Tensors: rank, entropy and entanglement

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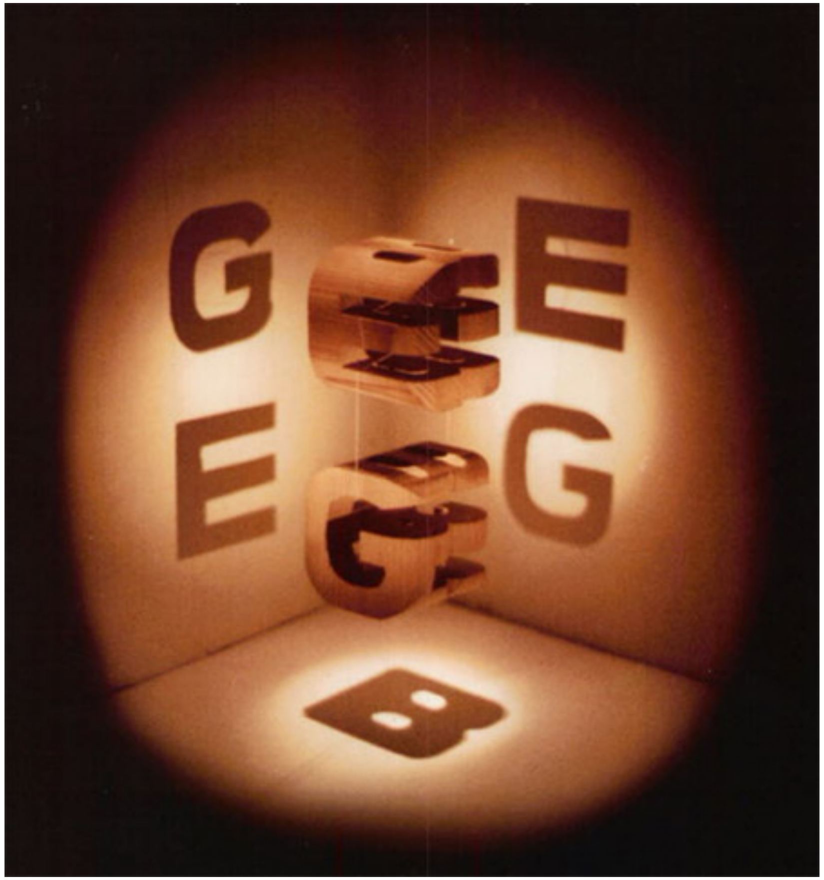
arXiv:1709.0781, Ref #9, Proc. STOC'18



Graeme Mitchison 1944-2018

”In 1930, Weyl observed with dry humour that the “group pest” seemed to be here to stay. The theory of representations of groups, which he did so much to develop, is indeed a firmly established component of modern physics, ….”

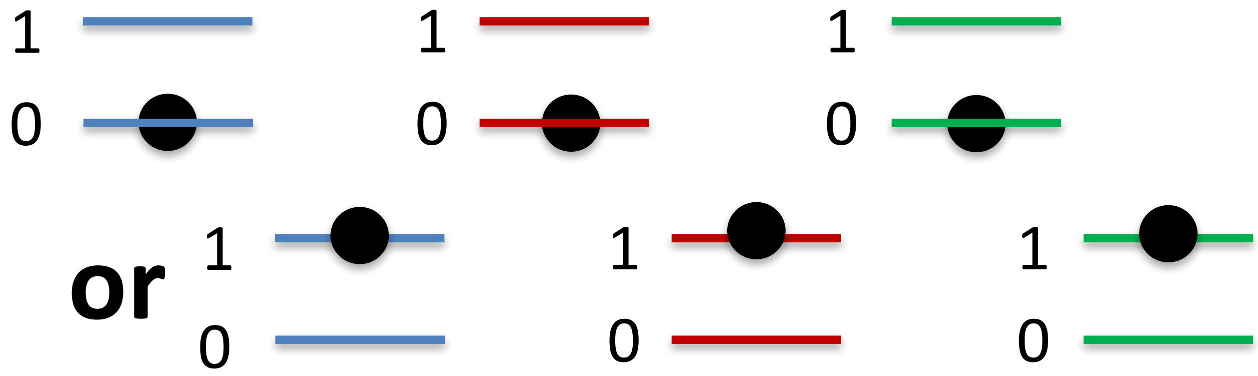
Outline

- Two motivations
- Resource theory of tensors
- Entanglement polytopes
- Tensor $\otimes$ tensor $\otimes$ …. $\otimes$ tensor
- Quantum functionals

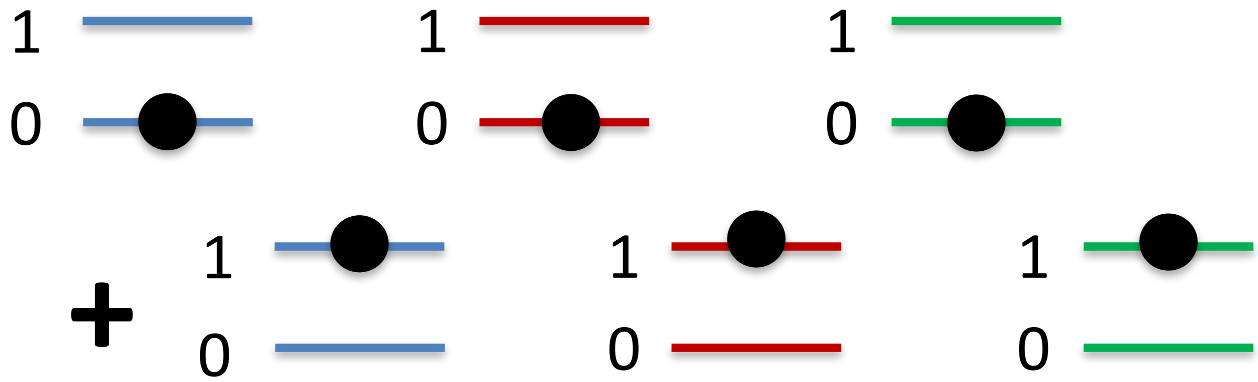
Two motivations

Quantum states

State of a classical system (3 bits)



State of a quantum system (3 qubits)



$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

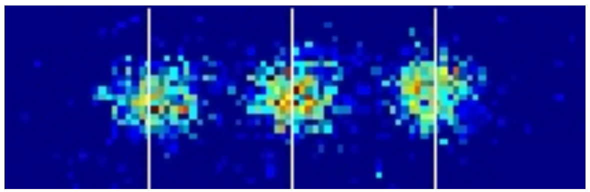
$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

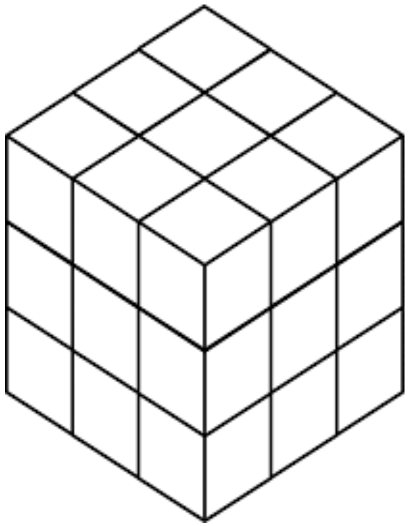
$$t = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

