Approximation Algorithms and Quadratic Form Maximization over Convex Sets

Vijay Bhattiprolu

Intro to Approximation Algorithms

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- What do such algorithms typically look like? (Today: Convex Programming)
- Can one prove a certain algorithm achieves the optimal constant? (Assuming $\mathrm{P} \neq \mathrm{NP}$ or similar hypotheses) (related to Probabilistically Checkable Proofs)

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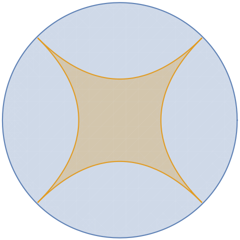
This Convex Programming algorithm achieves the optimal constant assuming the Unique Games Conjecture [Khot, Kindler, Mossel , O'Donnell, Oleszkiewicz]

Convex Relaxation + Rounding Paradigm

Given combinatorial optimization problem.

Convex Relaxation + Rounding Paradigm



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(1) "Relax" problem to convex program.

Interlude: Convex Relaxation

$$\sup_{x \in S_1} \langle a, x \rangle \leq \sup_{x \in S_2} \langle a, x \rangle$$

Relax the complicated set S_1 to a larger convex set S_2 (with a membership oracle).

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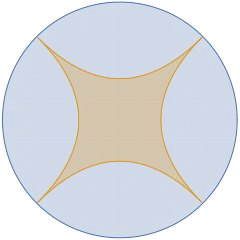

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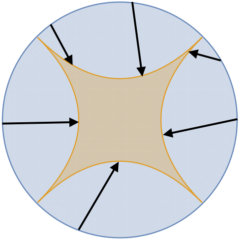

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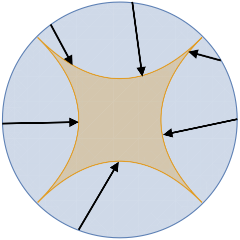

Often the best polytime approximation algorithm. (Under complexity assumptions.)

Convex Vector Relaxation for MAX-CUT

MAX-CUT reformulation:

$$\mathrm{OPT} = \ \mathsf{sup} \ \sum_{ij \in E} (1 - x_i x_j)/2 \quad \text{s.t. } \forall i, \ x_i \in \{\pm 1\}$$

Natural Vector Relaxation:

$$\mathrm{CP} = \ \mathsf{sup} \ \sum_{ij \in E} (1 - \langle u_i, u_j \rangle)/2 \ \text{ s.t. } \forall i, \ \|u_i\|_2 = 1,$$

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(Substituting $\mathbb{X}_{i,j} := \langle u_i,\ u_j \rangle$)

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Rounding Algorithm: Choose a random hyperplane through the origin and partition vectors according to it.
[Goemans Williamson 97]: Achieves ~ 1.14 approximation

Optimality of Relaxation + Rounding Paradigm

[Raghavendra 08]

- Given a Constraint Satisfaction Problem.
- A natural Convex Programming relaxation is the best polytime apx. alg. under Khot's Unique Games Conjecture.

My Interests: Quadratic Maximization

Goal: Polynomial time Approximation Algorithm

Input: $A \in \mathbb{R}^{n \times n}$ and an oracle computing the norm $(\| \cdot \|_X, \mathbb{R}^n)$.

Compute in polynomial time (approximately):

$$\sup_{\|x\|_X \leq 1} \langle x, Ax \rangle = \sum_{i,j} A_{i,j} \cdot x_i \cdot x_j \qquad \text{Quadratic Maximization}$$

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Very rich class. Captures tractable and highly intractable problems

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- ℓ_2: Maximum Eigenvalue. Exactly computable.

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- (Schatten) S_∞: $O(1)$ Apx.
  Non-commutative Grothendieck Inequality, Quantum Information Theory, etc.
  [Pisier 78, Haagerup 87] [Naor Regev Vidick 12].

Questions of Interest

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Hope: Theory can be used for both Algorithmic and Impossibility results for Continuous/Combinatorial Optimization.

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- When is Convex Programming the Optimal Algorithm?

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Generic Framework for Quadratic Maximization:

- Encompasses situations where $O(1)$-approximation algorithms were known.
- A rich family of new examples where $O(1)$-approximation is possible.
- Characterization (under complexity assumptions) for special families:
  (a) Norms invariant to permutations and sign-flips
  (b) Unitarily Invariant Matrix Norms.

Thank You. Questions?