3/4-Fractional superdiffusion of energy in a harmonic chain with bulk noise

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Motivation

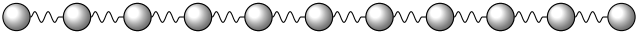
Prepare a macroscopic system at initial time with an inhomogeneous temperature $T_0(x)$. At some macroscopic time t, we expect that the temperature $T_t(x)$ at x is given by the solution of the **heat equation** (Fourier, 1822):

$$\partial_t T = \nabla[\kappa(T)\nabla T].$$

$\kappa(T)$ *is the diffusion coefficient.*

- It turns out that one dimensional systems (e.g. carbon nanotubes) can display anomalous energy diffusion *if momentum is conserved*. The heat equation is no longer valid: the diffusion coefficient is infinite.
- What shall replace the heat equation? There exists various controversial discussions about this problem in the physics litterature.

Microscopic models



Standard microscopic models of heat conduction are given by very long (=infinite) chains of coupled oscillators, i.e. infinite dimensional Hamiltonian system with Hamiltonian

$$\mathcal{H} = \sum_{x\in\mathbb{Z}} \left\{ \frac{p_x^2}{2} + V(r_x) \right\}, \quad r_x = q_{x+1} - q_x .$$

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2. The total momentum $\sum_x p_x$,
3. The compression of the chain $\sum_x r_x = \sum_x (q_{x+1} - q_x)$.

The problem of the existence (or not) of other conserved quantities is a challenging problem (ergodic problem).

Hydrodynamics: Euler equations

It is expected that in a Euler time scale the empirical energy $\mathfrak{e}(t,x)$, the empirical momentum $\mathfrak{p}(t,x)$ and the empirical compression $\mathfrak{r}(t,x)$ are given by a system of compressible Euler equations (hyperbolic system of conservation laws):

$$\begin{cases} \partial_t \mathfrak{r} = \partial_x \mathfrak{p}, \\ \partial_t \mathfrak{p} = \partial_x \tau, \\ \partial_t \mathfrak{e} = \partial_x (\mathfrak{p}\tau), \end{cases} \qquad \tau := \tau(\mathfrak{r}, \mathfrak{e} - \tfrac{\mathfrak{p}^2}{2}).$$

Nonlinear fluctuating hydrodynamics predictions

Recently, Spohn used the theory of *nonlinear fluctuating hydrodynamics* to predict the behavior of the long time behavior of the time-space correlation functions of the conserved fields $g(x,t) = (r_x(t), p_x(t), e_x(t))$

$$S_{\alpha\alpha'}(x,t) = \langle g_\alpha(x,t) g_{\alpha'}(0,0)\rangle_{\tau,\beta} - \langle g_\alpha\rangle_{\tau,\beta}\langle g_{\alpha'}\rangle_{\tau,\beta}$$

where $\langle\cdot\rangle_{\tau,\beta}$ is the (product) equilibrium Gibbs measure at temperature β^{-1} and pressure τ

$$\langle\cdot\rangle_{\tau,\beta} \sim \exp\{-\beta \sum_x (e_x + \tau r_x)\} drdp.$$

Nonlinear fluctuating hydrodynamics predictions

- The long time behavior of the correlation functions of the conserved fields depends on explicit relations between thermodynamic parameters (KPZ universality class and others).
- It is a *macroscopic* theory based on the validity of the hydrodynamics in the Euler time scale after some corse-graining procedure.
- Mutatis mutandis, it can be applied also for any conservative model whose conserved fields evolve in the Euler time scale according to a system of $n = 2, 3 \dots$ conservation laws. Similar universality classes appear.

Harmonic chain with bulk noise

- A rigorous proof of such predictions from Hamiltonian microscopic dynamics is out of the range of actual mathematics.
- Following ideas of [Olla-Varadhan-Yau'93] and [Fritz-Funaki-Lebowitz'94] we consider chains of oscillators perturbed by a bulk stochastic noise such that in the hyperbolic time scale Euler equations are valid.

- We start with a harmonic chain $\{(r_x(t), p_x(t))\,;\, x \in \mathbb{Z}\}$ and we use an equivalent dynamical variable $\{\eta_x(t)\,;\, x \in \mathbb{Z}\}$ defined by

$$\eta_{2x} = p_x, \quad \eta_{2x+1} = r_x.$$

- Newton's equations are

$$d\eta_x = (\eta_{x+1} - \eta_{x-1})dt, \quad x \in \mathbb{Z}.$$

- **Noise:** On each bond $\{x, x+1\}$ we have a Poisson process (clock). All are independent. When the clock of $\{x, x+1\}$ rings, η_x is exchanged with η_{x+1}. The dynamics between two successive rings of the clocks is given by the Hamiltonian dynamics.

- We obtain in this way a Markov process which conserves the total energy

$$\mathcal{H} = \sum_{x \in \mathbb{Z}} e_x = \sum_{x \in \mathbb{Z}} \eta_x^2 = \sum_{x \in \mathbb{Z}} \left\{ \frac{p_x^2}{2} + \frac{r_x^2}{2} \right\}.$$

- The noise destroys the conservation of the momentum and the conservation of the compression field.
- Nevertheless, the "volume" field η_x is conserved.

