m=1}^{\infty} \frac{(-1)^m 2^{m+1} \Gamma(\frac{2ik+2\tilde{w}}{\lambda})}{\Gamma(\frac{2ik+2\tilde{w}}{\lambda} + m)} e^{\lambda m y}$$

$$\phi_{+\infty}(k, y) = -2\theta(y) - \frac{1}{ik + \tilde{w}} \delta(y)$$

equivalent form of the Kernel

$$g_\infty(s) = Prob(\xi < s) = Det[I - \mathcal{K}]$$

$$\mathcal{K}(v_1, v_2) = \theta(v_1)\theta(v_2)\tilde{K}(v_1, v_2)$$

$$\tilde{K}(v_1, v_2) = K_{Ai}(v_1 + \sigma, v_2 + \sigma) + K_2^u(v_1 + \sigma, v_2 + \sigma)$$

$$\sigma = 2^{-2/3}(s + \frac{\tilde{x}^2}{16})$$

$$K_2^u(v_1, v_2) = \int_{y>0} dy Ai(v_1 + y) Ai(v_2 - y) e^{2yu}$$

$$u = -2^{2/3}(\tilde{w} + \frac{\tilde{x}}{8})$$

find same as (from TASEP) => universality

Borodin, Ferrari, Sasamoto (2008)

$$h = \ln Z = v_0 t + \lambda \xi_t$$

$$\lambda = (\bar{c}^2 t/4)^{1/3}$$

$\tilde{w} + \frac{\tilde{x}}{8} \to +\infty$ $\quad$ $\lambda\xi = \chi_2 t^{1/3} - \frac{x^2}{4t}$ $\quad$ GUE

$\tilde{w} + \frac{\tilde{x}}{8} \to -\infty$ $\quad$ $\lambda\xi = \lambda\chi_1 + wx + tw^2$ $\quad$ GOE

Transition process

$$\lambda\xi = -\frac{x^2}{4t} + t^{1/3}(\mathcal{A}_{2\to 1}(u) - \min(0, u)^2 + u^2)$$

Summary: we found

for droplet initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

Summary: we found

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$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3} (\frac{t}{t^*})^{1/3} \chi$$

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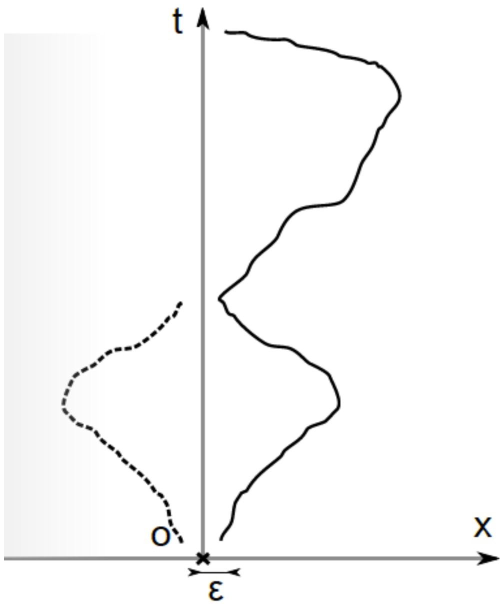
t^* is crossover time scale

GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



$$g(s) = \sqrt{\mathrm{Det}[I + \mathcal{K}]}$$

$$\mathcal{K}(v_1, v_2) = -2\theta(v_1)\theta(v_2)\partial_{v_1} f(v_1, v_2)$$

$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2) f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

