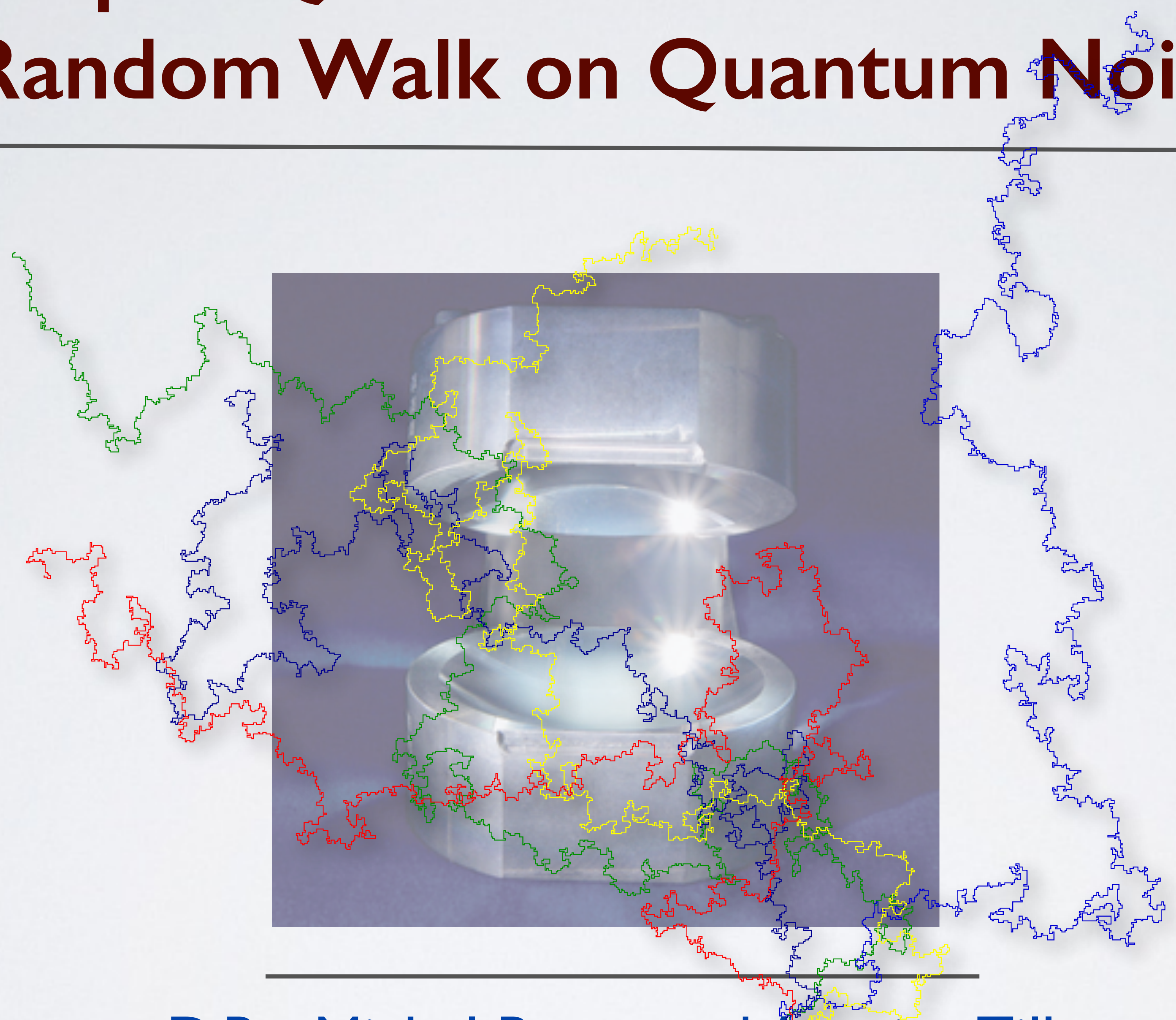
The Open Quantum Brownian Motion: a Random Walk on Quantum Noise.