- The energy $\sum_x \eta_x^2$ and the volume $\sum_x \eta_x$ are the **only** conserved quantities of the model (in a suitable sense which can be made precize).
- The Gibbs equilibrium measures $\langle \cdot \rangle_{\tau,\beta}$ are parameterized by two parameters $(\tau, \beta) \in \mathbb{R} \times [0, \infty)$ and are product of Gaussians

$$\langle \cdot \rangle_{\tau,\beta} \sim \exp\{-\beta \sum_x (\eta_x^2 + \tau \eta_x)\} d\eta.$$

Theorem (B., Stoltz'11)

In the Euler time scale, the empirical volume field $\mathfrak{v}(t,x)$ *and the empirical energy field* $\mathfrak{e}(t,x)$ *evolve according to*

$$\begin{cases} \partial_t \mathfrak{v} = 2\partial_x \mathfrak{v}, \\ \partial_t \mathfrak{e} = \partial_x \mathfrak{v}^2. \end{cases}$$

The proof is based on the ideas introduced in [Olla-Varadhan-Yau'93] and [Fritz-Funaki-Lebowitz'94]. The theorem is clearly false without the presence of the noise.

- We define

$$S_t(x) = \left\langle \left(\eta_0(0)^2 - \tfrac{1}{\beta}\right)\left(\eta_t(x)^2 - \tfrac{1}{\beta}\right) \right\rangle_{\tau=0,\beta}$$

- The case $\tau \neq 0$ can be recovered by considering the dynamics

$$\tilde{\eta}_t(x) = \eta_t(x) - \tau.$$

Theorem (B., Gonçalves, Jara'14)

Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions of compact support. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x,y\in\mathbb{Z}} f\left(\tfrac{x}{n}\right) g\left(\tfrac{y}{n}\right) S_{tn^{3/2}}(x-y) = \frac{2}{\beta^2} \iint f(x) g(y) P_t(x-y) dxdy,$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the fractional heat equation

$$\partial_t u = -\frac{1}{\sqrt{2}} \left\{ (-\Delta)^{3/4} - \nabla(-\Delta)^{1/4} \right\} u.$$

- One can also show that the correlation function of the volume field evolve in a diffusive time scale and that the limit is given by the fundamental solution of the standard heat equation.
- These results confirm the predictions of the nonlinear fluctuating hydrodynamics for this particular case.

Ideas of the proof ($\beta = 1$)

- The *energy field* is defined as

$$\mathcal{S}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}} f\left(\tfrac{y}{n}\right)\left(\eta_{tn^{3/2}}(y)^2 - \tfrac{1}{\beta}\right).$$

- The *quadratic field* is defined as

$$Q_t^n(h) = \frac{1}{n} \sum_{y \neq z \in \mathbb{Z}} h\left(\tfrac{y}{n}, \tfrac{z}{n}\right) \eta_{tn^{3/2}}(y)\eta_{tn^{3/2}}(z).$$

By Itô calculus,

$$d\mathcal{S}^n_t(f) \approx -2Q^n_t(f' \otimes \delta)dt + \frac{1}{\sqrt{n}}\mathcal{S}^n_t(f'')dt + \textit{martingale}.$$

$$dQ^n_t(h) \approx Q^n_t\big(L_n h\big)\,dt - 2\mathcal{S}^n_t\big([\mathbf{e}\cdot\nabla h](x,x)\big)\,dt$$
$$+ \frac{2}{\sqrt{n}}Q^n_t\big(\partial_y h(x,x) \otimes \delta\big)\,dt + \textit{martingale}.$$

where $(\varphi \otimes \delta)(x,y) = \varphi(x)\delta(x=y)$ (distribution) and $\mathbf{e} = (1,1)$.

The linear operator L_n is defined by

$$L_n h = n^{-1/2}\Delta h + 2n^{1/2}(\mathbf{e}\cdot\nabla)h.$$

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Choose h_n such that $L_n h_n = 2f' \otimes \delta$ and add the two equations.

Up to small terms, we get

$$d\mathcal{S}_t^n(f) \approx -2\mathcal{S}_t^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,dt - dQ_t^n(h_n)$$

Integrate in time and use Cauchy-Schwarz inequality to show that $Q_t^n(h_n), Q_0^n(h_n)$ vanish as $n \to \infty$. Then

$$\mathcal{S}_t^n(f) - \mathcal{S}_0^n(f) \approx -2\int_0^t \mathcal{S}_s^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,ds$$

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Recall that $h_n := h_n(f)$ is the solution of

$$L_n h_n = n^{-1/2}\Delta h_n + 2n^{1/2}(\mathbf{e}\cdot\nabla)h_n = 2f'\otimes\delta$$

The equation for $\mathcal{S}_t^n(\cdot)$ is closed.

It remains only to show (by Fourier transform, it's easy) that

$$\lim_{n\to\infty}[\mathbf{e}\cdot\nabla h_n](x,x) = \frac{1}{\sqrt{2}}\left[(-\frac{d^2}{dx^2})^{3/4} - \frac{d}{dx}(-\frac{d^2}{dx^2})^{1/4}\right] f.$$

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Remark: In [Cafarelli-Silvestre'08] such descriptions of fractional Laplacian (and generalizations) with various boundary conditions are given.

Related works, work in progress, open questions ...

- Jara, Komorowski and Olla obtained similar results for the harmonic chain perturbed by a different noise conserving energy, momentum and compression. Their proof is very different (Wigner function).
- With a bit of work, our proof can be applied to their model and we can recover their results.
- The nonlinear case is much more difficult (work in progress).

The evanescent flip noise limit

- Consider the same Markov process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity $\gamma_n = n^{-b}$, $b > 0$, which consists to flip independently on each site at Poissonian times the variable η_x into $-\eta_x$.
- The energy is conserved but the volume $\sum_x \eta_x$ is not (stricto sensu, only if $b = \infty$).
- We look at the system in the time scale tn^a, $a > 0$, such that the energy field has a non-trivial limit.

Some work in progress seems to indicate the following picture: