A gentle introduction to representation theory

Peter Bürgisser

TU Berlin

Georg Frobenius

Issai Schur

Hermann Weyl

Motivation

- Symmetries are a main source of unifying principles in mathematics and physics
- Groups are the appropriate math. concept for describing symmetries
- Representation theory studies linear transformations in the presence of symmetries
- Every representation of a group G is built up from "atomic" ones. The structure of those only depends on G. The way the "atoms" combine, explain the manifold phenomena observed.

- Role in computer science:
  - behind several fast algorithms (FFT, efficient polynomial arithmetic, matrix multiplication)
  - graph isomorphism problem
  - construction of expander graphs
  - Geometric complexity theory

Outline

Part I

1. Basic notions
2. Basic facts
3. Cyclic groups and Fourier transform
4. Schur's Lemma
5. Isotypical decomposition
6. Group algebra

General

Part II

7. Symmetric groups S_n
8. Tori
9. Weight space decomposition
10. General linear group GL_n
11. Highest weight vectors
12. Construction of Schur-Weyl reps
13. Schur-Weyl duality
14. GL_n-reps relevant in GCT

Understand reps of general linear groups through reps of tori

intriguing connection to symmetric groups

1. Basic notions

A representation of a group G is a group morphism

$$\rho : G \to \underbrace{GL(V)}_{\text{group of invertible linear transformation}}, \text{ where } V \text{ f.d. } \mathbb{C}\text{-vector space}$$

If $V = \mathbb{C}^m$, $\rho : G \to \overbrace{GL_m}^{\text{group of invertible complex } m\times m \text{ matrices}}$

represent abstract $g \in G$ by matrix $\rho(g)$

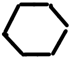
$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

abstract mult in G

matrix mult

Ex Cyclic group

$C_n = \{1, d, \dots, d^{n-1}\}$, $d^n = 1$



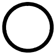
n = 6

$\rho(d^j) \in GL_2$

rotation by angle $j \cdot \frac{2\pi}{n}$

Ex $S^1 = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$

$\rho(e^{i\vartheta}) \in GL_2$



rotation by angle ϑ

If G is infinite, assume topology and ρ continuous

Subspace $W \leq V$ called subrepresentation if

$$\rho(g) = \begin{array}{|c|c|} \hline * & * \\ \hline 0 & * \\ \hline \end{array} \qquad \forall g \in G \quad \rho(g)(W) \leq W$$

Want to decompose into subreps W_i

$$\rho(g) = \begin{array}{|c|c|c|} \hline * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \\ \hline \end{array} \qquad V = W_1 \oplus \dots \oplus W_s$$

simultaneous block diagonalization

V called irreducible if it doesn't contain a proper subrep $0 \neq W \subsetneq V$

<u>Rem</u> $\dim V = 1 \Rightarrow V$ irreducible

$\rho : G \longrightarrow GL(V) \simeq \mathbb{C}^\times$ called character of G

2. Basic facts

rep $\rho : G \to GL(V)$ is called unitary if all $\rho(g)$ are unitary wrt some hermitian inner product

Prop If G is finite (or compact) then every rep of G is unitary.

Pf Averaging over G: $\langle v_1, v_2 \rangle_{av} := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)(v_1), \rho(g)(v_2) \rangle$ □

Thm (Maschke, Weyl)
If G is finite (or compact) then every rep $\rho : G \to GL(V)$ decomposes as
$V = W_1 \oplus \dots \oplus W_s$, W_i irred subreps. "Group G is reductive"

Pf ρ is unitary. Note: $W \subseteq V$ subrep $\Rightarrow W^\perp$ subrep. □

View the irreps W_i as the "atomic parts" of ρ

Warning $G = (\mathbb{R}, +)$ is not reductive: $\rho(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $\rho(x+y) = \rho(x)\rho(y)$
$\mathbb{C}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ only proper subrep of $\mathbb{C}^2$