D.B., Michel Bauer and Antoine Tilloy

IAS-Princeton, March 2014

Two facets of Quantum Noise:

— Observing, manipulating, controlling quantum systems.
These are probabilistic by nature,
and if time evolution is taken care of these are stochastic processes.

— Extending applications of probability theory by incorporating quantum effects,
e.g. quantum stochastic processes (as open quantum systems).

Aim:
Use the Open Quantum Brownian Motion (the definition will come)
to illustrate a few notions/properties of Quantum Noise.

— Progressive collapse in non-demolition quantum measurements
(as an illustration of the first point above)

— Transition in quantum random walks from diffusive to ballistic regimes.
(as an illustration of the second point above)

I) Progressive collapse in non-demolition measurements

Progressive field-state collapse and quantum non-demolition photon counting

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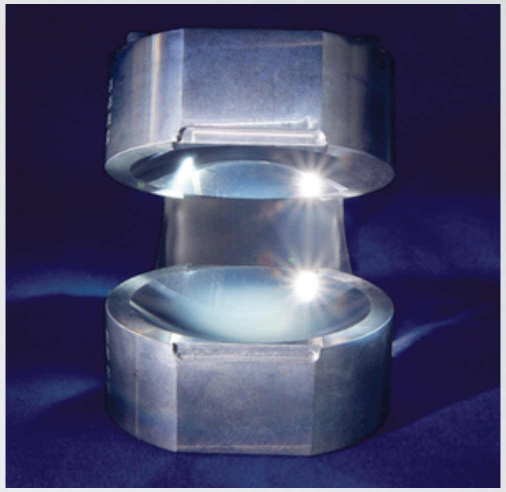
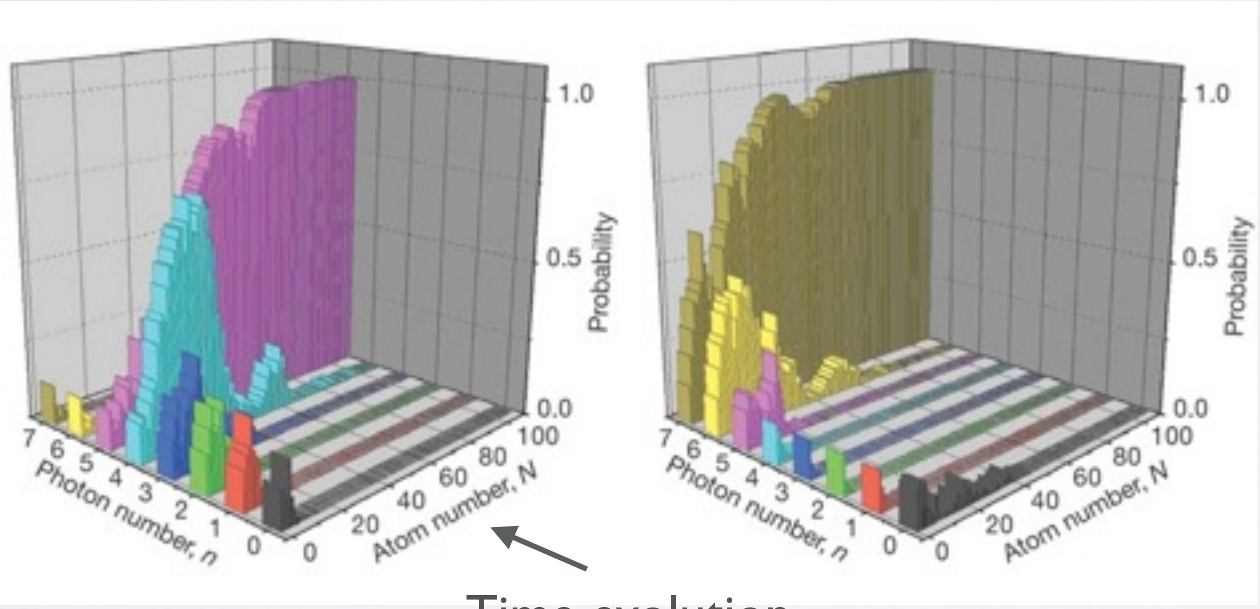


