Gap probabilities and Riemann-Hilbert problems in determinantal random point processes with or without outliers

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Abstract

It is well known that the gap probabilities for determinantal random point processes are computed by suitable Fredholm determinants of integral operators. For special type of kernels known as "integrable" (Its-Izergin-Korepin-Slavnov) the connection with a Riemann–Hilbert problem is also well known. On the other hand, in the case of processes, the kernels do not have this property but we will show how to still connect an appropriate RHP. The approach also yields a more straightforward proof of the well-known Tracy Widom distribution expressed in terms of Painleve' II solutions (a matrix version of the result produces solutions of the non-commutative PII equation). We will also show how gap probabilities of processes with outliers (Airy and other examples) relate to the notion of Schlesinger discrete transformations, a notion that originates in the theory of ODEs but can be extended to RHPs as well.

Outline

- Determinantal Random processes (and fields): gap probabilities.
- Fredholm determinants; IIKS (integrable) kernels
- RHP formulation and examples;
- Malgrange one-form and tau functions;
- Schlesinger-Darboux.
- Example: Baik's Painlevé formula and generalizations.

Determinantal Random Point Processes

We refer to the excellent review of A. Soshnikov ['00].

Definition

A Random Point Process is a probability on the space of configuration of $N \leqslant \infty$ points in a configuration measure space $(X, \mathrm{d}x)$ (e.g. $\mathbb{R}$). It is determined by the **correlation functions**

$$\rho_k(x_1, x_2, \dots, x_k) \prod \mathrm{d}x_j = \mathbb{E}\left(\text{Number of particles in each } [x_j, x_j + \mathrm{d}x_j]\right) \qquad (1)$$

It may depend on parameters (time $\Rightarrow$ nonstationary RPP)

If B_j are (Borel) subsets of X and $\#_j$ = number of points in B_j (an **integer-valued random variable**) then the above reads

$$\left\langle \prod_{j=1}^{m} \binom{\#_j}{k_j} \right\rangle = \frac{1}{\prod_{j=1}^m k_j!} \int_{B_1^{k_1} \times \dots B_m^{k_m}} \rho_k(x_1, \dots, x_{k_1}, x_{k_1+1} \dots) d^k x \qquad (2)$$

where $k = \sum_{j=1}^m k_j$.

Generating functions

In general

Definition

The generating functions of the occupation numbers in the sets B_j

$$F_{\vec{B}}(z_1, \dots, z_m) := \left\langle \prod_{j=1}^{m} (z_j)^{\#_j} \right\rangle = \sum_{\ell_1, \dots, \ell_m = 0}^{\infty} \left\langle \prod_{j=1}^{m} \binom{\#_j}{k_j} (1 - z_j)^{\ell_j} \right\rangle \qquad (3)$$

We take the simplest case of one set, for simplicity:

$$F_B(z) := \left\langle (z)^{\#_B} \right\rangle = \sum_{k=0}^{\infty} \left\langle \binom{\#_B}{k} (1 - z)^k \right\rangle = \sum_{k=0}^{\infty} \left\langle \binom{\#_B}{k} \right\rangle (1 - z)^k \qquad (4)$$

(then use (2)) which should be quite self evident by the Taylor formula

$$z^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (1 - z)^k \qquad (5)$$

Determinantal Random Point Fields

Definition

The RPP is **determinantal** (DRPP) if all corr. functions are determinants of a **Kernel**

$$K(x,y) : X^2 \to \mathbb{R} \tag{6}$$

$$\rho_k(x_1,\dots,x_k) = \det \begin{bmatrix} K(x_1,x_1) & K(x_1,x_2) & \dots & K(x_1,x_k) \\ K(x_2,x_1) & \dots & & \\ \vdots & & & \\ K(x_k,x_1) & \dots & & K(x_k,x_k) \end{bmatrix} \tag{7}$$

It is clear that a necessary condition for the well-definiteness is that the above determinants are all positive (**Total Positiveness (TP) of the kernel**). One then has

Lemma

The generating function $F_{\vec{B}}(\vec{z})$ admits the following representation

$$F_{\vec{B}}(\varkappa_1,\dots,\varkappa_m) := \left\langle \prod_{j=1}^{m} (1-\varkappa_j)^{\#_j} \right\rangle = \det \left[\mathrm{Id} - \sum_{j=1}^{m} \varkappa_j K\Big|_{B_j} \right] \tag{8}$$

Fredholm determinants

Given an integral operator $\mathcal{K} : L^2(X,\, \mathrm{d}x) \to L^2(X,\, \mathrm{d}x)$ then

$$(\mathcal{K}f)(x) = \int_X K(x,y)f(y)\,\mathrm{d}y \tag{9}$$

$$\det(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{10}$$

The series defines an entire function of z as long as $\mathcal{K}$ is **trace-class**. Other *trace-ideals* (e.g. Hilbert-Schmidt) have suitable regularized determinants (but carry anomalies). For sufficiently small z (less than the spectral radius of $\mathcal{K}$) then the following can be used equivalently

$$\ln\det(Id - z\mathcal{K}) = -\sum_{n=1}^{\infty} \frac{z^n}{n}\mathrm{Tr}\mathcal{K}^n \tag{11}$$

Integrable kernels: Its-Izergin-Korepin-Slavnov (IIKS) theory

This theory links certain types of integral operators to Riemann–Hilbert problems:
Let $\Sigma \subset \mathbb{C}$ be a collection of contours and

$$K(\lambda,\mu) := \frac{f^T(\lambda)\cdot g(\mu)}{\lambda-\mu}\ , \qquad f,g \in Mat(r\times p,\mathbb{C})\ ,\quad f^T(\lambda)\cdot g(\lambda)\equiv 0 \tag{12}$$

The integral operator with kernel $K(\lambda,\mu)$ acts on $L^2(\Sigma,\mathbb{C}^p)$.
We can get informations on the Fredholm determinant of K by using the