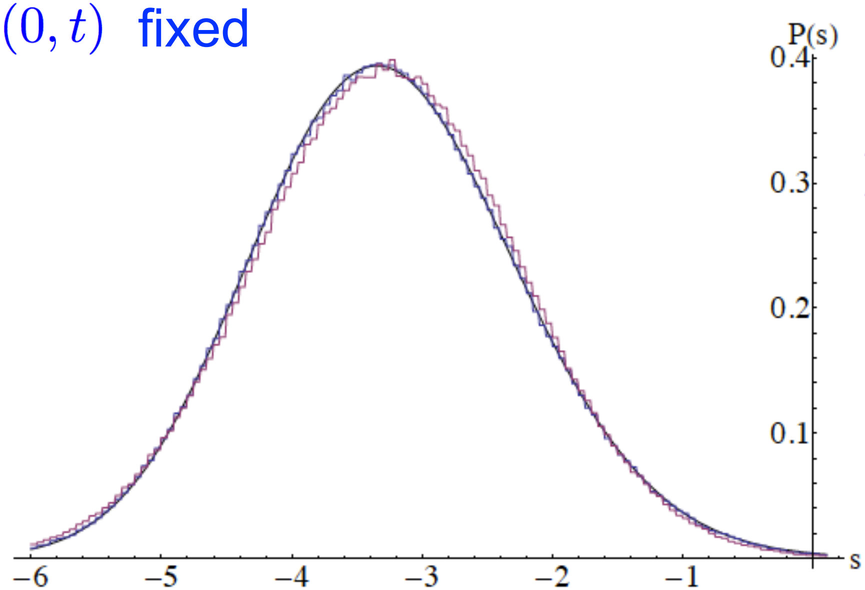
$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$



$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

χ_4 distributed as $F_4(s)$

Gaussian Symplectic Ensemble

Perspectives/other works

- replica BA method

Airy process

stationary KPZ Sasamoto Inamura $t \to \infty$ $A_2(y)$

2 space points $Prob(h(x_1,t),h(x_2,t))$ Prohlac-Spohn (2011), Dotsenko (2013)

2 times $Prob(h(0,t),h(0,t'))$ Dotsenko (2013)

endpoint distribution of DP Dotsenko (2012) Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP $q \to 1$ Bose gas

avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

moments as nested contour integrals

Universal distribution of conductance in 2D localized phase

Somoza, Ortuno, Prior (2007)

$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$\ln g = -\frac{2L}{\xi} + \alpha \left(\frac{L}{\xi} \right)^{1/3} \chi_2$$

ξ localization length

L system size

χ random variable with Tracy Widom distribution

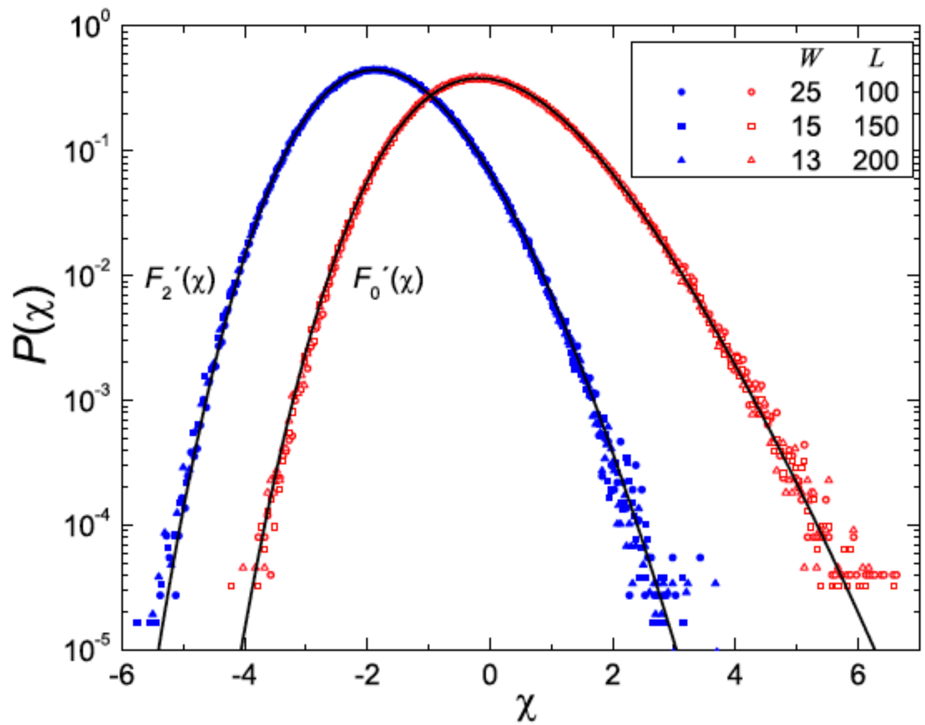


FIG. 1 (color online). Histograms of $\ln g$ versus the scaled variable χ for several sizes and disorders of the Anderson model with narrow (solid symbols) and wide (empty symbols) leads. The continuous lines correspond to $F_2'(\chi)$ and $F_0'(\chi)$.