P.d.f. of the photon numbers



Time evolution

Figure 2 | Progressive collapse of field into photon number state. [redacted] **c**, Photon number probabilities plotted versus photon and atom numbers n and N. The histograms evolve, as N increases from 0 to 110, from a flat distribution into $n = 5$ and $n = 7$ peaks.

Courtesy of LKB-ENS.

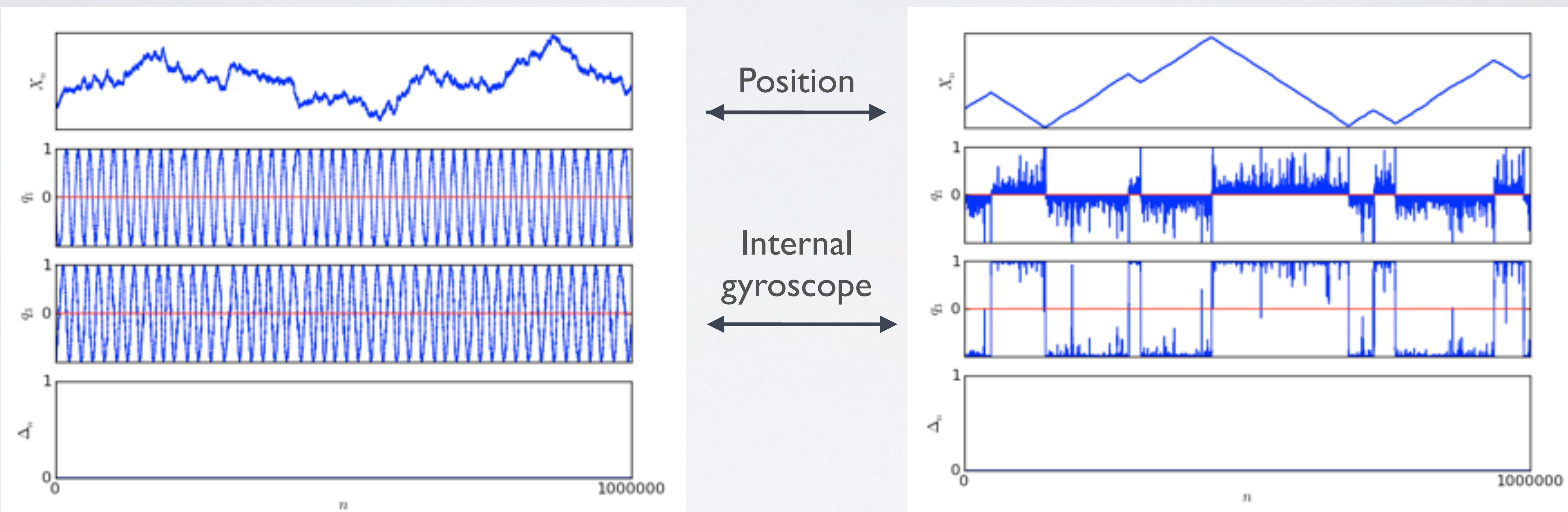
— *How to record the cavity states? Is it faithful representation?*
— *Why does the p.d.f. changes with time?*
— *How does it evolve? What's the dynamics? Why does it become peaked (collapsed)?*

→ *Use (quantum) random walks to illustrate the universality of this phenomenon.*

II) Quantum trajectories and a diffusive to ballistic transition in open quantum Brownian motion.

— Open Quantum Brownian Motion:
A quantum random motions on the line for a « quantum walker » with internal and orbital degrees of freedom.