Jacobi variational formula

$$\partial \ln\det(\mathrm{Id}-K) = -\mathrm{Tr}_{\,L^2(\Sigma)}\left((\mathrm{Id}+R)\circ\partial K\right) \tag{13}$$

where R is the **resolvent operator**:

$$R := K\circ(Id-K)^{-1} \tag{14}$$

Thus it is of interest to characterize R

The **resolvent** operator

$$R(\lambda,\mu) := K \circ (\mathrm{Id} - K)^{-1}(\lambda,\mu) = \frac{f^T(\lambda)\Gamma^T(\lambda)\Gamma^{-T}(\mu)g(\mu)}{\lambda-\mu} \tag{15}$$

where $\Gamma(\lambda)$ solves the RHP

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)\left(\mathbf{1}_r - 2i\pi f(\lambda)g^T(\lambda)\right), \quad \lambda \in \Sigma \tag{16}$$

$$\Gamma(\lambda) = \mathbf{1}_r + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty \tag{17}$$

The reason of interest in the resolvent is Jacobi's formula for the variation of the determinant

Theorem (B.-Cafasso 2011)

Let $f(\lambda;\vec{s}), g(\lambda;\vec{s}) : \Sigma \times S \longrightarrow \mathrm{Mat}_{p\times k}(\mathbb{C})$ and consider the IIKS RHP. Given any vector field ∂ in the space of the parameters S of the integrable kernel we have the equality

$$\begin{aligned} \partial \ln \det(\mathrm{Id}_{L^2(\Sigma)} - K) &= \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-' \partial M M^{-1}\right) \frac{\mathrm{d}\lambda}{2i\pi} + \\ &+ \int_{\mathcal{C}} \partial\, \mathrm{Tr}\left({f'}^T g\right) d\lambda + 2\pi i \int_{\mathcal{C}} \mathrm{Tr}(g^T f' \partial g^T f) d\lambda \qquad (18) \\ M &:= \mathbf{1} + 2i\pi f(\lambda;\vec{s}) g^T(\lambda;\vec{s}) \qquad (19) \\ K(\lambda,\mu) &= \frac{f^T(\lambda)\cdot g(\mu)}{\lambda-\mu} \qquad (20) \end{aligned}$$

Many determinantal random processes have kernels of this form; the main case is the random point field of **eigenvalues** of a (Hermitean) random matrix. For example, rescaling near the edge of the Gaussian random matrix model one has

$$K_{\mathrm{Ai}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{21}$$

$$\text{Airy random point field} \;\Rightarrow\; \text{Tracy-Widom distribution} \tag{22}$$

Example (Example of multimatrix "integrable" kernel (B. Gekhtman Szmigielski '13))

The Meijer-G random point field on $\mathbb{R}_+ \sqcup \mathbb{R}_+$ (universality near hard-edge of two-matrix Cauchy model) (M_j Hermitean, +ve def. matrices)

$$\mathrm{d}\mu(M_1, M_2) = \frac{\mathrm{e}^{-\frac{N}{T}\mathrm{Tr}(V_1(M_1)+V_2(M_2))}}{\det(M_1+M_2)^N}\,\mathrm{d}M_1\,\mathrm{d}M_2 \tag{23}$$

Problem (Notable exceptions)

- *Multi-time random fields (processes) are* **not** *of this form.*
- *Convolution operators, e.g. GOE and Airy (Ferrari-Spohn '05).*

Equivalence of determinants

Let $\mathcal{C}$ be the **matrix** convolution operator on $L^2(\mathbb{R}_+)$ with symbol

$$\mathbf{C}_s(z) = -i\int_{\gamma_+} e^{iz\mu}\mathbf{r}(\mu,s)\,\mathrm{d}\mu\ , \quad \mathbf{r}(\mu,s) := e^{i\mu s}E_1(\mu)E_2^T(\mu) \tag{24}$$

$$e^{i\mu s/2}E_j(\mu) \in L^2 \cap L^\infty(\gamma_+, Mat(r\times p)) \tag{25}$$

Here γ_+ is a (collection of) contour(s) in the upper half plane.

Theorem (B.-Cafasso, 2011)

The two Fredholm determinants below (exist!) are equal

$$\det\left[\mathrm{Id}_{L^2(\mathbb{R}_+,\mathbb{C}^r)} + \mathcal{C}_s\right] = \det\left[\mathrm{Id}_{L^2(\gamma_+,\mathbb{C}^p)} + \mathcal{K}_s\right] \tag{26}$$

with $\mathcal{K}_s : L^2(\gamma_+,\mathbb{C}^p) \circlearrowleft$ *having kernel*

$$\mathcal{K}_s(\lambda,\mu) = \frac{e^{\frac{i(\lambda+\mu)s}{2}}E_1^T(\lambda)E_2(\mu)}{\lambda+\mu}\ . \tag{27}$$

We shall study kernels of the form $\mathcal{K}$, which is **not** of the IIKS form.

Resolvents

We want to find the (kernels of the) resolvent operators

$$\mathcal{S} := -\mathcal{K} \circ (\mathrm{Id}_{\gamma_+} + \mathcal{K})^{-1} , \qquad \mathcal{R} := \mathcal{K}^2 \circ (\mathrm{Id}_{\gamma_+} - \mathcal{K}^2)^{-1} \tag{28}$$

Theorem (B.-Cafasso 2011)

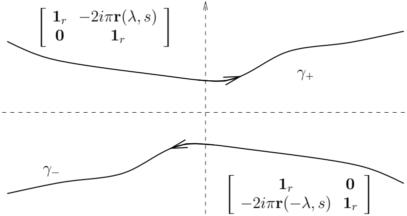
$$\mathcal{S}(\lambda, \mu) = \frac{2\mu \left[E_1^T(\lambda), \mathbf{0}_{p\times r}\right] \Gamma^T(\lambda)\Gamma^{-T}(\mu) \left[\begin{array}{c} \mathbf{0}_{r\times p} \\ E_2(\mu) \end{array}\right]}{\lambda^2 - \mu^2} \tag{29}$$

$$\mathcal{R}(\lambda, \mu) = [E_1^T(\lambda), \mathbf{0}_{p\times r}] \frac{\Xi^T(\lambda)\Xi^{-T}(\mu)}{\lambda - \mu} \left[\begin{array}{c} \mathbf{0}_{r\times p} \\ E_2(\mu) \end{array}\right] \tag{30}$$

where $\Gamma(\lambda)$, $\Xi(\lambda)$ are $2r \times 2r$ matrix solutions of two (related) Riemann–Hilbert problems on $\gamma_+ \cup \gamma_-$ ($\gamma_- = -\gamma_+$) described below.

Example

- GOE stands to GUE like Airy convolution stands to Airy kernel.
- KdV to mKdV (KdV= Fred. det of convolution op.: mKdV = Fred det of square)



$$\mathbf{r}(\lambda,s) := \mathrm{e}^{i\lambda s} E_1(\lambda) E_2^T(\lambda)$$

Problem 1	**Problem 2**
$\Xi_+(\lambda) = \Xi_-(\lambda) M(\lambda)$ $\Xi(\lambda) = \mathbf{1}_{2r} + \frac{\Xi_1}{\lambda} + \ldots$	$\Gamma_+(\lambda) = \Gamma_-(\lambda) M(\lambda)$ $\Gamma(\lambda) = \begin{bmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r & i\lambda\mathbf{1}_r \end{bmatrix} \left(\mathbf{1}_{2r} + \frac{Q \otimes \sigma_3}{\lambda} + \ldots\right)$ (31) $\Gamma(\lambda) \begin{bmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r & i\lambda\mathbf{1}_r \end{bmatrix}^{-1} = \mathcal{O}(1) \ \lambda \to 0$ $\Gamma(\lambda) = \widehat{\sigma}_1 \Gamma(-\lambda) \widehat{\sigma}_1$

Airy process: Multi-layer PolyNuclear Growth (PNG) model

The Airy process was introduced by Praehofer and Spohn in the study of the fluctuations around the top layer of the growth model.

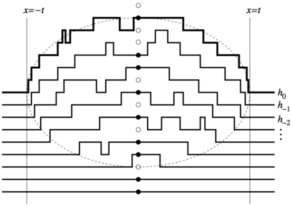
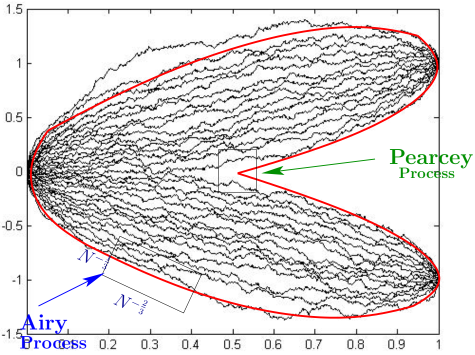


Figure : A snapshot of a multi-layer PNG configuration at time t. Asymptotic droplet is also marked. From Praehofer-Spohn, 2001

We will consider it as a scaling limit of Dyson brownian motioan.

It also occurs in the study of fluctuations around the edge in the model of self-avoiding brownian motions in the limit $N \to \infty$

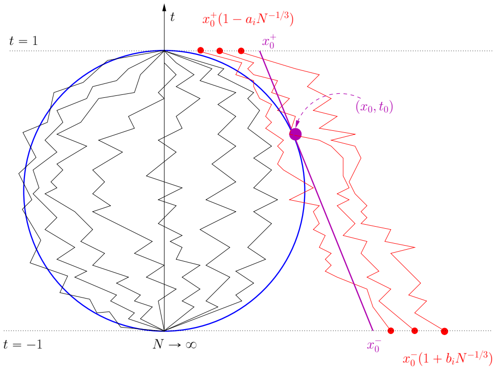


Simulation with $N = 30$ non-intersecting Brownian particles starting at $x = 0$ and ending at $x = 1$, $x = -1$. Courtesy of P.M. Roman, S. Delvaux.

$N \longrightarrow \infty$

Transition probability: $p_N(\Delta t, x, y) := Ce^{-N\frac{(x-y)^2}{2\Delta t}}$

The Airy kernel with outliers



N non–intersecting Brownian particles $\{\lambda_i(t)\}$ with transition probability

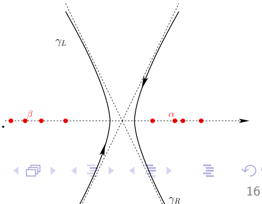
$$p(x,y;t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

Theorem (Adler–Ferrari–van Moerbeke, 2010) [also multi–time and multi–interval cases]

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{max}(t_0) < x_0\left(1+\frac{s}{2N^{2/3}}\right)\right) = F_{\mathrm{Ai}}(s;\alpha;\beta)$$

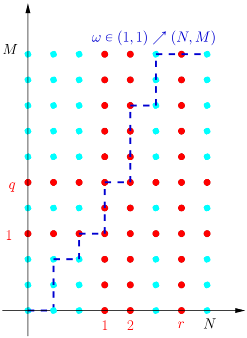
$$K_{\mathrm{Ai}}^{(\alpha,\beta)}(x,y) :=$$

$$\frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3}-\frac{w^3}{3}-zx+wy}}{w-z} \prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{j=1}^{q}\left(\frac{w-a_j}{z-a_j}\right).$$



The Airy kernel with outliers

First appeared in problems of percolation:



Now take

$$\begin{cases} \{i_1, i_2, \dots, i_r\} & \subseteq & \{1,\dots,N\}, \\ \{j_1, j_2, \dots, j_q\} & \subseteq & \{1,\dots,M\}, \end{cases}$$

$X_{(i_k, j_t)}$ exponential random variable of mean $(\ell_k + m_t)/M,\ k = 1,\dots,r; \quad t = 1,\dots,q.$