Quantum state=tensor

$$t \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$

$$t = \sum_{i,j,k=1}^{d} t_{ijk} e_i \otimes e_j \otimes e_k$$

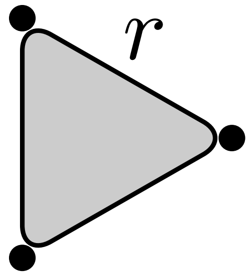


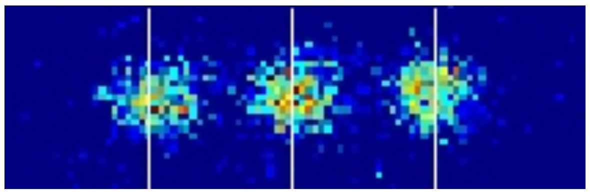


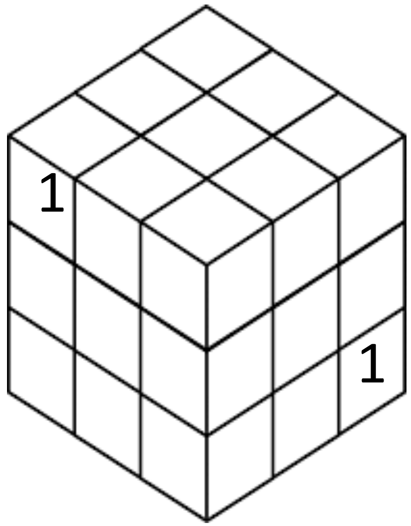
GHZ state = unit tensor

Greenberger-Horne-Zeilinger

$$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

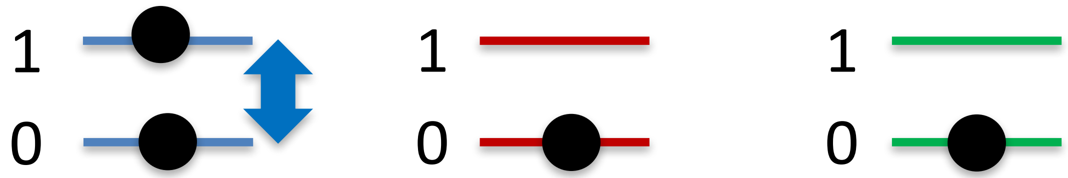




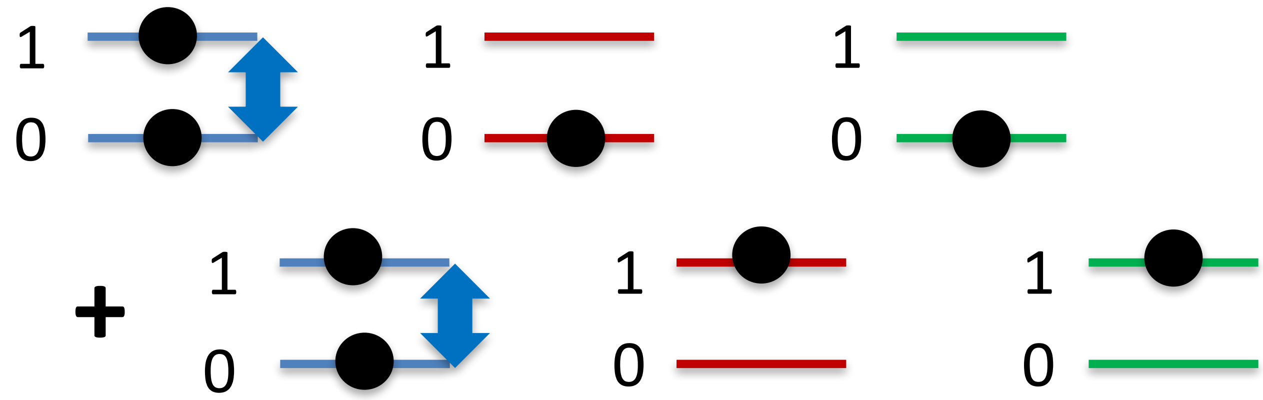


Local operations

Local trans-formation: Flip first bit



Local trans-formation: Flip first qubit



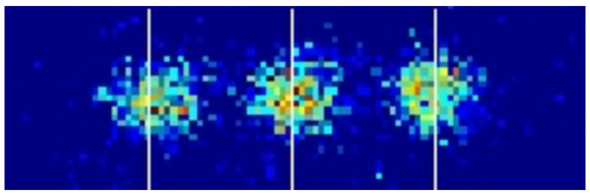
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

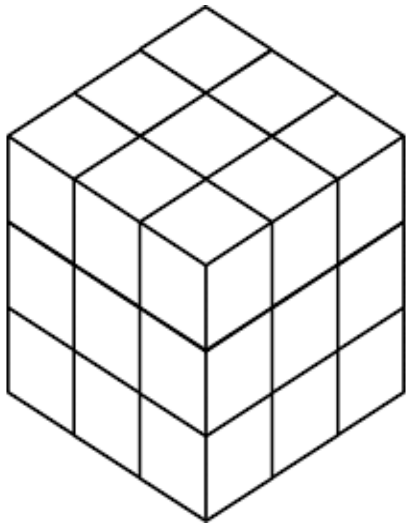
$$t = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

Local operations=restrictions

$$t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$$

$$\text{for some matrices } a, b, c$$





Linear combination of slices

3 qubits

Greenberger-Horne-Zeilinger
GHZ-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

Einstein-Podolsky-Rosen
(EPR)-state

W-state

$$\approx e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 \qquad e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0$$

unentangled state

Entanglement cost=tensor rank

in terms of GHZ states

$$R(t) := \min\{r : \langle r \rangle \geq t\}$$

$$= \min\{r : t = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i\}$$

Strassen: ”cost” and ”value”

$$Q(t) := \max\{r : t \geq \langle r \rangle\}$$

distillable entanglement= subrank

3 qubits

Greenberger-Horne-Zeilinger GHZ-state

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

Einstein-Podolsky-Rosen (EPR)-state

W-state

$$\approx e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0$$

$$e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

$$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1$$

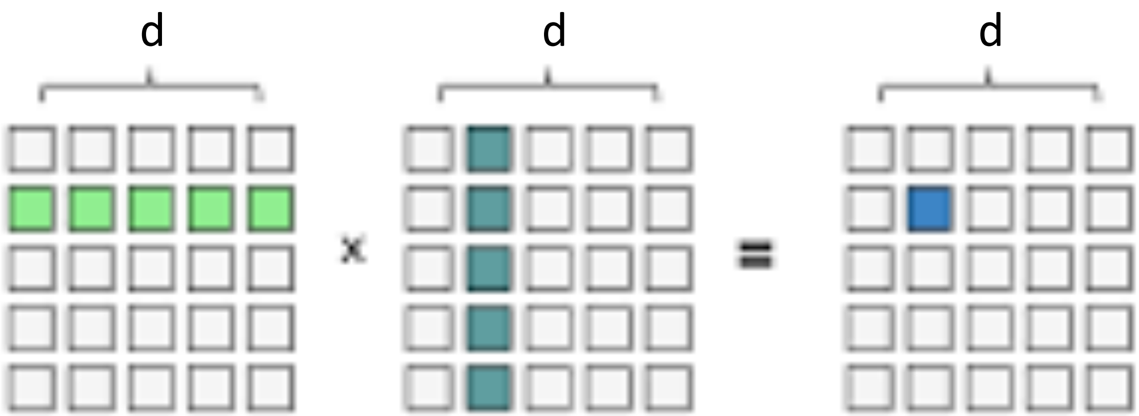
$$e_0 \otimes e_0 \otimes e_0$$

unentangled state

Algebraic Complexity

$$M(d) = \text{ algebra of } d \times d \text{ complex matrices}$$

$$\begin{aligned} Mamu(d) : M(d) \times M(d) &\to M(d) \qquad \text{bilinear} \\ (A, B) &\mapsto A \cdot B \end{aligned}$$