3. Cyclic group and Fourier transform

characters of $C_n = \{1, d, \dots, d^{n-1}\}$ [Gauss] : $\omega = e^{\frac{2\pi i}{n}}$

$\chi_k : C_n \to \mathbb{C}^\times, \; d^j \mapsto \omega^{kj}$ group morphisms, $0 \le k < n$

$\mathbb{C}[C_n] := \{ f : C_n \to \mathbb{C} \}$ $\quad C_n$ acts on $\mathbb{C}[C_n]$ via

$\rho(g)(f) := f \circ$ mult with g^{-1}

$$\begin{tikzcd} C_n \arrow[r] \arrow[rr, bend left, "\rho(g)(f)"] & C_n \arrow[r, "f"] & \mathbb{C} \\ h \arrow[r, mapsto] & g^{-1}h & \end{tikzcd}$$

$\mathbb{C}\chi_k$ is subrep of $\mathbb{C}[C_n]$:

$\rho(g)(\chi)(h) = \chi(g^{-1}h) = \chi(g)^{-1}\chi(h)$, $\quad \rho(g)(\chi_k) = \underset{\text{scalar mult}}{\chi_k(g)^{-1}\cdot}\chi_k$

$\chi_0, \dots, \chi_{n-1}$ orthonormal basis w.r.t. $\langle f_1, f_2 \rangle := \frac{1}{n} \sum_{g \in C_n} f_1(g)\overline{f_2(g)}$

$\mathbb{C}[C_n] = \bigoplus_{k=0}^{n-1} \mathbb{C}\chi_k \qquad f = \sum_{k=0}^{n-1} c_k \chi_k$, $\quad c_k = \langle f, \chi_k \rangle$ Fourier coeff

Fourier decomposition of f

View $C_n \subseteq \mathbb{C}[C_n]$ by $g \mapsto$ indicator fct of g

$\mathbb{C}[C_n]$ has basis C_n

$$\Big(\sum_g a_g g\Big)\Big(\sum_h b_h h\Big) = \sum_k \Big(\sum_{gh=k} a_g b_h\Big) k$$ Convolution product

$\mathbb{C}[C_n]$ becomes a $\mathbb{C}$-algebra wrt convolution, called group algebra of C_n

Thm

$$\mathbb{C}[C_n] \longrightarrow \mathbb{C}^n, \quad g \longmapsto (\chi_0(g), \dots, \chi_{n-1}(g))$$

defines algebra isomorphism, called Discrete Fourier Transform.

Fast algorithm for it: FFT

Applications in CS

Fast polynomial arithmetic (multiplication)

Fast integer multiplication (Schönhage & Strassen)

Similar reasonings for group S^1: leads to Fourier decomposition of "periodic" functions $f: S^1 \to \mathbb{C}$

4. Schur's Lemma

$\rho_i : G \to GL(V_i)$ reps, $i=1,2$

Def linear map $\varphi : V_1 \to V_2$ called G-equivariant if

$$\varphi(\rho_1(g)(v)) = \rho_2(g)(\varphi(v))$$ "group action commutes with φ"

Denote $\mathrm{Hom}_G(V_1, V_2) = \{\varphi : V_1 \to V_2 \text{ equivariant}\}$

Call V_1 and V_2 equivalent, $V_1 \simeq V_2$, if there is invertible G-equivariant $V_1 \to V_2$

Rem $\varphi \in \mathrm{Hom}_G(V_1, V_2) \Rightarrow \ker\varphi$, $\operatorname{im}\varphi$ are subreps

Thm (Schur's Lemma) If V_1, V_2 irred

$$\dim \mathrm{Hom}_G(V_1, V_2) = \begin{cases} 0 & \text{if } V_1 \not\simeq V_2 \\ 1 & \text{if } V_1 \simeq V_2 \end{cases}$$