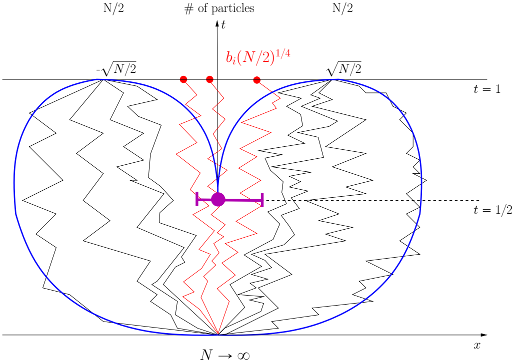
$$\begin{cases} \ell_j & := & 1+\gamma^{-1} - \frac{(1+\gamma)^{2/3} b_j}{\gamma M^{1/3}},\ j = k_1,\dots,k_r; \\ m_t & := & 1+\gamma^{-1} + \frac{(1+\gamma)^{2/3} a_t}{\gamma M^{1/3}},\ t = s_1,\dots,s_q. \end{cases}$$

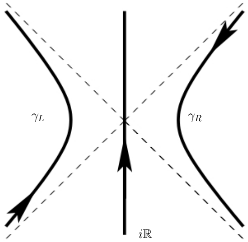
Theorem (Baik-Ben Arous-Peché 2005 for sample-covariance matrices, Borodin–Peché, 2008) [also multi–time case]

$$\mathbb{P}\left[\left(L(N,M) - \left(\frac{1+\gamma}{\gamma}\right)^2\right)\frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leqslant s\right] \longrightarrow F_{\mathrm{Ai}}(s;\alpha;\beta) = \det(\mathrm{Id} - K_{\mathrm{Ai}}^{(\alpha,\beta)}\chi_{[s,\infty)})$$

$$K_{\mathrm{Ai}}^{(\alpha,\beta)}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{\mathrm{e}^{\frac{z^3}{3} - \frac{w^3}{3} - zx + wy}}{w - z} \prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{j=1}^{q}\left(\frac{w-a_j}{z-a_j}\right).$$

The Pearcey kernel with inliers





$$\lim_{N\to\infty} \mathbb{P}\left(x_i\left(\frac{1}{2}+\frac{\tau}{4\sqrt{2N}}\right) \notin \frac{E}{4(N/2)^{1/4}};\ i=1,\dots,N\right) = \det\left(\mathrm{Id} - K_P^{(\beta)}\chi_E\right)$$

$$K_P^{(\beta)}(x,y) := \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}} dw \int_\gamma dz \frac{e^{\theta_\tau(x;z)-\theta_\tau(y;w)}}{w-z}\prod_{k=1}^{r}\left(\frac{w-b_k}{z-b_k}\right);\ \theta_\tau(x;z) := \frac{z^4}{4} - \tau\frac{z^2}{2} - xz.$$

(Adler–Delépine–van Moerbeke–Vanhaecke, 2011)

Example: 2-time Airy gap probability

For example the two–times Airy process (without outliers) has a matrix kernel

$$A(x,y) = \begin{bmatrix} \widetilde{A}_{11}(x,y) & \widetilde{A}_{12}(x,y) - B_{12}(x,y) \\ \widetilde{A}_{21}(x,y) & \widetilde{A}_{22}(x,y) \end{bmatrix} \tag{32}$$

One verifies that $A_{jj}(x,y) = K_{\mathrm{Ai}}(x,y)$ **does** have the IIKS form: however all the other (off-diagonal) entries **do not**.
Yet, we want to characterize the Fredholm determinants describing the **gap probabilities**; the simplest example of which is

$$Pr\left(\begin{array}{c} \text{no particle in } (a,\infty) \text{ at time } \tau_1 \\ \text{no particle in } (b,\infty) \text{ at time } \tau_2 > \tau_1 \end{array}\right) = \det\left(Id_{\mathbb{R}^2} - A(\bullet;\tau_1,\tau_2)_{\substack{(a,\infty)\\(b,\infty)}}\right) \tag{33}$$

Problem

Can the IIKS theory be applied? Can we obtain a Lax representation?

Note that by different methods, Tracy and Widom (2004) do obtain PDEs for the gap probabilities, but no Lax representation.

The Airy process

This is a determinantal point field with configuration space

$$X = \mathbb{R} \times \{\tau_1 < \tau_2 < \cdots < \tau_n\} \simeq \mathbb{R} \times \{1, 2, \ldots, n\} \tag{34}$$

$$A_{ij}(x,y) := \tilde{A}_{ij}(x,y) - B_{ij}(x,y) \ , 1 \leqslant i,j \leqslant n \tag{35}$$

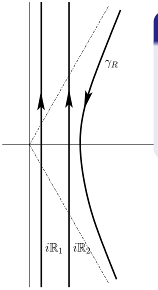
$$\widetilde{A}_{ij}(x,y) := \frac{1}{(2\pi i)^2} \int_{\gamma_{R_i}} \mathrm{d}\mu \int_{i\mathbb{R}} \mathrm{d}\lambda \frac{\mathrm{e}^{\theta(x,\mu)-\theta(y,\lambda)}}{\lambda + \tau_j - \mu - \tau_i} \tag{36}$$

$$\theta(x,\mu) := \frac{\mu^3}{3} - x\mu. \tag{37}$$

$$B_{ij}(x,y) := \chi_{\tau_i < \tau_j} \frac{1}{\sqrt{4\pi(\tau_j - \tau_i)}} \mathrm{e}^{\frac{(\tau_j-\tau_i)^3}{12} - \frac{(x-y)^2}{4(\tau_j-\tau_i)} - \frac{(\tau_j-\tau_i)(x+y)}{2}} \tag{38}$$

It represents a field of ∞'ly many particles undergoing **mutually avoiding Brownian motions**.