d^3 multiplications

Bilinear maps=tensors

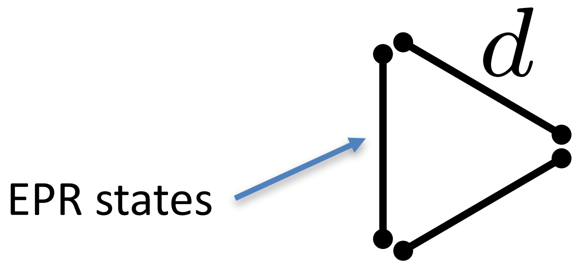
$$Mamu(d) : M(d) \times M(d) \times M(d)^* \to \mathbf{C}$$

$$(A, B, C) \mapsto trA \cdot B \cdot C$$

$$Mamu(d) = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki}$$

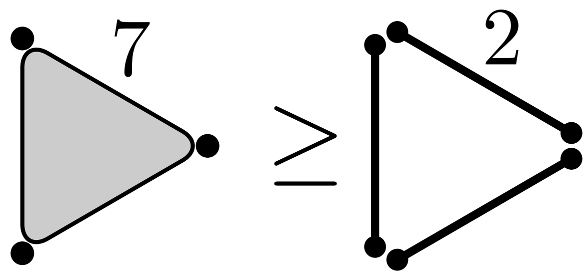
$$e_{ij} = e_i \otimes e_j$$

$$= \sum_{i,j,k=1}^{d} (e_i \otimes e_j) \otimes (e_j \otimes e_k) \otimes (e_k \otimes e_i)$$



Complexity=Tensor rank

Strassen: # elementary multiplications = tensor rank



Do you like Strassen's decomposition?
Then you might want to look at some tensor surgery next!
arXiv:1606.04085

$$e_{00} \otimes e_{00} \otimes e_{00} + e_{11} \otimes e_{11} \otimes e_{11}$$

$$e_{01} \otimes e_{10} \otimes e_{00} + e_{10} \otimes e_{01} \otimes e_{11}$$

$$e_{01} \otimes e_{11} \otimes e_{10} + e_{10} \otimes e_{00} \otimes e_{01}$$

$$e_{00} \otimes e_{01} \otimes e_{10} + e_{11} \otimes e_{10} \otimes e_{01}$$

$$e_{\pm} := e_0 \pm e_1$$

$$\begin{aligned} =& e_{-1} \otimes e_{1+} \otimes e_{00} + e_{1+} \otimes e_{00} \otimes e_{-1} + e_{00} \otimes e_{-1} \otimes e_{1+} \\ & - e_{-0} \otimes e_{0+} \otimes e_{11} - e_{0+} \otimes e_{11} \otimes e_{-0} - e_{11} \otimes e_{-0} \otimes e_{0+} \\ & + (e_{00} + e_{11}) \otimes (e_{00} + e_{11}) \otimes (e_{00} + e_{11}) \end{aligned}$$

Resource theory of tensors

free operations

Resource theory of tensors

valuable resource

- Restriction $\quad t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$ for some matrices a, b, c
- Unit $\quad \langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$
- Rank $\quad R(t) = \min\{r : \langle r \rangle \geq t\}$
- Subrank $\quad Q(t) = \max\{r : t \geq \langle r \rangle\}$

Restriction

$$t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$$

for some matrices a, b, c

$$t \cong t' \text{ if } t \geq t' \text{ and } t' \geq t$$

$$\text{iff } (a \otimes b \otimes c)\ t = t'$$

for invertible a, b, c

$$\text{iff } G.t = G.t'$$

$G = GL(d) \times GL(d) \times GL(d)$

Deciding restriction $\Longleftrightarrow$ Classifying orbits and their relations

GHZ state

Degeneration

W state

$$(e_0 + \epsilon e_1)^{\otimes 3} - e_0^{\otimes 3}$$
$$= \epsilon(e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0) + O(\epsilon^2)$$

$$t \trianglerighteq t' \text{ if } t_\epsilon \underset{\epsilon \mapsto 0}{\longrightarrow} t', t \geq t_\epsilon$$

Deciding degeneration ⟺ Classifying orbit closures and their relations

Deciding degeneration

- Orbit closures are G-invariant algebraic varieties

$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G - \text{covariant polynomial } f : f(t) \neq f(t')$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

- Example: $e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$

$$\downarrow \quad \not\uparrow$$

f=Cayley hyperdeterminant

$$\approx \; e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Summary

- Resource theory of tensors
  - entanglement in quantum information theory
  - bilinear complexity in algebraic complexity theory
- Nontrivial results
  - Strassen's 2x2 Mamu
  - degeneration from GHZ to W-state
    (but not the other way around)
  - Difficult in general (wild, NP-hard, …)
  - Is the answer useful?
- How to proceed?

Be happy with partial information

- Quantum information
  - focus on local information (easier to access experimentally)

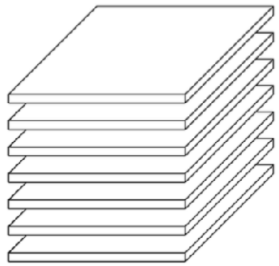
- Algebraic complexity
  - focus on partial information about orbit closures (obstructions in Geometric Complexity Theory)

Entanglement polytopes

Local spectra

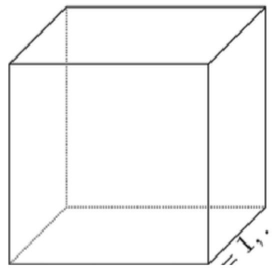
$$t'_A \in \mathbf{C}^d \otimes \left(\mathbf{C}^d \otimes \mathbf{C}^d\right)$$

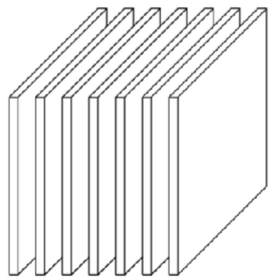
$$\lambda_A = \text{ singular values } (t'_A)^2$$



normalised

$$t' \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$





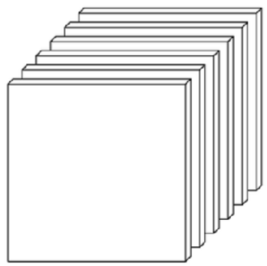
$$t'_B \in \ldots$$

$$\lambda_B = \text{ singular values } (t'_B)^2$$

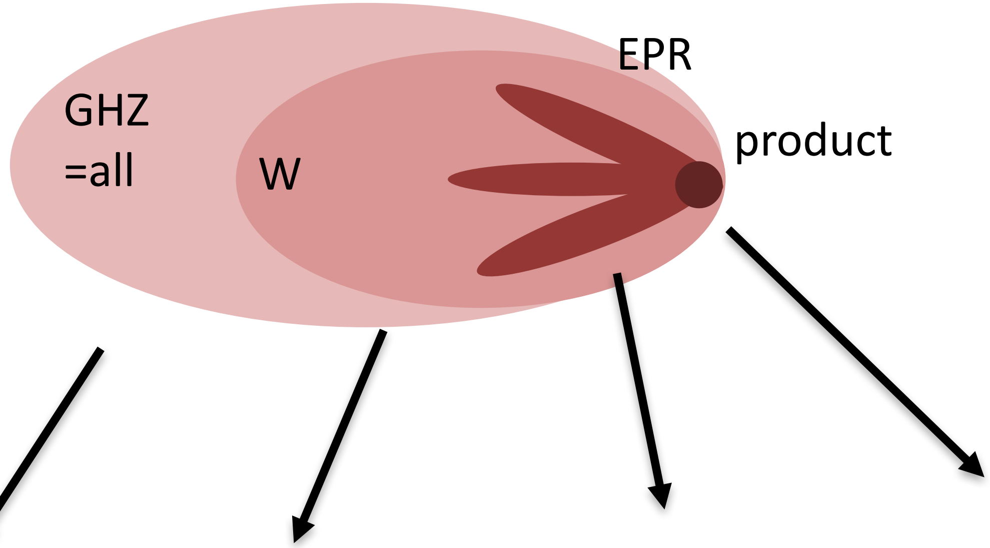
ordered probability distribution
=spectrum of reduced density operator

