We have the following

PROPOSITION Let $K, \delta > 0$ be given and D sufficiently large. In fact $K = o(\log D)$ suffices. Given numbers $\{A(k);\ 0 \leq k \leq K\}$ satisfying

$$|A(k)| \lesssim D^{k+1} \qquad (0 \leq k \leq K) \qquad (1)$$

there exist $x_d \in [-\frac{1}{2}, \frac{1}{2}]$ for $D < d < D^{1+\delta}$ s.t

$$A(k) = \sum x_d\, d^k \quad \text{for all } 0 \leq k \leq K \qquad (2)$$

In fact we will only use $d \sim D^{1+\delta}$.

Denote for $\frac{D}{2} < \alpha < D$ a prime by $\Delta_\alpha = \{\frac{1}{2} D^{1+\delta} < d < D^{1+\delta};\ \alpha | d\}$.

This gives $\frac{D}{\log D}$ disjoint sets Δ_α of size $\sim D^\delta$. In order to prove (2), it clearly suffices to show that if $y = (y_0, \dots, y_K)$ satisfies

$$|y_k| < D^k \log D \qquad (0 \leq k \leq K) \qquad (3)$$

then, for each α, there is a representation

$$y_k = \sum_{d \in \Delta_\alpha} x_d\, d^k \qquad (0 \leq k \leq K) \qquad (5)$$

with $x_d \in [-\frac{1}{2}, \frac{1}{2}]$.

Fix α and set $\Delta = \Delta_\alpha$

For each $d \in \Delta$, denote ν_d the probability measure on $\mathbb{R}^{K+1}$ induced by the map $[-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}^{K+1} : t \to t.(1, d, d^2, \dots, d^K)$. Thus ν_d is supported by a line segment in $\mathbb{R}^{K+1}$. Set

$$\mu = \mathop{*}_{d \in \Delta} (\nu_d * \nu_d * \nu_d * \nu_d) \qquad (5)$$

Since $\hat{\mu}(\xi) = \prod_{d \in \Delta} |\hat{\nu}_d(\xi)|^4$, $\hat{\mu}(\xi) \geq 0$ and $\|\mu\|_\infty = \mu(0)$

Also $\operatorname{supp} \mu \subset B_{D^\delta \cdot D^{K(1+\delta)}}$ implying

$$\mu(0) > D^{-2K^2} \qquad (6)$$

For $k = 0, 1, \dots, K$, let

$$I_k = [-D^{-k-\frac{\delta}{4}}, D^{-k-\frac{\delta}{4}}]$$

and

$$Q = \sum_{k=0}^{K} I_k e_k = \text{box in } \mathbb{R}^{K+1}$$

By construction

$$\prod_{d \in \Delta} |\hat{\nu}_d(\xi)| = \prod_{d \in \Delta} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(t \left(\sum \xi_k d^k\right)\right) dt \right|$$

$$\leq \prod_{d\in\Delta} \min\left(1, \frac{|\sin\pi(\sum \xi_k d^k)|}{\pi|\sum \xi_k d^k|}\right) \tag{7}$$

Let $\Delta' = \left\{ d \in \Delta;\ \frac{1}{\lambda(K+1)} D^{1+\delta} < d < \frac{1}{K+1} D^{1+\delta} \right\}$. Hence $j\Delta' \subset \Delta$ for $j \leq K+1$

We claim that for each $d \in \Delta'$

$$\max_{1\leq j\leq K+1} \left| \sum_{k=0}^{K} \xi_k (jd)^k \right| > c^{-K} \max_k d^k |\xi_k| \tag{8}$$

To see this, set $\xi'_k = d^k \xi_k$ and assume

$$\max_{1\leq j\leq K+1} \left| \sum_{k=0}^{K} j^k \xi'_k \right| < \theta$$

Using VdM determinants, it follows that

$$\max |\xi'_k| < c^K \theta$$

implying (8).