Equivalence of determinants: an example of relation (non-IIKS) $\leftrightarrow$ IIKS



Theorem

The following determinants are equal

$$\det\left(\mathrm{Id}_{\mathbb{R}^2} - A(\bullet;\tau_1,\tau_2)\,{}_{(a,\infty)}^{(b,\infty)}\right) = \det(\mathrm{Id} - K) \tag{39}$$

where K acts on $L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2 \cup \gamma_R, \mathbb{C}^2)$ with kernel ($i\mathbb{R}_j := i\mathbb{R} + \tau_j$, $\lambda_j := \lambda - \tau_j$, $\mu_j := \mu - \tau_j$)

$$K(\lambda,\mu) = \frac{f^T(\lambda)g(\mu)}{\lambda - \mu} \tag{40}$$

$$f(\lambda) := \begin{bmatrix} e^{\frac{\lambda_1^3}{6}}\chi_{\gamma_R} & e^{\frac{\lambda_2^3}{6}}\chi_{\gamma_R} \\ e^{a\lambda_1}\chi_{i\mathbb{R}_1} & 0 \\ 0 & e^{b\lambda_2}\chi_{i\mathbb{R}_2} \end{bmatrix}, \quad g(\mu) := \begin{bmatrix} e^{-\frac{\mu_1^3}{3}}\chi_{i\mathbb{R}_1} & e^{-\frac{\mu_2^3}{3}}\chi_{i\mathbb{R}_2} \\ e^{\frac{\mu_1^3}{6} - a\mu_1}\chi_{\gamma_R} & e^{\frac{\mu_1^3 - \mu_2^3}{3} - a\mu_1}\chi_{i\mathbb{R}_2} \\ 0 & e^{\frac{\mu_2^3}{6} - b\mu_2}\chi_{\gamma_R} \end{bmatrix} \tag{41}$$

Thsi is an integrable kernel (one has to check $f(\lambda)\cdot g^T(\lambda)\equiv 0$)
It has a Ψ–function:

$$\Psi(\lambda) := \Gamma(\lambda)\mathrm{e}^{T} \tag{42}$$

$$T(\lambda;\tau_1,\tau_2,a,b) := \mathrm{diag}\left(\frac{\frac{\lambda_1^3+\lambda_2^3}{3}+a\lambda_1+b\lambda_2}{3}, \frac{\frac{\lambda_2^3-2\lambda_1^3}{3}+b\lambda_2-2a\lambda_1}{3}, \cdots\right) \tag{43}$$

(RHP with **constant** jumps) solves an ODE in λ (which can be easily written) as well as isomonodromic deformations in a, b, τ_1, τ_2. It can be also shown that

Proposition

The Jimbo-Miwa-Ueno isomonodromic tau function coincides with the Fredholm determinant(s)

$$\partial \ln \tau_{JMU} = -\text{“}\operatorname*{res}_{\lambda=\infty}\text{“}\ \mathrm{Tr}\left(\Gamma^{-1}\Gamma'(\lambda)\partial T\right)\mathrm{d}\lambda \tag{44}$$

Similar approach works for any gap probability of

- extended Pearcey (B. Cafasso 2012);
- extended (and generalized) Bessel (Girotti 2013);
- extended tacnode [Johansson, Adler-VanMoerbeke] (B. Cafasso Girotti, in progress);

Some details on the proof

The equivalence of determinants is actually unitary ($a_1 = a, a_2 = b$, $\chi_{I_j} := [a_j, \infty)$)

$$A_{ij}(x,y)\chi_{I_i}(x) = \int_{i\mathbb{R}+\tau_i} \frac{d\xi}{2\pi i} e^{\xi_i(a_i - x)} \times \left[\int_{i\mathbb{R}+\tau_j} \frac{d\lambda}{2\pi i} \int_{\gamma_R} \frac{d\mu}{2\pi i} \frac{e^{\theta(a_i,\mu_i)-\theta(0,\lambda_j)+y\lambda_j}}{(\xi-\mu)(\mu-\lambda)} + \right. \left. + \chi_{\tau_i<\tau_j} \int_{i\mathbb{R}+\tau_j} \frac{d\mu}{2\pi i} \frac{e^{\theta(a_i,\mu_i)-\theta(0,\mu_j)+y\mu_j}}{\xi-\mu}\right]$$

After Fourier transform (some care to be paid) one has an unitarily equivalent operator on $L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2, \mathbb{C}^2)$ with kernel

$$(\mathfrak{K})_{ij}(\xi,\lambda) = = \chi_{i\mathbb{R}_i}(\xi)\chi_{i\mathbb{R}_j}(\lambda)\left(\underbrace{\int_{\gamma_R} \frac{d\mu}{2\pi i} \frac{e^{\theta(a_i,\mu_i)-\theta(0,\lambda_j)+a_i\xi_i}}{(\xi-\mu)(\mu-\lambda)}}_{\mathcal{G}\circ\mathcal{F}} + \underbrace{\chi_{\tau_i<\tau_j}\frac{e^{\theta(a_i,\lambda_i)-\theta(0,\lambda_j)+a_i\xi_i}}{\xi-\lambda}}_{\mathcal{H}}\right).$$

$$\overset{\circlearrowleft \mathcal{H}}{L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2, \mathbb{C}^2)} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} L^2(\gamma_R, \mathbb{C}^2) \tag{45}$$

So we have the determinant of

$$\det(Id - \mathcal{G}\circ\mathcal{F} - \mathcal{H}) \tag{46}$$

$$\overset{\circlearrowleft\mathcal{H}}{L^2(i\mathbb{R}_1)\oplus L^2(i\mathbb{R}_2)} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} L^2(\gamma_R,\mathbb{C}^2) \tag{47}$$

Note that all three operators are **Hilbert-Schmidt** so that $\mathcal{G}\circ\mathcal{F}$ is trace-class but $\mathcal{H}$ is not (at least we cannot prove it directly).
However the matrix kernel of $\mathcal{H}$ is upper-triangular so that it is "traceless" (it is not, technically)
But then the series of $\det_2$ for HS operators (well-defined) coincides with the series of $\det$ for trace-class (ill-defined here); thus, the correct definition is

$$\text{“}\det\text{”}(\mathrm{Id} - \mathcal{G}\circ\mathcal{F} - \mathcal{H}) := \det{}_2(\mathrm{Id} - \mathcal{G}\circ\mathcal{F} - \mathcal{H})\mathrm{e}^{-\mathrm{Tr}\mathcal{G}\circ\mathcal{F}} \tag{48}$$

Finally one uses the identity

$$\text{“}\overset{\prime\prime}{\det}(Id - \mathcal{G}\circ\mathcal{F} - \mathcal{H}) = \det{}_2\left(Id - \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{G} & \mathcal{H}\end{bmatrix}\right) \tag{49}$$

and then recognize that the last operator on $L^2(i\mathbb{R}_1\cup i\mathbb{R}_2\cup\gamma_R,\mathbb{C}^2)$ has the postulated kernel.