i

Mapping to directed polymers with non-positive weights

$$G_{ij}(E) = < i|\frac{1}{E-H}|j >$$

$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$g = \sum_{i\in a, j \in b} G_{ij} G_{ji}$$

Nguyen, Spivak, Shklovski (85)

$$\epsilon_i = \eta_i W$$
$$\eta_i = \pm 1$$

$$G_{ij}(E) = \sum_{\gamma \in \Gamma_{ij}} \prod_{\ell \in \gamma} \frac{t}{\epsilon_\ell - E}$$

Γ_{ij} restricted to directed paths from i to j

NSS model

$$\sim (\frac{t}{W})^L \sum_{\gamma \in \Gamma_{ij}^{directed}} \prod_{\ell\in\gamma} \eta_\ell$$

Directed Polymer + random sign weights

complex weights

Derrida et al. (93)

Kardar Medina (92)

A. Dobrinevski, PLD, K. Wiese
PRE 83 061116 (2011)

$\overline{\ln|Z|} \sim \ln|\overline{Z}|$ phase I

$\sim \frac{1}{2} \ln \overline{ZZ^*}$ phase III

$\overline{(ZZ^*)^n} \sim \overline{Z_{DP}^n}$ phase II similar to positive weights .. d=1+1 expect TW

M Mueller (2011) hard core bosons: DP w. positive weights

Calculation of F1(s)

$$\sum_{j=1}^{m} w_j f(x_j) \approx \int_a^b f(x)\,dx$$

Gauss Legendre quadrature rule

$$d(z) = \det(I - zK\restriction_{L^2(a,b)})$$

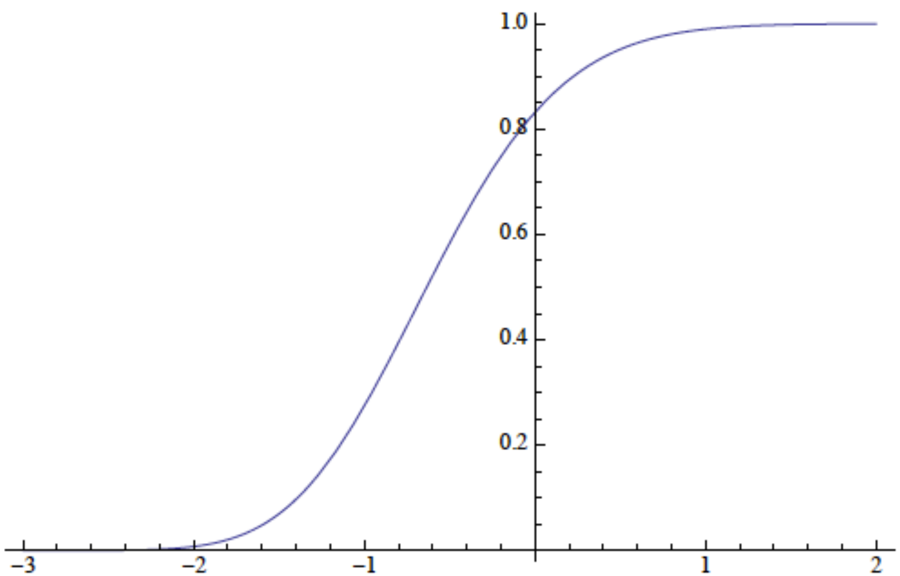
$$d_m(z) = \det\left(\delta_{ij} - z\, w_i^{1/2} K(x_i, x_j) w_j^{1/2}\right)_{i,j=1}^{m}$$