$$t'_C \in \left(\mathbf{C}^d \otimes \mathbf{C}^d\right) \otimes \mathbf{C}^d$$

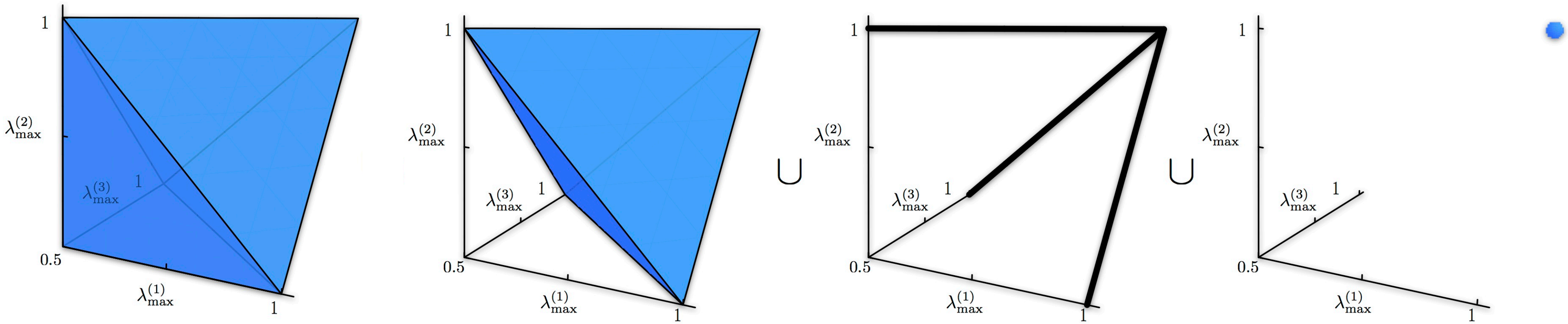
$$\lambda_C = \text{ singular values } (t'_C)^2$$



Entanglement polytopes



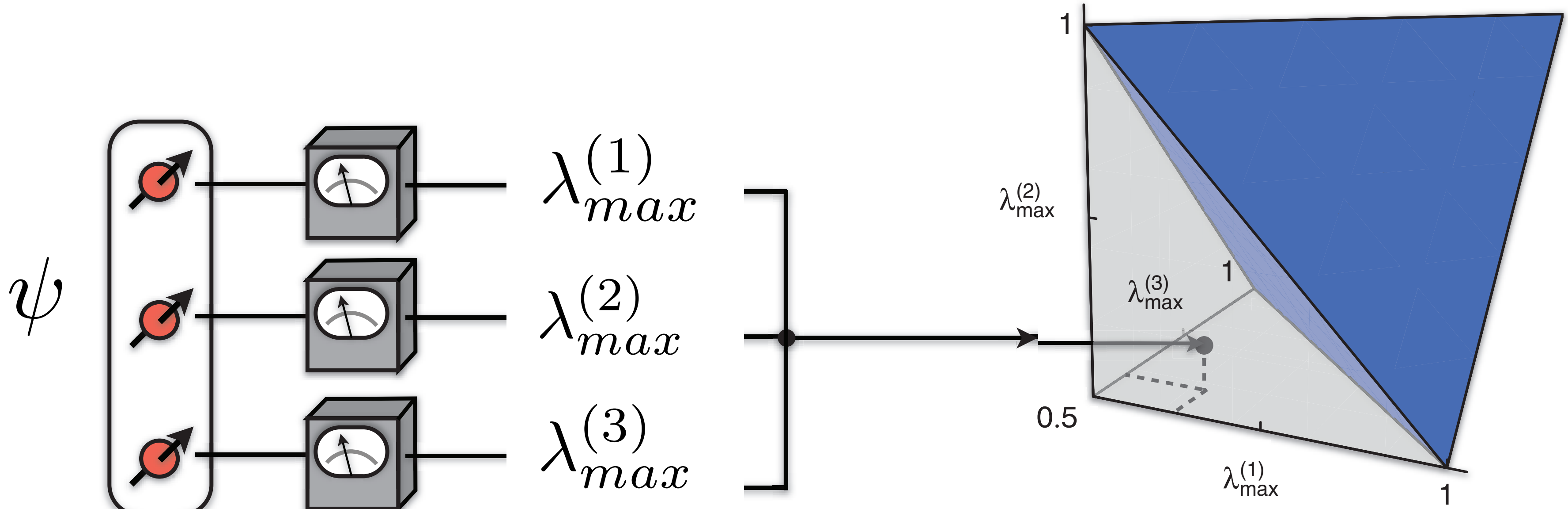
marginal
polytope



Ch-Mitchison, Klyachko,
Daftuar-Hayden (2004)
based in part on Kirwan

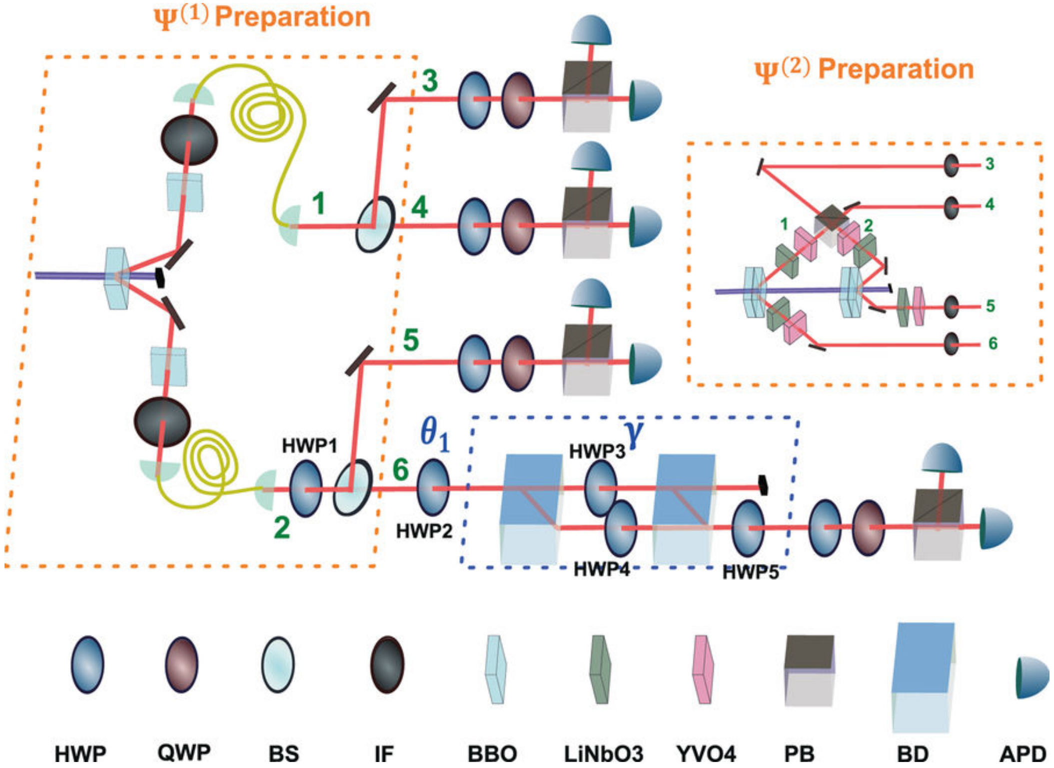
Walter-Doran-Gross-Ch,
Sawicki-Oszmaniec-Kus (2010) based on Brion

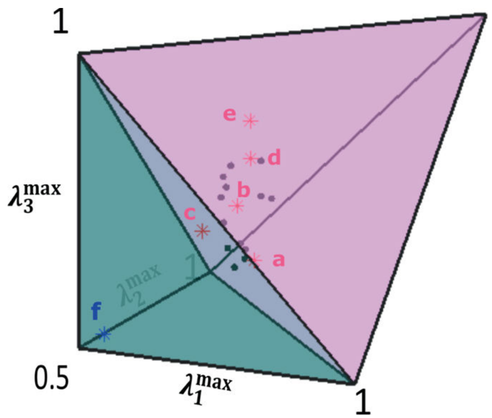
Experimental Detection



- if measured value
  - not in W-polytope
  - Then must be in GHZ-class!
- easy test for entanglement!

Experimental Detection





- G.H.Aguilar *et al.* PRX 2015
  Y-Y.Zhao *et al.* npj Quantum Information 2017

A little more partial information?

- Orbit closures are G-invariant algebraic varieties

$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f : f(t) \neq f(t')$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

- f's come in types indexed by 3 Young diagrams

$$\lambda_A = \ydiagram{4,3,1}.$$

boxes=degree

Weyl's construction

- Schur-Weyl duality

S_n acts $\quad$ $GL(d)$ acts

$$(\mathbf{C}^d)^{\otimes n} \cong \bigoplus_{\lambda} [\lambda] \otimes V_\lambda$$

- P_{λ_A} orthogonal projector onto λ_A component

$$\underbrace{\left(P_{\lambda_A} \otimes P_{\lambda_B} \otimes P_{\lambda_C}\right)}_{=:P_\lambda} t^{\otimes n}$$

$$= \left(\sum_i v_i v_i^*\right) t^{\otimes n} = \sum_i v_i^* f_i(t)$$

Relaxation

- Orbit closures are G-invariant algebraic varieties

$$t \not\trianglerighteq t' \text{ iff there exists}$$

$$G - \text{covariant polynomial } f : f(t) \neq f(t')$$

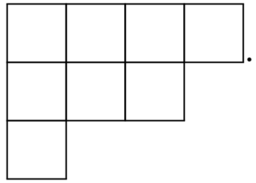
$$f(t) = 0, \text{ but } f(t') \neq 0$$

$$\text{if there is } \lambda \text{ s.th.}$$

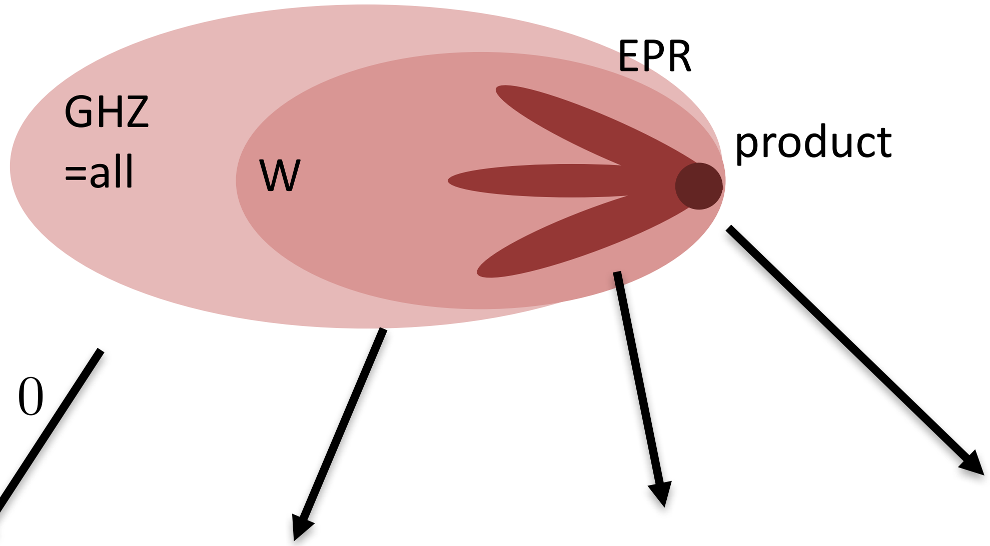
$$P_\lambda t^{\otimes n} = 0 \text{ but } P_\lambda t'^{\otimes n} \neq 0$$

occurrence obstructions (Geometric Complexity Theory)
Mulmuley-Sohoni, Strassen, Bürgisser-Ikenmeyer, …

Entanglement polytopes

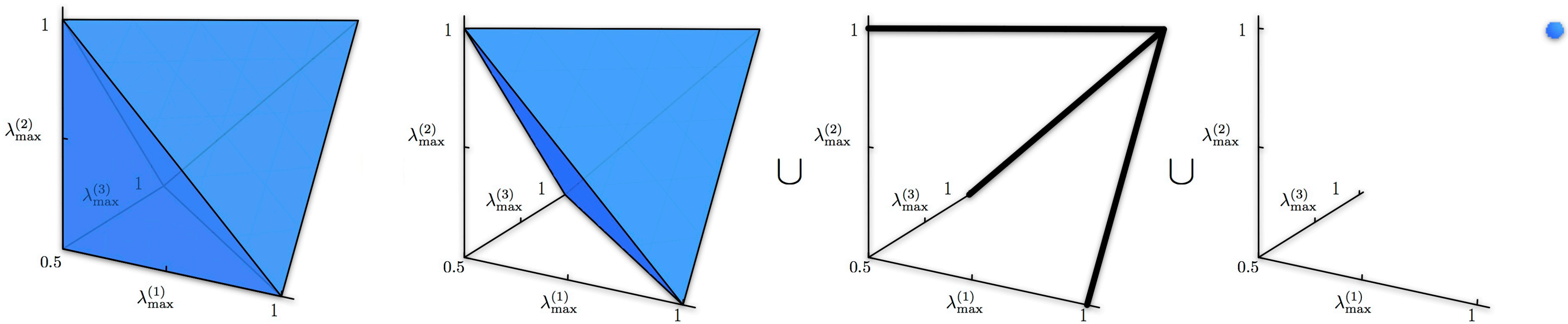


$$\longrightarrow \left(\frac{4}{8}, \frac{3}{8}, \frac{1}{8}\right)$$



$g_\lambda \neq 0$
Kronecker
= marginal
polytope

$P_\lambda t^{\otimes n} \neq 0$



Entanglement Polytopes

- Quantum information
  - focus on local information (easier to access experimentally)
- Algebraic complexity
  - focus on partial information about orbit closures (obstructions in Geometric Complexity Theory)
- Return to main question: $t \geq t'$