A Brownian like regime. A ballistic like regime.



Transition in the behavior of the internal « gyroscope »:
from oscillation to bi-stability with jumps (analogous to Bohr quantum jumps)

— The ballistic flips are not due to collision but to the flips of the internal gyroscope.

Outline:

— Quantum parallelism on (classical) random walks and measurements.

(progressive collapse or `quantum computation' with random walks)

— A glance at Quantum Noise and quantum trajectories.

(how analogue to classical processes is it?)

— Open quantum random walk trajectories.

(a random walk but with an internal quantum gyroscope)

— Two regimes and bi-stability (diffusion or ballistics).

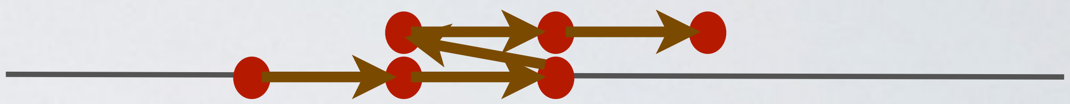
(gyroscope quantum jumps and flips a la Kramer's)

— How to make OQBM conformally invariant?

(if time permits….)

Quantum Parallelism on Random Walks.

— Consider a random walk on the line:



Events: $\omega = (+, +, -, +, \cdots) = (\epsilon_1, \epsilon_2, \epsilon_2, \cdots)$

Position: $X_n(\omega) = X_0 + \epsilon_1 + \cdots + \epsilon_n$

— We can use Quantum Mechanics to describe it, keeping all possible walks at once:

Let $\mathbb{C}^{\mathbb{Z}}$ be the walker Hilbert space, with basis $|x\rangle$, $x \in \mathbb{Z}$.

Let $\mathbb{C}^2$ be an auxiliary "coin" Hilbert space, with basis $|\pm\rangle$.

$$|\psi_n\rangle := \frac{1}{2^{n/2}} \sum_{[\omega_n]} |X_n(\omega])\rangle \otimes |[\omega_n]\rangle$$

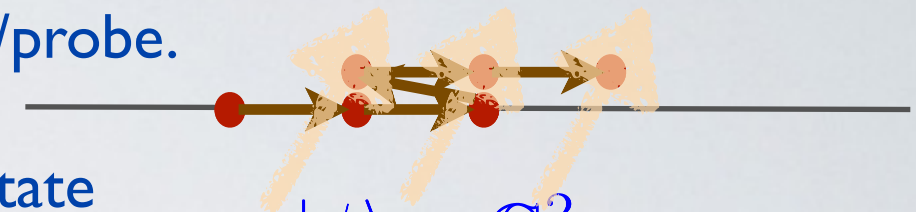
with a sum over all walks of n steps. $|[\omega_n]\rangle = |\epsilon_1\rangle \otimes \cdots \otimes |\epsilon_n\rangle \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

— There is no more information than in the usual description, all walks appear in the sum with equal weight:

simple « book keeping » or « quantum parallelism »,
keeping track of the position « X », of the history « w ».

Quantum Parallelism as dynamical process.

— We view the auxiliary Hilbert space of an auxiliary coin/probe.
We consider a collection of probes.
Assume that each probes are all initially prepared in a state



$|\phi\rangle \in \mathbb{C}^2$

— Consider the elementary (unitary) evolution (walker+one probe):

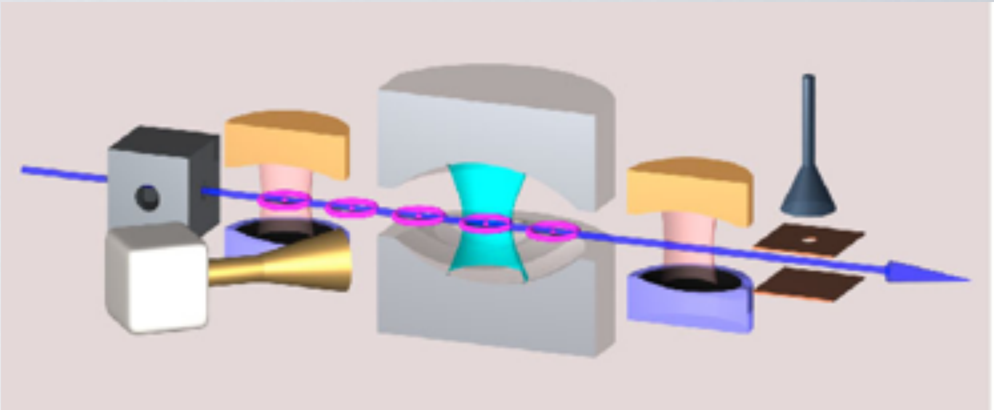
$$|x\rangle \otimes |\phi\rangle \to \frac{1}{\sqrt{2}}|x+1\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}}|x-1\rangle \otimes |-\rangle$$

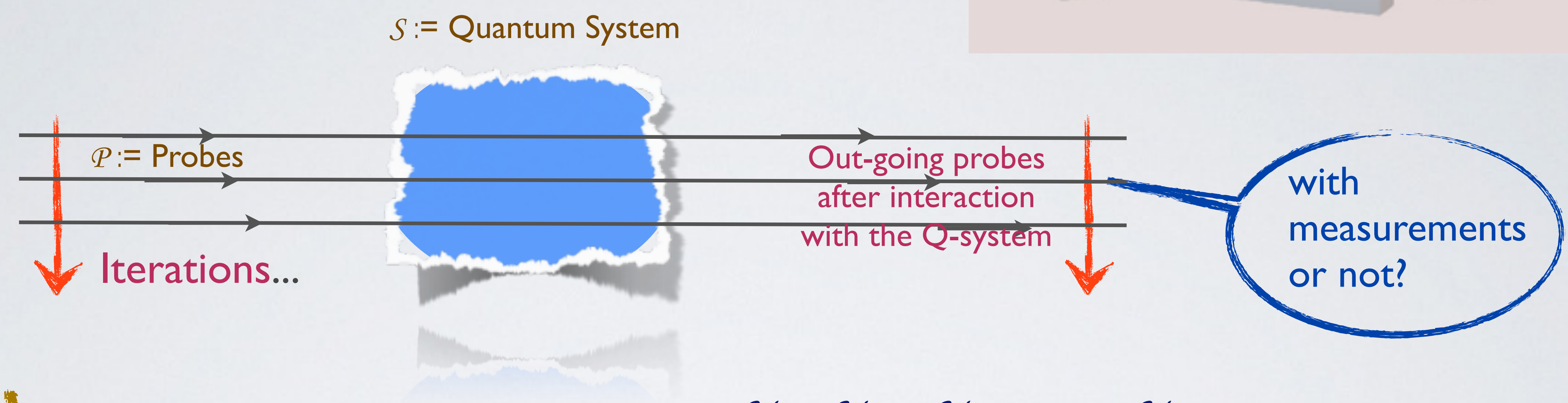
After e.g. there iterations:

$$\begin{aligned} 2^{-3/2}\Big(& |x_0+3\rangle \otimes |+++\rangle \\ & + |x_0+1\rangle \otimes (|++-\rangle + |+-+\rangle + |-++\rangle) \\ & + |x_0-1\rangle \otimes (|-+-\rangle + |--+\rangle + |+--\rangle) \\ & + |x_0-3\rangle \otimes |---\rangle \Big) \end{aligned}$$

— The occurrence probability of a given walk is the modulus square of the amplitude (« quantum measurement »).
A given output measurement projects on a given realization of the random walk.

Quantum noise and repeated quantum interaction.