Pf almost trivial

Cor G abelian, V irred $\Rightarrow \dim V = 1$

Pf $\mathrm{Hom}_G(V, V) = \mathbb{C}\, id_V$

$\rho(h) \in \mathrm{Hom}_G(V, V)$ since G abelian $\square$

5. Isotypical decomposition

Decompose rep $\rho: G \to GL(V)$ and group together equivalent subreps

$$V = \bigoplus_{i=1}^{r} \underbrace{\left(\bigoplus_{j=1}^{m_i} V_{ij} \right)}_{=: V_i}, \quad V_{ij} \simeq W_i, \quad W_1, \dots, W_r \text{ nonequivalent irreps}$$

Rem $W' \leq V$, $W' \simeq W_i \Rightarrow W' \leq V_i$

Pf $i' \neq i$, projection $\begin{matrix} V & \longrightarrow & V_{i'j} \\ \cup\!\shortmid & & \\ W' & \dashrightarrow & \end{matrix}$ is G-equivariant $\Rightarrow W' \leq V_i$ $\square$

0 by Schur's Lemma

We call $V = V_1 \oplus \dots \oplus V_r$ the isotypical decomposition

It is unique: V_i is sum of all subreps $W' \leq V$ s.t. $W' \simeq W_i$

The decomposition of $V_i = V_{i1} \oplus \dots \oplus V_{im_i}$ is not unique, but

$$m_i \overset{\uparrow}{=} \dim \mathrm{Hom}_G(W_i, V)$$ multiplicity of W_i in V, indep of decomp

Schur's Lemma

Rem Computing multiplicities is often computationally hard, e.g. for decomposition of tensor products $G \to GL(U_1 \times U_2)$, $g \mapsto \rho_1(g) \otimes \rho_2(g)$

Kronecker coefficients $(G = S_n)$, Littlewood-Richardson coefficients (GL_n)

6. Group algebra

Where (and how) to find the ireps of a finite group G?

We already introduced the group algebra $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$

Multiplication in $\mathbb{C}[G]$ is convolution $(f_1 * f_2)(g) = \sum_h f_1(h) f_2(h^{-1}g)$

$$\rho(g)(f) := f \circ \text{left mult with } g^{-1} \qquad \begin{array}{c} G \longrightarrow G \xrightarrow{f} \mathbb{C} \\ h \longmapsto g^{-1}h \end{array}$$

Thm (Frobenius)

Each irep W_i occurs as subrep of $\mathbb{C}[G]$ with multiplicity $\dim W_i$.

$\mathrm{End}(W_i) := \{\text{linear maps } W_i \to W_i\}$ is a $\mathbb{C}$-algebra (mult = composition)

The irep $\rho_i : G \to GL(W_i)$ extends to algebra morphism $\mathbb{C}[G] \xrightarrow{\rho_i} \mathrm{End}(W_i)$.

Thm If $W_1, \ldots, W_r$ are all G-ireps, up to equivalence (no repetition), then

$$\mathbb{C}[G] \longrightarrow \bigoplus_{i=1}^{r} \mathrm{End}(W_i), \quad g \longmapsto (\rho_1(g), \ldots, \rho_r(g))$$

is a $\mathbb{C}$-algebra isomorphism. [Special case of Wedderburn's Thm]

This generalizes the Discrete Fourier transform to nonabelian groups!

There is an analogous result for compact G

$$\mathbb{C}[G] \simeq \bigoplus_{i=0}^{\infty} \mathrm{End}(W_i)$$

Peter-Weyl Thm (harmonic analysis)

Ex $G = SO(3)$ rotations in $\mathbb{R}^3$. Irreps ρ_ℓ naturally labeled by $\ell \in \mathbb{N}$.