A small observation

$$d = 2^n$$

$$e_i = e_{i_1 i_2 \cdots i_n} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

$$\sum_{i=1}^{d} e_i \otimes e_i = \left(\sum_{i_1=1}^{2} e_{i_1} \otimes e_{i_1}\right) \otimes \left(\sum_{i_2=1}^{2} e_{i_2} \otimes e_{i_2}\right) \otimes \cdots \otimes \left(\sum_{i_n=1}^{2} e_{i_n} \otimes e_{i_n}\right)$$

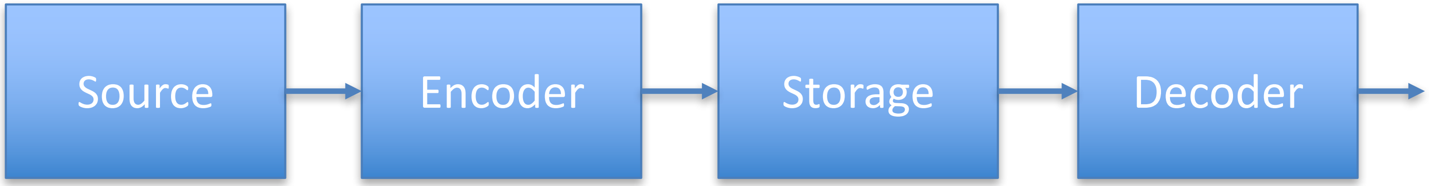
$$= (e_0 \otimes e_0 + e_1 \otimes e_1)^{\otimes n}$$

$$\langle d \rangle = \sum_{i=1}^{d} e_i \otimes e_i \otimes e_i = (e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)^{\otimes n} = \langle 2 \rangle^{\otimes n}$$

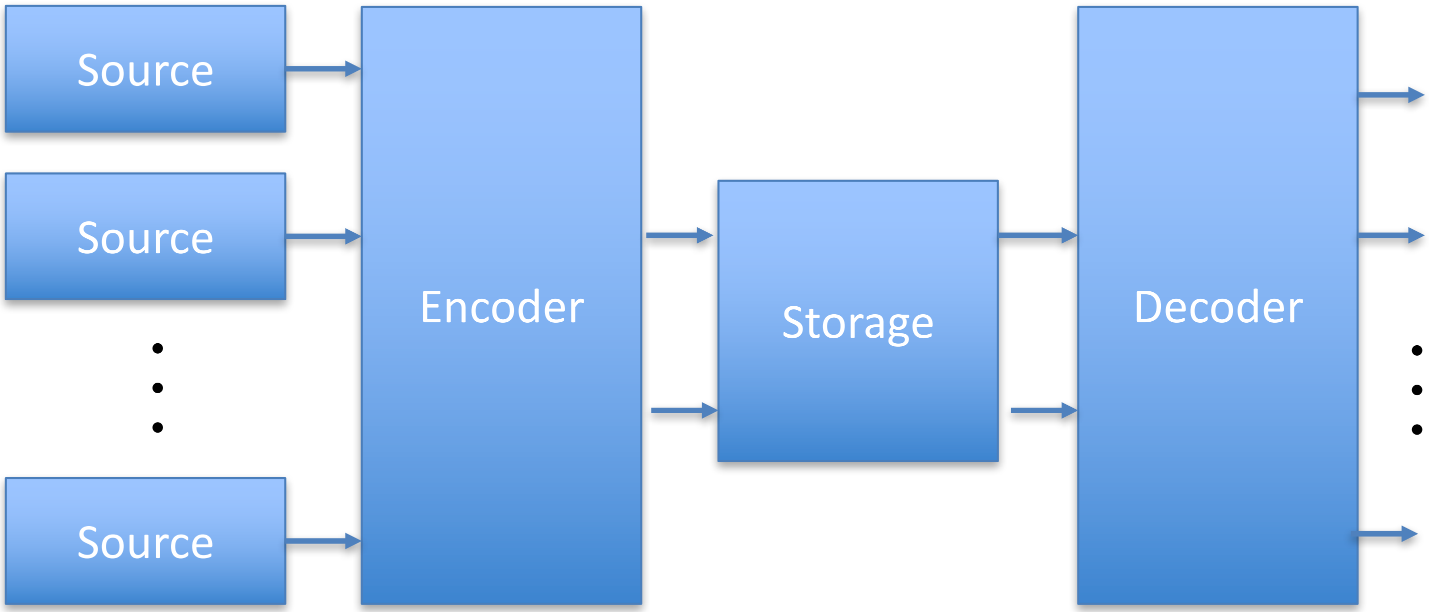
$$Mamu(d) = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki} = \left(\sum_{i,j,k=1}^{2} e_{ij} \otimes e_{jk} \otimes e_{ki}\right)^{\otimes n} = Mamu(2)^{\otimes n}$$

$$\text{tensor} \otimes \text{tensor} \otimes \ldots \otimes \text{tensor}$$

(Quantum) information theory

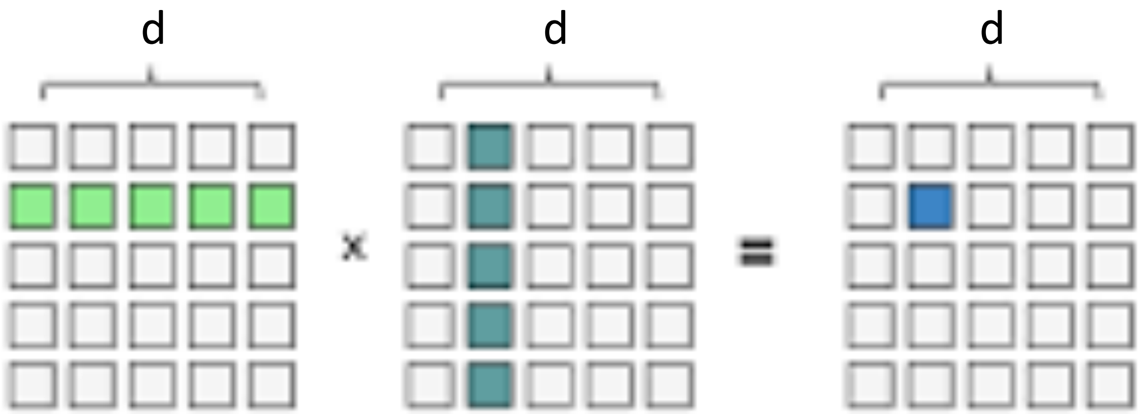


Shannon: storage cost= all bits



Shannon: storage cost= H(X) bits/symbol

Algebraic complexity theory



d^3 multiplications

- Exponent of matrix multiplication $O(d^\omega)$

$$2 \leq 2.38 \leq \cdots \leq 2.8 \leq 3$$

…, Coppersmith-Winograd Strassen

$$\omega = \inf\{r : \langle 2 \rangle^{\otimes(nr+o(n))} \geq Mamu(2)^{\otimes n}\}$$