— Hilbert space: $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \cdots$

— Algebras of observable: $\mathcal{B}_n := \mathcal{A}_s \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \otimes \mathbb{I}$

— (Quantum) Filtration: $\mathcal{A}_s \subset \mathcal{B}_n \subset \mathcal{B}_m, \quad \text{for } n < m$

— Gain of information : by testing output observable on the n-th first probes,
but a probabilistic gain because of Q.M.

--- Notion of noise (with a measure, the probe state).
--- Notion of arrow of time (information on the probes)
--- Reservoir, by summing over the probes.

— Measurement of some observables on the output probes (not on the Q-system):

⟶ The quantum filtration is reduced to a classical filtration.

Quantum trajectories= (classical random process, with events the out-put measurements)

Random walk with tilted measurements.

— Once the auxiliary space is viewed as that of probes we may imagine measuring the « spin » probes in another direction, say at an angle « theta ».

$$|\pm^u\rangle = \cos\vartheta/2\,|\pm\rangle \pm \sin\vartheta/2\,|\mp\rangle$$

Measuring the probes (successively) given a random output +/-,
(with probability given by the (square of the) projection of the state on the above vectors).
Hence the state of the walker after the probe measurement is random.

— What is this random state evolution?

At time t: $|\psi_t\rangle = \sum_x \psi_{x,t}\,|x\rangle,\ \text{with}\ \sum_x |\psi_{x,t}|^2 = 1$

Elementary evolution: $|\psi_t\rangle \to \frac{1}{\sqrt{2}} \sum_x \psi_{x,t}\big(|x+1\rangle \otimes |+\rangle + |x-1\rangle \otimes |-\rangle\big).$

Measuring the spin in the « u » direction, we have to project the probe on the above states, and the probability of +/- is the modulus square of the amplitude:

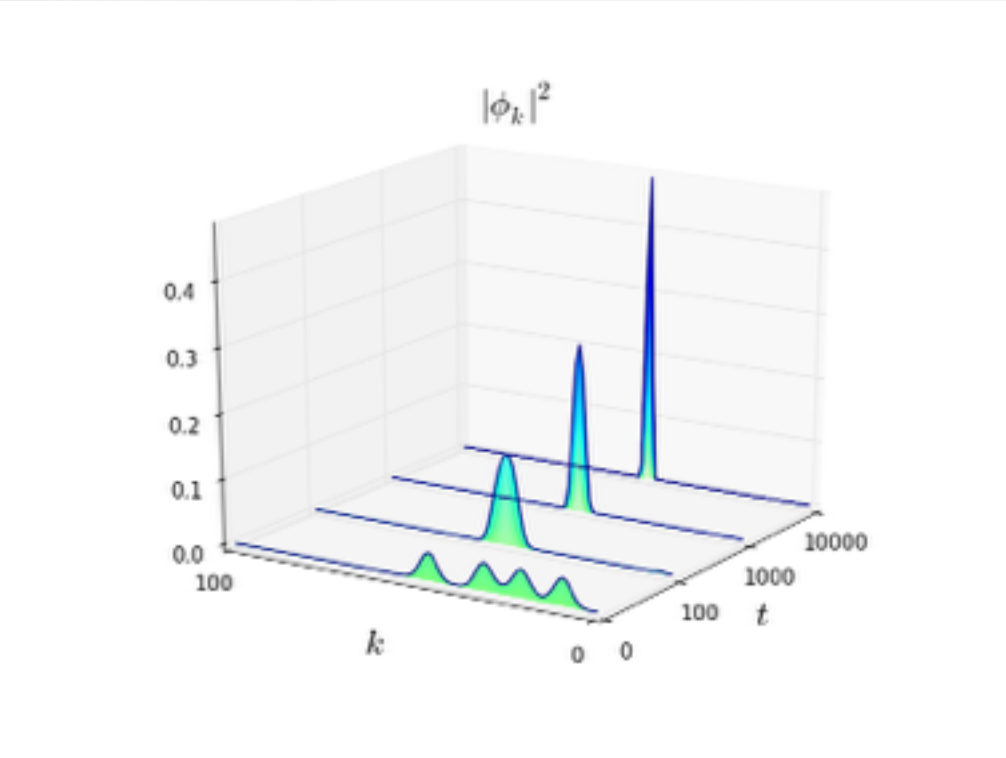
Probe measurement: $|\psi_{t+1}\rangle = \frac{1}{\sqrt{2p_\pm(t)}} \sum_x \big(\psi_{x-1,t}\,\langle\pm^u|+\rangle + \psi_{x+1,t}\,\langle\pm^u|-\rangle\big)\,|x\rangle$