They ρ_ℓ transform the spherical harmonics $Y_\ell^{-\ell}, \dots, Y_\ell^{\ell}$

- Angular momentum in quantum mechanics
- Solution of Schrödinger's equation for H-atom

Back to finite G: center $Z(\mathbb{C}[G]) \xrightarrow{\sim} \bigoplus_{i=1}^{r} Z(\mathrm{End}(W_i)) \simeq \mathbb{C}^r$

$Z(\mathbb{C}[G]) = \{ f : G \to \mathbb{C} \mid f$ commutes with left-multiplications of $G\}$

$f \in \mathbb{C}[G] \iff \forall g \; g\left(\sum_h f(h) h\right) = \left(\sum_h f(h) h\right) g \iff \forall g \; \sum_h f(g^{-1}h) h = \sum_h f(hg^{-1}) h$

$\iff f$ constant on conjugacy classes

Cor $\#$ irreps of $G = r = \dim Z(\mathbb{C}[G]) = \#$ conjugacy classes of G

Part II

not finite

- Representions of general linear groups GL_n
  via reps of subgroup of diagonal matrices (tori)
- Highest weights and highest weight vectors
- Intriguing "duality" between GL_n reps
  and reps of symmetric groups finite

Schur-Weyl duality

7. Symmetric groups S_n: construction of irreps

Conjugacy classes given by cycle decomp. of permutations

Ex S_3 $\quad \underbrace{(123),\ (132)}_{3} \quad \underbrace{(12)(3),\ (13)(2),\ (23)(1)}_{2+1} \quad \underbrace{(1)(2)(3)}_{1+1+1}$ cycle lengths

partitions of n = # conjugacy classes of S_n = # equ. classes of S_n-irreps

Fact $\mathbb{1}$, sgn : $S_n \to \mathbb{C}^\times$ are the only 1-dim reps of S_n

"bosons" "fermions"

Partition λ of n $\longleftrightarrow$ Young diagram, e.g. $5 = 3+2$, $3 \geq 2$ $\longleftrightarrow$ $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$

tableau $t = \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$ of shape λ

row equivalent tableau $\begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \simeq \begin{array}{|c|c|c|}\hline 2 & 4 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}$

$$(12) . \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 4 & 1 & 3 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} \simeq \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} = (12) . \begin{array}{|c|c|c|}\hline 2 & 4 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}$$

S_n action well-defined on equivalence classes $[t]$ of tableau

Consider formal linear combinations of tableau classes $[t]$

Young symmetrization (Alfred Young) tableau t

$$\sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\, \sigma . [t] \quad , \quad C_t := \{ \sigma \in S_n \text{ respecting columns of } t \}$$

The span of the Young symmetrizations of all tableau t of shape λ gives an irrep S_λ "of type λ"

Specht module

(Wilhelm Specht)

Ex

$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$ $\quad S_\lambda = \mathbb{C}$ trivial rep

$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} - \dots$ $\quad S_\lambda = \mathbb{C}$ sgn rep

$v = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad w = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$

$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} = -v$ $\qquad$ v, w basis of S_λ

8. Tori

torus $T \simeq (\mathbb{C}^\times)^n$ infinite abelian group

all irreps are 1-dim, since T abelian

they are given by the group morphisms (characters)

$\chi : T \to \mathbb{C}^\times$, $\chi(st) = \chi(s)\chi(t)$

assume χ is rational function

Ex $a \in \mathbb{Z}^n$, $\chi_a(t) := t_1^{a_1} \dots t_n^{a_n}$

Prop All (rational) characters of T are of this form.

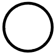
Character group $\underset{\substack{\uparrow \\ \text{mult}}}{\chi(T)} \simeq \underset{\substack{\uparrow \\ \text{addition}}}{\mathbb{Z}^n}$, $\chi_a \cdot \chi_b = \chi_{a+b}$

Thm Every (rational) representation of a torus T decomposes into a direct sum of 1-dim reps.