It follows from (8) that if $\xi \notin Q$, then for each $d \in \Delta'$

$$\max_{1\leq j\leq K+1} \left| \sum_{k=0}^{K} \xi_k (jd)^k \right| > \min_k \left\{ c^{-K} \left(\frac{D^{1+\delta}}{2(K+1)} \right)^k D^{-k-\frac{\delta}{4}} \right\} > D^{-\frac{\delta}{3}} \tag{9}$$

(since $K = o(\log D)$)

Therefore

$$(7) < \left(1 - c\, D^{-\frac{2\delta}{3}}\right)^{\Delta'} < e^{-cD^{-\frac{2\delta}{3}} \frac{D^\delta}{2K}} < e^{-D^{\delta/4}} \qquad (10)$$

Define

$$\mu_1(x) = \int_Q \hat{\mu}(\xi)\, e(x\xi)\, d\xi$$

Then, by (10) and (5)

$$\|\mu - \mu_1\|_\infty \leq \int_{\mathbb{R}^{K+1} \setminus Q} |\hat{\mu}(\xi)|$$

$$< e^{-D^{\delta/4}} \int_{\mathbb{R}^{K+1}} \prod_{d \in \Delta} |\hat{\nu}_d(\xi)|^2$$

$$= e^{-D^{\delta/4}} \left\| \mathop{*}_{d \in \Delta} \nu_d \right\|_2^2 \qquad \text{(Parseval)}$$

$$\leq e^{-D^{\delta/4}} \left\| \mathop{*}_{d \in \Delta} \nu_d \right\|_\infty < e^{-D^{\delta/4}} \qquad (11)$$

Note indeed that if we have some $d_1 \in \Delta'$, then

$$\left\| \mathop{*}_{d \in \Delta} \nu_d \right\|_\infty \leq \| \nu_{d_1} * \nu_{2d_1} * \dots * \nu_{(K+1)d_1} \|_\infty < 1$$

To prove (4) for y satisfying (3), it suffices to show that $\mu(y) \neq 0$ and by (6), (11) this will follow from $\mu_1(y) \sim \mu_1(0)$. It follows from definition of Q that if $\xi \in Q$ and y as above

$$|\xi . y| \leq \sum D^{-k-\frac{\varepsilon}{4}} D^k \log D < D^{-\frac{\varepsilon}{5}}$$

Using 1-point spectral synthesis, write for $|\psi| < D^{-\frac{\varepsilon}{5}}$

$$1 - e^{2\pi i \psi} = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \psi} \qquad (12)$$

with

$$\sum |c_n| \lesssim D^{-\frac{\varepsilon}{5}}$$

Applying (12) with $\psi = \xi . y$ gives

$$|\mu_1(0) - \mu_1(y)| = \left| \int_Q (1 - e(y.\xi))\, \hat{\mu}(d\xi) \right|$$

$$\leq \sum_{n \in \mathbb{Z}} |c_n| \left| \int_Q e(ny.\xi)\, \hat{\mu}(d\xi) \right|$$

$$= \sum_{n \in \mathbb{Z}} |c_n| \, \mu_1(ny)$$

$$\leq D^{-\frac{\delta}{5}} \, \|\mu_1\|_\infty$$

$$= D^{-\frac{\delta}{5}} \, \mu_1(0)$$

Thus $\mu_1(g) \sim \mu_1(0)$ as required.

This proves the Prop

REMARKS

(1) Inspecting the argument, in particular (9), we see that δ, K, D should satisfy

$$\delta \log D > cK + c \log\log D \qquad (13)$$

(2) Performing an additional randomization, we see that we may take $x_d \in \{1,-1\}$, replacing (2) by

$$(14) \qquad \Big|A(k) - \sum x_d \, d^k\Big| < D^{k+\frac{1}{2}+\delta(k+\frac{1}{2})} \qquad (0 \leq k \leq K)$$