Why is this useful?

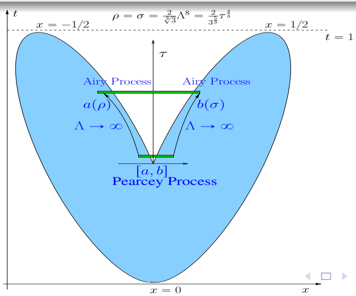
Example: study asymptotics.

Theorem (Pearcey $\longrightarrow$ (Tracy–Widom)^2)

$$F_{\mathrm{P}}([a_p, b_p]; \tau) = \mathbb{P}\left\{\mathcal{P}(\tau) \notin [a_p, b_p]\right\} \ , \qquad F_{TW}(\sigma) = \mathbb{P}\left\{\mathcal{A} \notin [\sigma, \infty)\right\} \tag{50}$$

In (B. Cafasso 2012) it was shown by using the Deift–Zhou nonlinear steepest descent method that, in particular

$$F_{\mathrm{P}}\left(\left[-2\left(\frac{\tau}{3}\right)^{\frac{3}{2}} + (3\tau)^{\frac{1}{6}}\rho, 2\left(\frac{\tau}{3}\right)^{\frac{3}{2}} - (3\tau)^{\frac{1}{6}}\sigma, \right]; \tau\right) \underset{\tau\to\infty}{\longrightarrow} F_{TW}(\sigma)F_{TW}(\rho) \tag{51}$$



Usefulness/2

A RHP formulation allows use of Hirota bilinear method to extract nonlinear PDEs.

Example (Pearcey gap prob)

The logarithm of the Fredholm determinant $g(a,b,\tau) := \log\det\left(\mathrm{Id} - K_P\big|_{[a,b]}\right)$ satisfies the differential equations in $\partial_E := \partial_a + \partial_b$, ∂_τ, $\epsilon := a\partial_a + b\partial_b$

$$\partial_E^4 g + 6(\partial_E^2 g)^2 - 4\tau\partial_E^2 g + 12\partial_\tau^2 g = 0$$

$$\left(-3\epsilon - 2\tau\partial_\tau + 2\partial_\tau\partial_E^2 + 1\right)\partial_E g + 12(\partial_E^2 g)(\partial_\tau\partial_E g) = 0$$

$$\begin{aligned}\epsilon\ &\left(12\partial_\tau g - 2\partial_E^2 g\right) + \left(8\partial_\tau^2 g + 4\partial_\tau\partial_E^2 g - 4\partial_E^4 g - 8\left(\partial_E^2 g\right)^2\right)\tau + \\ &+4\partial_E^2 g + 16\left(\partial_E^2 g\right)^3 + 8\left(\partial_E\partial_\tau g\right)\partial_E^3 g + 10\left(\partial_E^3 g\right)^2 + 16\left(\partial_E^4 g\right)\partial_E^2 g + \\ &+\partial_E^6 g - 16\,\partial_\tau^3 g + 4\,\partial_\tau^2\partial_E^2 g - 24\left(\partial_E\partial_\tau g\right)^2 - 8\left(\partial_\tau\partial_E^2 g\right)\partial_E^2 g - 8\partial_\tau g = 0\end{aligned}$$

The first two were found by Adler-VanMoerbeke-Cafasso using vertex operators: the third found by the isomonodromic approach by B. Cafasso.
We now know that they all are reduction of a simple KP-like bilinear formulation, ipso-facto thanks to the RHP formulation.

Outlier insertion and Schlesinger transformations

The kernels (Pearcey/Airy + outlier) are *finite rank perturbations*
$\Longrightarrow$ ratio of gap prob. are finite determinants of the Schur complement.

Let R be finite rank, A any operator ("determinantable" without anomaly, i.e. $Id + tr.Cl.$ (rather than $Id + HS$, e.g.)

$$\det(A + R) = \det(A)\det(I + A^{-1}R) \tag{52}$$

The second is a finite-rank perturbation of the identity, hence can be written as a finite det.

Incidentally: the Tacnode

The gap probs. of the tacnode process are (non finite-rank) cases of the above obtained by a (formal) restriction of a simpler determinantal point process (see B.Cafasso 2013 (appendix) and also Bufetov 2012)

Dyson b.m.: Tacnode gap probabilities

Theorem (B. Cafasso 2013)

$$\mathrm{Prob}\left\{\mathcal{T}_\sigma(\tau_i) \notin E^{(i)}, \quad i = 1, \ldots, r\right\} = \frac{\det\left[\mathrm{Id} - \mathbb{H}_E\right]}{F_{TW}(2^{\frac{2}{3}}\sigma)} \qquad (53)$$

$$\mathbb{H}((x, \tau_j); (y, \tau_i)) = \qquad (54)$$

$$= \left[\begin{array}{c|c|c} 0 & -\mathrm{Ai}(x+y) & \mathrm{Ai}^{(-\tau_j)}(x\sqrt[3]{2} + \sigma - y) \\ \hline -\mathrm{Ai}(x+y) & 0 & \mathrm{Ai}^{(-\tau_j)}(x\sqrt[3]{2} + y - \sigma) \\ \hline \mathrm{Ai}^{(\tau_i)}(\sigma - x + y\sqrt[3]{2}) & \mathrm{Ai}^{(\tau_i)}(x - \sigma + y\sqrt[3]{2}) & -p(\tau_i - \tau_j; x, y)\chi_{i>j} \end{array}\right],$$

where $\mathrm{Ai}^{(\tau)}(x) = 2^{\frac{1}{6}} \mathrm{e}^{\tau x + \frac{2}{3}\tau^3} \mathrm{Ai}(x + \tau^2)$ and τ, σ are parameters of the tacnode process, respectively the time and the "pressure". A RHP formulation can thus be obtained because it is a convolution operator.