"tori are reductive"

Pf Via Weyl's unitarian trick

$S^1 \leq \mathbb{C}^\times$



S^1

$U := (S^1)^n \leq (\mathbb{C}^\times)^n = T$ compact subgroup

U is Zariski dense in T (think of $n = 1$)

$\rho : T \to GL(V)$ rep, entries $\rho_{ij} : T \to \mathbb{C}$ wrt basis of V are rational functions

$\rho(U)$ is Zariski dense in $\rho(T)$

There is $\rho(U)$-invariant $\langle -, - \rangle$ on V

(average over compact group U w.r.t. invariant (Haar) measure)

$W \subseteq V$ subrep $\Rightarrow$ $W^\perp$ subrep

Hence V decomposes into irred subreps. $\square$

9. Weight space decomposition

$T \simeq (\mathbb{C}^\times)^n$, $\mathcal{X}(T) \simeq \mathbb{Z}^n$, $\rho: T \to GL(V)$ rat rep

Define **weight space** of character $\chi \in \mathcal{X}(T)$

$$V_\chi := \{ v \in V \mid \forall t \in T \quad \rho(t)(v) = \chi(t)\, v \}$$

v is eigenvector of $\rho(t)$, for all $t \in T$

V_χ is the isotypical component of 1-dim irrep corr. to χ

weight space decomposition

$$V = \bigoplus_\chi V_\chi$$

Weights of V: χ s.t. $V_\chi \neq 0$

Embed $T = (\mathbb{C}^\times)^n \hookrightarrow GL_n$, $t \mapsto \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}$

Ex 1 $V = \mathbb{C}^n = \bigoplus_{i=1}^n \mathbb{C} e_i$

$\rho(t) e_i := t_i e_i = \chi_{\varepsilon_i}(t) e_i$

$\varepsilon_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \dots, 0)$

Ex 2 $V = \mathbb{C}[x_1, \dots, x_n]_d$ homog degree d polynomials

$\rho(t)(v) := v(t_1 x_1, \dots, t_n x_n)$

$$V = \bigoplus_{|a| = d} \mathbb{C}\, x^a, \quad \rho(t)(x^a) = t^a x^a$$

weight χ_a

10. General linear group

Focus now on rational representations $\rho : \overbrace{GL_n(\mathbb{C})}^{GL_n} \longrightarrow GL(V)$

Thm (Hilbert) Every rational representation of GL_n decomposes into a direct sum of irreps. "GL_n is reductive"

Pf idea Weyl's unitarian trick: unitary group is compact and Zariski dense in GL_n (nontrivial). Otherwise argue as for tori. ($\square$)

View ρ as rep of subgroup $T_n := \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\} \cong (\mathbb{C}^\times)^n$ of GL_n.

weight space decomposition

$$V = \bigoplus_{\chi \in \mathfrak{X}(T_n)} V_\chi$$

Subgroup $S_n \hookrightarrow GL_n$ permutes the weight spaces:

$$\rho(\hat{\sigma})(V_\chi) = V_{\sigma\chi}$$

$\hat{\sigma}$ permutation matrix of σ, $(\sigma\chi)(t) := \chi(\hat{\sigma}^{-1} t \hat{\sigma})$, $\sigma \chi_a = \chi_{\sigma a}$

action of

$$b_{ij}(x) = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} & i \\ \end{array}$$

$$b_{ij}(x) = \begin{bmatrix} 1 & & x \\ & \ddots & \\ & & 1 \end{bmatrix} \leftarrow j \qquad i \neq j \qquad b_{ij} : (\mathbb{C}, +) \longrightarrow Gl_n \text{ group morphism}$$

$t \in T_n$

$$t\, b_{ij}(x)\, t^{-1} = b_{ij}(t_i x t_j^{-1}) = b_{ij}\big(\overbrace{(\varepsilon_i - \varepsilon_j)(t)}^{t_i t_j^{-1}}\, x\big) \qquad \varepsilon_i - \varepsilon_j = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \dots, 0, \underset{\substack{\uparrow \\ j}}{-1}, 0, \dots, 0)$$

Lemma $\forall v \in V_\chi \quad \exists v_k \in V_{\chi + k(\varepsilon_i - \varepsilon_j)}$

$$\forall x \qquad b_{ij}(x)\, v = v + \sum_{k \geq 1} x^k v_k$$

Dominance order on $X(T_n) \simeq \mathbb{Z}^n$ $\qquad\qquad \trianglelefteq$ is partial order

$$\sum_i a_i \varepsilon_i \trianglelefteq \sum_i b_i \varepsilon_i \iff \forall s \quad \sum_{i=1}^{s} a_i \leq \sum_{i=1}^{s} b_i$$