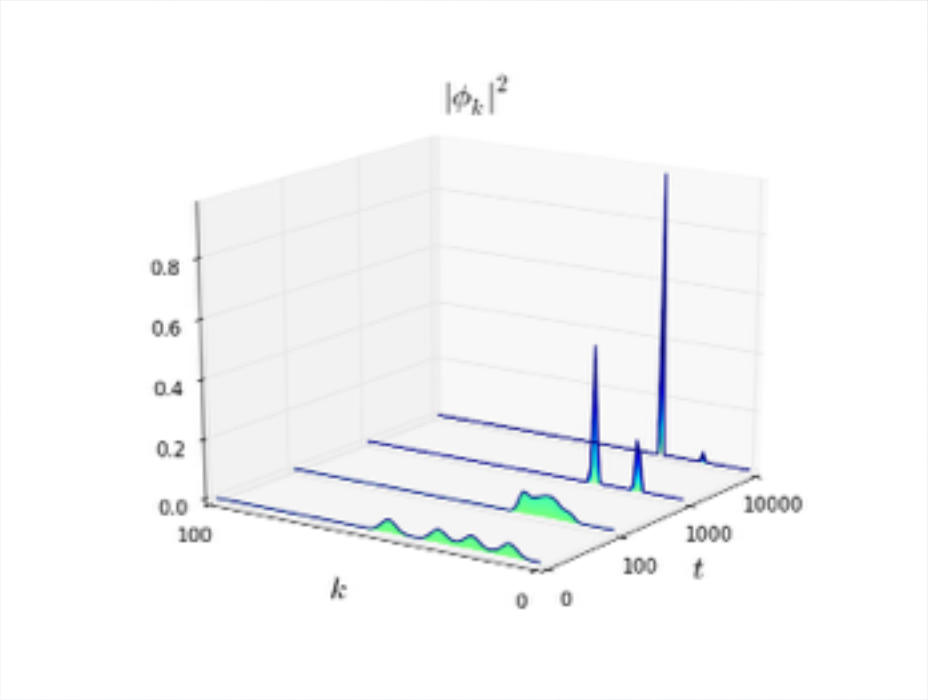
with probability: $p_\pm(t) := \frac{1}{2}\sum_x \Big|\psi_{x-1,t}\,\langle\pm^u|+\rangle + \psi_{x+1,t}\,\langle\pm^u|-\rangle\Big|^2.$

— The probability distribution of the walker position is then always delocalized.
« But » this distribution in momentum space collapses to a peaked measure !!
i.e. the Fourier component of the wave function collapse almost surely.

— In momentum space $\psi_{x,t} = \frac{1}{\sqrt{N}} \sum_k \phi_{k,t}\, e^{2i\pi kx/N}$, with $\sum_k |\phi_{k,t}|^2 = 1$

for a walk on $\mathbb{Z}_N$





— The momentum distribution $|\phi_{k,t}|^2$ are random but:

Claim:

$$\lim_{t\to\infty} |\phi_{k,t}|^2 = \delta_{k;k_\infty}, \text{ with } k_\infty \text{ random}$$

$$\mathbb{P}[k_\infty = p] = |\phi_{p,t=0}|^2$$

— Open Quantum Random Walks but wit tilted probe measurements define « progressive » quantum measurements of the momentum.