Useful (e.g.) for numerical evaluations.

Airy with outliers: RHP

Theorem

The Fredholm determinant $F^{(\alpha,\beta)}(\sigma) = \det(\mathrm{Id} - K^{(\alpha,\beta)}\chi_E)$ *associated to the kernel*

$$K^{(\alpha,\beta)}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta(x;z)-\theta(y;w)}}{w-z} \prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right).$$

coincides with the isomonodromic tau function of the following RH problem

(General RH(α,β))

$$\begin{cases} \Gamma_+^{(\alpha,\beta)}(\lambda) = \Gamma_-^{(\alpha,\beta)}(\lambda) \begin{bmatrix} 1 & -e^{\theta(s_1;\lambda)}C(\lambda)\chi_1 & \dots & (-)^N e^{\theta(s_N;\lambda)}C(\lambda)\chi_1 \\ -\frac{e^{-\theta(s_1;\lambda)}}{C(\lambda)}\chi_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{e^{-\theta(s_N;\lambda)}}{C(\lambda)}\chi_2 & 0 & \dots & 1 \end{bmatrix} \\ \Gamma^{(\alpha,\beta)}(\lambda) \sim \mathbf{1} + \Gamma_1^{(\alpha,\beta)}\lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad \lambda\to\infty; \quad C(\lambda) := \frac{\prod_{k=1}^r(\lambda-b_k)}{\prod_{j=1}^q(\lambda-a_j)}. \end{cases}$$

In particular $\partial_{s_i}\log F^{(\alpha,\beta)} = -\left(\Gamma_1^{(\alpha,\beta)}\right)_{(i+1,i+1)}, \quad \forall i = 1,\dots,N.$

Malgrange differential and variation of determinants

Given a RHP with jumps on Σ (collection of contours) for an $n \times n$ matrix $\Gamma(z;\underline{t})$

(glossing over details)

$$\Gamma_+(z;\underline{t}) = \Gamma_-(z;\underline{t})M(z;\underline{t}) \ , \quad \Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1}),\ z \to \infty. \tag{55}$$

Definition (Malgrange one-form)

The Malgrange-one form ("Liouville" form)[a] is

$$\Omega(\partial;[M]) = \frac{1}{2}\int_\Sigma \frac{\mathrm{d}z}{2i\pi}\mathrm{Tr}\Big(\Gamma_-^{-1}\Gamma_-'\partial M M^{-1} + \Gamma_+^{-1}\Gamma_+' M^{-1}\partial M\Big) \tag{56}$$

$$\mathrm{d}\Omega(\partial,\tilde\partial;[M]) = \frac{1}{2}\int_\Sigma \frac{\mathrm{d}z}{2i\pi}\mathrm{Tr}\Big(M^{-1}M'\left[M^{-1}\partial M, M^{-1}\tilde\partial M^{-1}\right]\Big) \tag{57}$$

[a] (almost like this) appears in Malgrange, 1983

- If the form is closed then there is a (local) function τ: in certain cases it coincides with the "isomonodromic" τ function of the Japanese school [B. 2010] and allows to find deformations wrt *monodromy* data.
- If $M = \mathbf{1} + N$ with $N^2 \equiv 0$ then it is a (Carleman) Fredholm determinant of a IIKS (integrable) operator (up to an anomaly) [B. Cafasso 2011].

General theorem (B 2013.9, but see also JMU 1980)

Let $D(z)$ be a **diagonal, rational matrix**, Consider two RHPs

$$\begin{array}{c|c} \Gamma_+ = \Gamma_- M \quad z \in \Sigma & \widehat{\Gamma}_+ = \widehat{\Gamma}_- D^{-1} M D \quad z \in \Sigma \\ \Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \,, \quad z \to \infty & \widehat{\Gamma}(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \,, \quad z \to \infty \end{array} \tag{58}$$

Let ∂ be any deformation of $\underline{t}$ or position of poles/zeroes of $D(z)$: the variation of the *Malgrange differential* ("differential of $\ln \tau$" + anomaly)

$$\Omega(\partial; [D^{-1} M D]) - \Omega(\partial; [M]) =$$

$$= \partial \ln \det G_{\left\{ \mathcal{A} \, \mathcal{B} \atop K \, L \right\}} + \partial \ln \prod_{\mu=1}^{n} \frac{\prod_{a \in \mathcal{A}' \atop b \in \mathcal{B}'} (a-b)^{k_{a,\mu} \ell_{b,\mu}}}{\prod_{a<a'} (a-a')^{k_{a,\mu} k_{a',\mu}} \prod_{b<b'} (b-b')^{\ell_{b,\mu} \ell_{b',\mu}}} +$$

$$+ \frac{1}{2} \int_\Sigma \frac{\mathrm{d}z}{2i\pi} \mathrm{Tr} \Big(D' D^{-1} \left(\partial M M^{-1} + M^{-1} \partial M + M \partial D D^{-1} M^{-1} - M^{-1} \partial D D^{-1} M \right) +$$

$$- \partial D D^{-1} \left(M^{-1} M' + M' M^{-1} \right) \Big)$$

where $\mathcal{A}, \mathcal{B}$ are the poles/zeroes of $D(z)$ of multiplicities $k_{a,\mu}, \ell_{b,\mu}$ on the μ–entry.