Bornemann (2009)

```
In[1]:= Needs["NumericalCalculus`"]
GaussLegendre[a_, b_, m_] :=
 Module[{beta, T, V, c, d, e}, beta = Table[i / Sqrt[(2 i - 1) (2 i + 1)], {i, 1, m - 1}];
  T = DiagonalMatrix[beta, -1] + DiagonalMatrix[beta, 1];
  V = Eigensystem[N[T, 10]]; e = V[[2]]; d = Table[e[[i, 1]], {i, 1, m}];
  c = (V[[1]] + 1) / 2; {d^2 (b - a), (1 - c) a + b c}]
FredholmDet[K_, z_, a_, b_, m_] := Module[{w, x}, {w, x} = GaussLegendre[a, b, m];
  w = Sqrt[w] ; Det[IdentityMatrix[m] + (Transpose[{w}].{w}) Outer[K, x, x]]]

In[24]:= K[x_, y_] = -AiryAi[x + y];
g[x0_, b_, m_] := N[FredholmDet[K, x0, x0, b, m]]
g2 = Interpolation[Table[{x, g[x, 8, 20]}, {x, -10, 5, 0.2}]];
Plot[g2[s], {s, -3, 2}]
```

Out[27]=



Exact results for height distributions for some discrete models in KPZ class

- PNG model <=> LIS problem
<=> discrete directed path

Baik, Deft, Johansson (1999)

$$l_N \to 2\sqrt{N} + N^{1/6}\chi$$

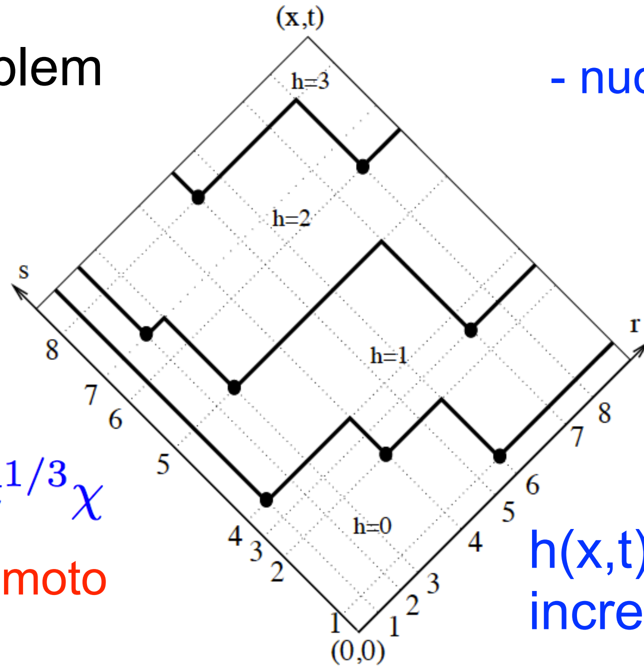
$\mathrm{Prob}(\chi \leq x) = F_2(x)$ GUE

$N \sim t^2$ $\quad h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$

Prahofer, Spohn, Ferrari, Sasamoto (2000+)

flat IC $\quad \chi = \chi_1 \quad$ GOE

multi-point correlations
Airy processes



- nucleation events in PNG are poisson points

- each PNG history maps to a permutation p

$$(4,7,5,2,8,1,3,6)$$

h(x,t) = length of longest increasing subsequence of p

also=largest number points collected by up-going directed path 0,0 -> (x,t)

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

$A_2(y)$ GUE

$A_1(y)$ GOE

- similar results in TASEP $\quad$ Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model <=> LIS problem <=> optimal discrete directed path

Baik, Deft, Johansson (1999)

h(0,t) = length of longest increasing subsequence of a random permutation

$$l_N \to 2\sqrt{N} + N^{1/6}\chi \qquad \mathrm{Prob}(\chi \leq x) = F_2(x)$$

GUE

$N \sim t^2$ $\qquad h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$

Prahofer, Spohn, Ferrari, Sasamoto,.. (2000+)

flat IC $\qquad \chi = \chi_1 \qquad$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3}A_n(y)$$

- similar results for TASEP $\qquad$ Johansson (1999), ...