- Conjecture: $\langle 2 \rangle^{\otimes 2n+o(n)} \geq Mamu(2)^{\otimes n}$

Asymptotic resource theory

- Asymp. restriction $t \gtrsim t'$ if $t^{\otimes n + o(n)} \geq t'^{\otimes n}$
- Unit $$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$
- Asymp. rank $$\tilde{R}(t) := \lim_{n \to \infty} R(t^{\otimes n})^{\frac{1}{n}}$$
- Asymp. subrank $$\tilde{Q}(t) := \lim_{n \to \infty} Q(t^{\otimes n})^{\frac{1}{n}}$$

$\tilde{R}(Mamu(2)) = 2^{\omega}$

Strassen's spectral theorem

$$t \gtrsim t' \text{ iff } F(t) \geq F(t') \text{ for all } F:$$

F monotone — under restriction $F(s) \geq F(s')$ for all $s \geq s'$

F normalised — $F(\langle r \rangle) = r$

F multiplicative — $F(s \otimes s') = F(s) \cdot F(s')$

F additive — $F(s \oplus s') = F(s) + F(s')$

$$\tilde{R}(t) = \max_F F(t)$$

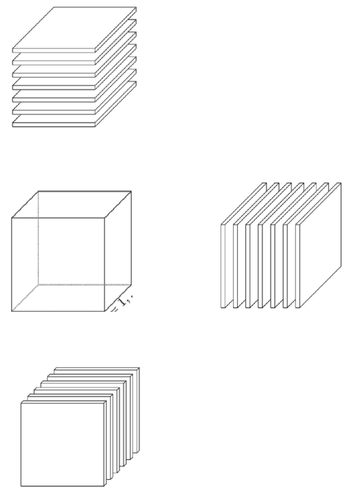
$\Rightarrow$ easy

$\Leftarrow$ difficult

$$\tilde{Q}(t) = \min_F F(t)$$

every *F* is an obstruction

What are the F's?



- Existence non-constructive
  - Compact space worth of them
  - Gauge points: ranks of slicings
  - What are the others?

- Theorem also true for subclasses of tensors
  - Oblique tensor
  - Strassen's support functionals
  - Conjecture (Strassen): they are all

Quantum functionals

$$\theta = (\theta_A, \theta_B, \theta_C) \text{ probability distribution e.g. } \theta_A = \theta_B = \theta_C = \frac{1}{3}$$

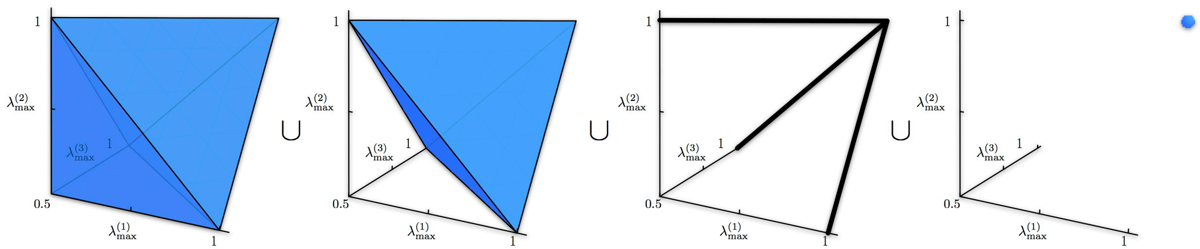
$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

entanglement polytope

$$F_\theta(t) := 2^{E_\theta(t)}$$

quantum functionals

Measures distance to origin (relative entropy distance)



$E_{(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$	1	$h\left(\frac{1}{3}\right) \approx 0.92$	$\frac{2}{3}$	0

Quantum functionals

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

easy, since polytope gets smaller under restriction
quantum functional gets smaller

F_θ monotone

easy, since polytope of unit tensor
contains uniform point $F(\langle r \rangle) = r$

F_θ normalised

F_θ multiplicative

similar to multiplicativity, see paper

F_θ additive

Multiplicativity

$$F_\theta(t \otimes t') = F_\theta(t) \cdot F_\theta(t')$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

$$E_\theta(t \otimes t') = E_\theta(t) + E_\theta(t')$$

$\geq$	$\leq$
easy	more difficult

$$E_\theta(t \otimes t') \geq E_\theta(t) + E_\theta(t')$$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

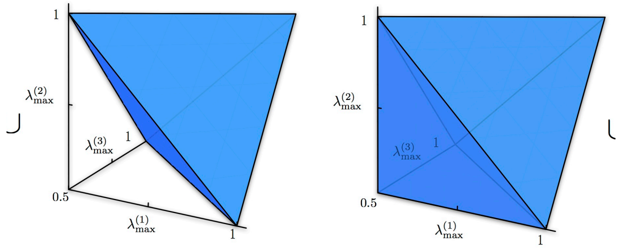
Lemma: $\Delta(t \otimes t') \supseteq \Delta(t) \otimes \Delta(t')$

Proof:
$$\begin{aligned}
\Delta(t \otimes t') &= \{\lambda(\tau) : t \otimes t' \trianglerighteq \tau\} \\
&\supseteq \{\lambda(s \otimes s') : t \otimes t' \trianglerighteq s \otimes s'\} \\
&= \{\lambda(s) \otimes \lambda(s') : t \trianglerighteq s, t' \trianglerighteq s'\} \\
&= \Delta(t) \otimes \Delta(t')
\end{aligned}$$

product distribution

qed

$$E_\theta(t \otimes t') \leq E_\theta(t) + E_\theta(t')$$

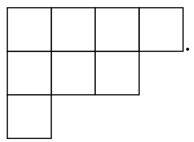


Lemma:

$$\Delta(t \otimes t') \subseteq \Delta(t) \otimes_{\mathrm{Kron}} \Delta(t')$$
$$:= \{(\alpha, \beta, \gamma) : (a, b, c) \in \Delta(t), (a', b', c') \in \Delta(t'),$$
$$(a, a', \alpha) \& (b, b', \beta) \& (c, c', \gamma) \in \mathrm{Kron}\}$$

Proof: $0 \neq (P_{n\alpha} \otimes P_{n\beta} \otimes P_{n\gamma})\, t^{\otimes n} \otimes t'^{\otimes n}$

$$= [P_{n\alpha} \otimes P_{n\beta} \otimes P_{n\gamma}] \left[\underbrace{(\sum P_{na})}_{=id} \otimes (\sum P_{nb}) \otimes (\sum P_{nc}) \otimes (\sum P_{na'}) \otimes (\sum P_{nb'}) \otimes (\sum P_{nc'})\right] [t^{\otimes n} \otimes t'^{\otimes n}]$$



$\Rightarrow$

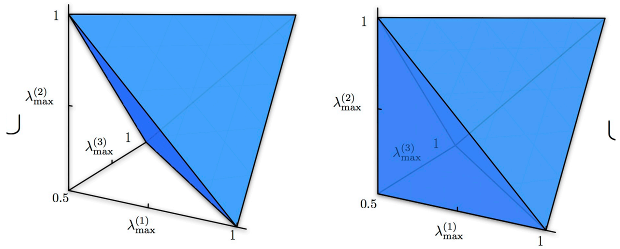
$$P_{n\alpha}(P_{na} \otimes P_{na'}) \neq 0$$
$$P_{n\beta}(P_{nb} \otimes P_{nb'}) \neq 0$$
$$P_{n\gamma}(P_{nc} \otimes P_{nc'}) \neq 0$$

$$(P_{na} \otimes P_{nb} \otimes P_{nc})t^{\otimes n} \neq 0$$
$$(P_{na'} \otimes P_{nb'} \otimes P_{nc'})t'^{\otimes n} \neq 0$$

qed

$$E_\theta(t \otimes t') \leq E_\theta(t) + E_\theta(t')$$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$