Note
$$\chi \trianglelefteq \chi + k(\varepsilon_i - \varepsilon_j) \quad \text{if } i < j,\ k > 0$$

Thus for upper triangular $b_{ij}(x)$ $\quad (i<j)$

$b_{ij}(x)\, v$ has only contributions in weight spaces of weight larger than weight χ of v

11. Highest weight vectors

$V = \bigoplus_{\chi} V_\chi$ weight space decomposition of $\rho: GL_n \to GL(V)$

Let χ_{max} be maximal wrt $\trianglelefteq$ among the weights χ st $V_\chi \neq 0$

By Lemma, for $v \in V_{\chi_{max}}$, $i<j$,

$$b_{ij}(x)v = v$$

Thus v is invariant under subgroup $U_n := \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

Line $\mathbb{C}v$ is invariant under

$$B_n := \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \quad \text{Borel subgroup}$$

Such v is called highest weight vector of ρ.

hwv

Thm Any rat. GL_n-irrep has exactly one B_n-invariant line $\mathbb{C}v$.

Let $\chi_{max} = \sum_i \lambda_i \varepsilon_i$ be its weight. Then:

(1) $V_{\chi_{max}} = \mathbb{C}v$

(2) $\forall \chi \quad V_\chi \neq 0 \Rightarrow \chi \leq \chi_{max}$

(3) $\lambda_1 \geq \dots \geq \lambda_n$

(4) $V = \mathrm{span}\{\rho(g)(v) \mid g \in GL_n\}$

" Properties of highest weight vector v "

Cor ∘ To any rat GL_n-irrep can assign a unique highest weight $\lambda \in \mathbb{Z}^n$, $\lambda_1 \geq \dots \geq \lambda_n$.

∘ Two GL_n-irreps are equivalent iff they have the same highest weight.

Ex 1 $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$

highest weight $\varepsilon_1 = (1,0,\dots,0)$

highest weight vector e_1

Ex 2 $\mathbb{C}[x_1,\dots,x_n]_d = \bigoplus_{|a|=d} \mathbb{C}x^a$

highest weight $d\varepsilon_1 = (d,0,\dots,0)$

highest weight vector x_1^d

Ex 3 Skew symmetric tensors "noncommutative polynomials" in x_i, $x_j \wedge x_i = -x_i \wedge x_j$

$x_i \wedge x_i = 0$

$$\Lambda^d \mathbb{C}^n = \bigoplus_{i_1 < \dots < i_d} \mathbb{C}\, \underbrace{e_{i_1}}_{x_{i_1}} \wedge \dots \wedge \underbrace{e_{i_d}}_{x_{i_d}} \qquad \text{weights } \varepsilon_{i_1} + \dots + \varepsilon_{i_d}$$

highest weight $\varepsilon_1 + \dots + \varepsilon_d = (1, \dots, 1, 0, \dots, 0)$

highest weight vector $e_1 \wedge \dots \wedge e_d$

Ex 4 $V = \mathbb{C}^{n \times n}$ conjugation action $\rho(g)(A) = g A g^{-1}$ $\quad g \in GL_n$, $A \in \mathbb{C}^{n \times n}$

$V_0 := \{ A \in \mathbb{C}^{n \times n} \mid \operatorname{trace}(A) = 0 \}$ subrep, $V = V_0 \oplus \mathbb{C} I_n$

$$V_0 = \bigoplus_{i \neq j} \mathbb{C} e_{ij} \oplus \left\{ \sum_{i=1}^n x_i e_{ii} \;\middle|\; \sum_i x_i = 0 \right\} \qquad \text{weight space decomp}$$

unique highest weight $\varepsilon_1 - \varepsilon_n = (1, 0, \dots, 0, -1)$

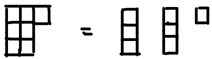
$\Rightarrow$ V_0 irreducible

12. Construction of Schur-Weyl rep W_λ

$\lambda_1 \geq \dots \geq \lambda_n \geq 0$ partition, size $|\lambda| = \lambda_1 + \dots + \lambda_n$