Definition

The **characteristic matrix**[a] $G_{\left\{ \begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ K & L \end{smallmatrix} \right\}}$ is the following matrix

$$G_{(a,\nu,k);(b,\mu,\ell)} = \operatorname*{res}_{z=a} \operatorname*{res}_{\zeta=b} \frac{\mathbf{e}_\mu^t \Gamma^{-1}(z)\Gamma(\zeta)\mathbf{e}_\nu \, \mathrm{d}z_b \, \mathrm{d}\zeta_a}{(z_b)^{\ell-\delta_{b\infty}}(z-\zeta)(\zeta_a)^{k_{a,\nu}+1-k-\delta_{a\infty}}} \tag{59}$$

where $z_c = (z-c)$ if c is a finite point and $z_\infty = 1/z$ if $c = \infty$, and where $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $1 \leqslant k \leqslant k_{a,\nu} = (K_a)_{\nu\nu}$, $1 \leqslant \ell \leqslant \ell_{b,\nu} = (L_b)_{\nu\nu}$, $1 \leqslant \mu, \nu \leqslant n$. This matrix has size $\sum_{a\in\mathcal{A}} \sum_{\nu=1}^n k_{a,\nu}$ (note that the total number of poles/zeroes is equal, counting the poles /zeroes at infinity).

[a] We borrow the name from [JMU2].

Remark

The matrix appears in [JMU2] in the context of monodromy/spectrum preserving deformations (with a proof by induction). No anomaly appears in that context.

In all the cases of outliers $D(z) = A_1(z)^{E_{11}} A_2(z)^{E_{22}}$, $A_1(z) = \prod_{j=1}^{q}(z-a_j)$, $A_2(z) = \prod_{k=1}^{r}(z-b_j)$,

$$\mathcal{A} = \{\infty\},\ \mathcal{B} = \{a_1, \dots, a_q, b_1, \dots, b_r\} \tag{60}$$

$$K_\infty = rE_{11} + qE_{22},\ L_{a_j} = E_{11}; L_{b_j} = E_{22}, \tag{61}$$

and

$$G_{\left\{ {\{\infty\} \atop K} {\mathcal{B} \atop L} \right\}} = \left[\begin{array}{c|c} \underset{\infty}{\operatorname{res}} \dfrac{\left[z^{\ell-1}\Gamma^{-1}(a_j)\Gamma(z)\right]_{11}}{z-a_j}_{1\leqslant \ell, j\leqslant q} & \underset{\infty}{\operatorname{res}} \dfrac{\left[z^{\ell-1}\Gamma^{-1}(a_j)\Gamma(z)\right]_{21}}{z-a_j}_{1\leqslant j\leqslant q; 1\leqslant \ell\leqslant r} \\ \hline \underset{\infty}{\operatorname{res}} \dfrac{\left[z^{\ell-1}\Gamma^{-1}(b_j)\Gamma(z)\right]_{12}}{z-b_j}_{1\leqslant \ell\leqslant q; 1\leqslant j\leqslant r} & \underset{\infty}{\operatorname{res}} \dfrac{\left[z^{\ell-1}\Gamma^{-1}(b_j)\Gamma(z)\right]_{22}}{z-b_j}_{1\leqslant j, \ell\leqslant r} \end{array}\right] \tag{62}$$

Here $\Gamma(z)$ is the solution of the ordinary Hastings–McLeod RHP. Using the isomonodromic equation for the Psi-function and elementary row operations we obtain:

Example I: Airy kernel with two sets of parameters

Let $F_{\mathrm{A}i}^{(\alpha,\beta)}(s) := \det(\mathrm{Id} - K_{\mathrm{A}i}^{(\alpha,\beta)}\chi_{[s,\infty)})$ with

$$K_{\mathrm{A}i}^{(\alpha,\beta)}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3}-\frac{w^3}{3}-zx+wy}}{w-z} \prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right).$$

Then, for arbitrary sets of parameters α and β,

$$F^{(\alpha,\beta)}(s) = F_{TW}(s)\frac{\det\left[\begin{array}{c|c} \underset{\ell,j\leqslant q}{(-\partial_s + a_j)^{\ell-1}\Gamma_{2,2}(a_j)} & \underset{\ell\leqslant r;j\leqslant q}{\partial_s^{\ell-1}\Gamma_{1,2}(a_j)} \\ \hline \underset{\ell\leqslant q;j\leqslant r}{\partial_s^{\ell-1}\Gamma_{2,1}(b_j)} & \underset{\ell,j\leqslant r}{(-\partial_s + b_j)^{\ell-1}\Gamma_{1,1}(b_j)} \end{array}\right]}{\Delta(\alpha)\Delta(\beta)}.$$

where Γ is the solution of the Riemann–Hilbert problem associated to the Hasting–McLeod solution of Painlevé II.

Baik formula (2005) is the case $r = 0$. A remark in the paper reads: “P. Deift and A. Its pointed out that this formula resembles the Darboux transformation in the theory of integrable systems (see, e.g., 2). It would be interesting to identify the formula in terms of a Darboux transformation of an integrable system.”
Yes it was!

Conclusion and perspective

- The key point is not so much that the kernel is "integrable", but that the *Fourier/Laplace* transform of its restriction to subintervals is; the main signal is the double-integral representation with a denominator, e.g.

$$\widetilde{A}_{ij}(x,y) := \frac{1}{(2\pi i)^2} \int_{\gamma_{R_i}} \mathrm{d}\mu \int_{i\mathbb{R}} \mathrm{d}\lambda \frac{\mathrm{e}^{\theta(x,\mu)-\theta(y,\lambda)}}{\lambda + \tau_j - \mu - \tau_i} \tag{63}$$

  The off-diagonal part of the kernel contains also a **convolution operator**, typically a Gaussian (Pearcey), possibly with a drift (Airy), depending on the underlying diffusion process, or a Bessel function ("self-avoiding squared Bessel paths").
- The Riemann–Hilbert formulation allows for effective asymptotic study using Deift-Zhou steepest descent approach.
- Relation with noncommutative Painlevé (type) equations.
- From the RHP one can deduce bilinear Hirota–type relations ⇒ nonlinear PDEs
- Even for IIKS kernels (e.g. Airy field), the endpoints enter usually as *poles* in the associated Ψ–function; in our approach they enter *exponentially* (but the size of the RHP depends on the number of endpoints): this is possibly a manifestation of **duality** of isomonodromic deformations (Its-Harnad).

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