Lemma:

$$\Delta(t \otimes t') \subseteq \Delta(t) \otimes_{\text{Kron}} \Delta(t')$$

$$:= \{(\alpha, \beta, \gamma) : (a, b, c) \in \Delta(t), (a', b', c') \in \Delta(t'),$$
$$(a, a', \alpha) \& (b, b', \beta) \& (c, c', \gamma) \in \text{Kron}\}$$

Subadditivity
v. Neumann entropy

Lemma: If $(a, a', \alpha) \in \text{Kron}$, then $H(\alpha) \leq H(a) + H(a')$

Proof: $$\theta_A H(\alpha) + \theta_B H(\beta) + \theta_C H(\gamma) \leq \theta_A (H(a) + H(a')) + \theta_B (H(b) + H(b')) + \theta_C (H(c) + H(c'))$$

Subadditivity of E

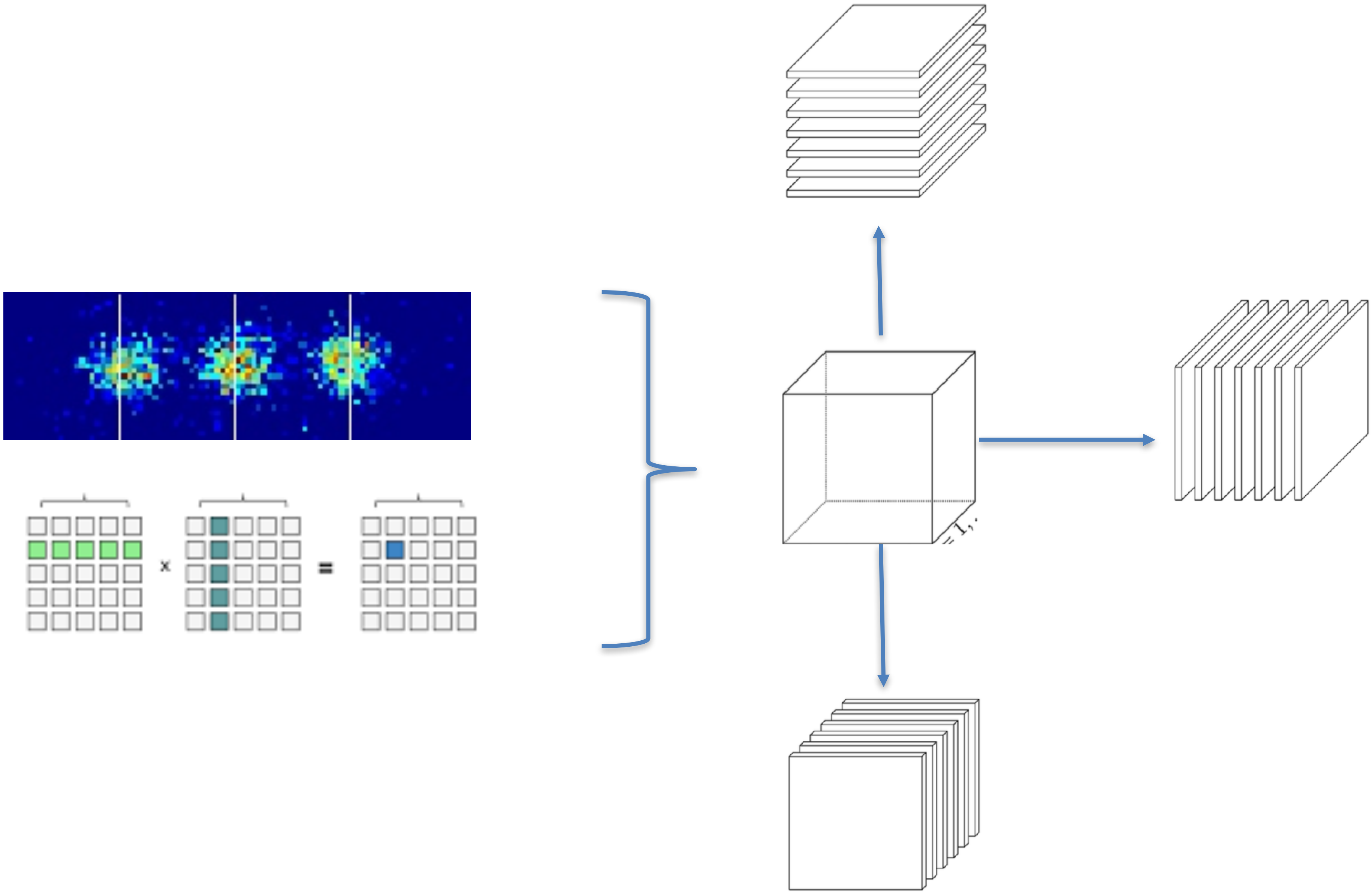
optimal

qed

Quantum functionals

- Extend Strassen's support functionals
- Are they complete?
- If complete, then $\omega = 2$
- Characterise slice-rank
- General setting of tensors of order k
- Connect Strassen's framework to capset

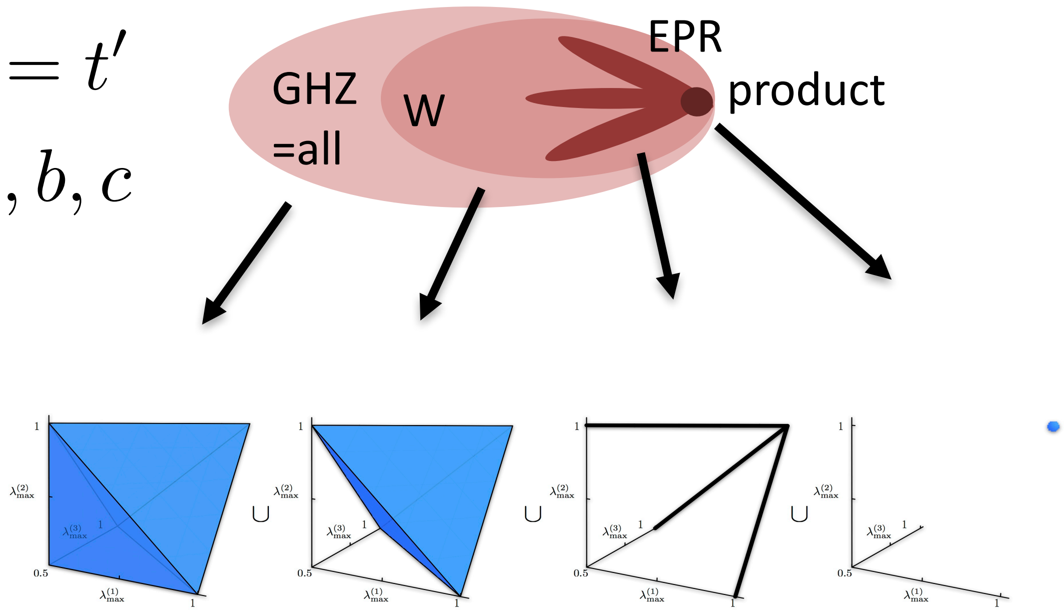
Summary



Summary

$t \geq t'$ if $(a \otimes b \otimes c)\ t = t'$

for some matrices a, b, c



$t \gtrsim t'$ if $t^{\otimes n + o(n)} \geq t'^{\otimes n}$

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$F_\theta(t) := 2^{E_\theta(t)}$

If all, then $\omega = 2$