$\omega_k := \varepsilon_1 + \dots + \varepsilon_k = (\overbrace{1, \dots, 1}^{k}, 0, \dots, 0)$

Write $\lambda = \sum_{k=1}^{n} d_k \omega_k$



size $|\lambda| = 7$

$\Rightarrow |\lambda| = \sum_{k=1}^{n} d_k k$

$(3, 2, 2) = 2 \cdot (1, 1, 1) + (1, 0, 0)$

$$v_\lambda := \bigotimes_{k=1}^{n} (e_1 \wedge \dots \wedge e_k)^{\otimes d_k} \in \bigotimes_{k=1}^{n} \left(\Lambda^k \mathbb{C}^n\right)^{\otimes d_k} \subseteq \bigotimes_{k=1}^{n} (\mathbb{C}^n)^{\otimes k d_k} = (\mathbb{C}^n)^{|\lambda|}$$

Easy to check:

- v_λ is U_n-invariant
- v_λ is weight vector of weight λ

$\xRightarrow{\text{Thm}}$ v_λ is highest weight vector

v_λ generates GL_n-irrep W_λ

13. Schur-Weyl duality

$TP := (\mathbb{C}^n)^{\otimes d}$ has action ρ of symmetric group S_d and Gl_n-action $g \mapsto g^{\otimes d}$ } commuting actions

Thm Every Gl_n-equivariant linear map $TP \to TP$ is a lin. combination of $\rho(\pi)$'s.
Every S_d-equivariant linear map $TP \to TP$ is a lin. combination of $g^{\otimes d}$'s

Thm (Schur)
$(\mathbb{C}^n)^{\otimes d}$ has the same isotypical decomp wrt. S_d and Gl_n action.

Ex

$$(\mathbb{C}^n)^{\otimes 2} = \overbrace{S^2\mathbb{C}^n}^{\text{symm matrices}} \oplus \overbrace{\Lambda^2\mathbb{C}^n}^{\text{antisymm matrices}}$$

Gl_n	irrep	irrep
S_2		
	1	sgn

Ex

$$(\mathbb{C}^n)^{\otimes 3} = \overbrace{S^3\mathbb{C}^n}^{\mathbb{C}[x_1,\dots,x_n]_3 \,\cong} \oplus \Lambda^3\mathbb{C}^n \oplus (T_1 \oplus T_2)$$

Gl_n	irrep	irrep	equivalent irreps
S_3			
	1	sgn	

14. GL_n-reps relevant in GCT

$\rho: GL_n \to GL(V)$ rat rep

$\mathbb{C}[V] := \{ f: V \to \mathbb{C} \text{ polyn. fct} \} = \mathbb{C}[Y_1, \ldots, Y_m]$ if $V = \mathbb{C}^m$

GL_n acts on $\mathbb{C}[V]$: $(gf)(v) := f(\rho(g)^{-1}(v))$

irred subreps of $\mathbb{C}[V]$ determined by hwv $f: V \to \mathbb{C}$

Note: f_i hwv, weight λ_i $\Rightarrow$ $f_1 \cdot f_2$ hwv, weight $\lambda_1 + \lambda_2$

Assume $V = \mathbb{C}[x_1, \ldots, x_n]_d$, $f: V \to \mathbb{C}$, $v \mapsto f(v)$ polynomial in coefficients of v

Isotypical decomp of $\mathbb{C}[V]$?

multiplicities = plethysm coefficients

More generally: $Z \subseteq V$ Zariski closed, GL_n-invariant (e.g. orbit closure)

Isotypical decomp. of $\mathbb{C}[Z]$?