Progressive: the collapse/measurement takes some time,
but the collapse is exponential in time.

Proof:

— The random state evolution in momentum space

$$|\phi_{k,t+1}|^2 = \frac{1}{2p_\pm(t)} \left[1 \pm \sin\vartheta\, \cos\left(4\pi k/N\right)\right] \left|\phi_{k,t}\right|^2$$

$$p_\pm(t) = \frac{1}{2}\sum_l \Big(1 \pm \sin\vartheta\, \big[\cos(4\pi l/N)\, |\phi_{l,t}|^2\big]\Big).$$

preserving the normalisation of the momentum distribution.

— Why does the momentum distribution converges and collapse?

$t \to |\phi_{k,t}|^2$ are bounded martingales.

as such they converge almost surely and in L1

A martingale is a random process which is conserved in mean when conditioned on the past:
And here it is tautologically true.

$$\mathbb{E}[|\phi_{p,t+1}|^2|\mathcal{F}_t] = |\phi_{p,t}|^2$$

— Bayesian interpretation.
The above formula is actually the formula for conditional probability:

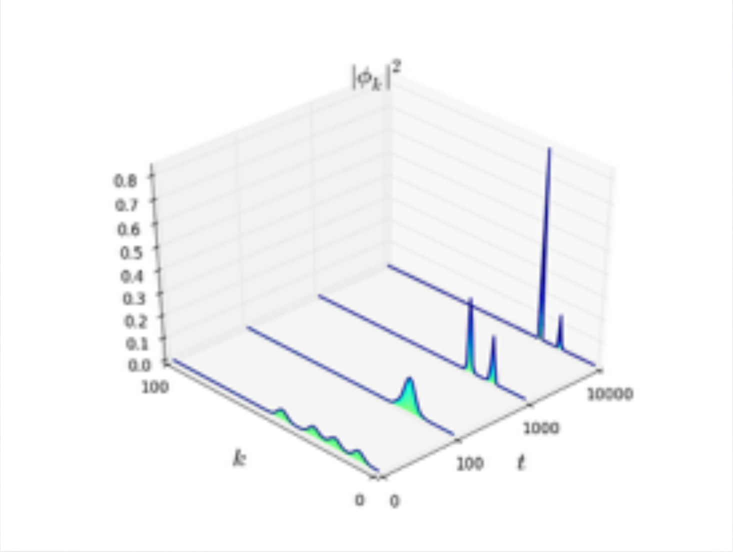
$$|\phi_{k,t+1}|^2 = \frac{p(\pm|k)\,|\phi_{k,t}|^2}{p_\pm(t)}, \text{ with } p(\pm|k) := \frac{1}{2}[1 \pm \sin\theta \cos(4\pi k/N)]$$

Quantum mechanics code for the Bayesian rules.
Collapse is « classical »: An echo of random Bayesian updating

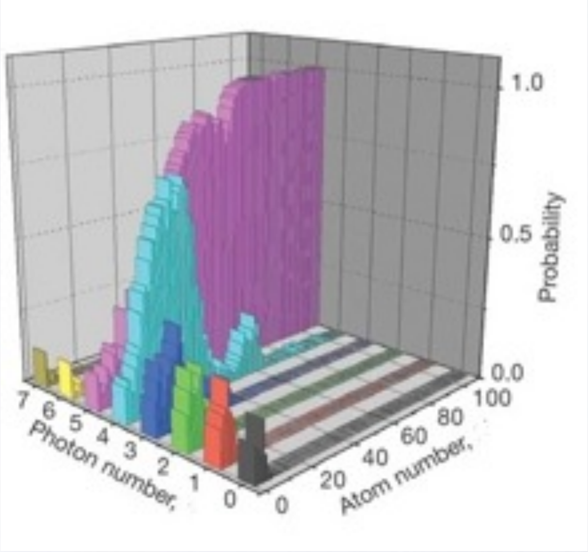
Generalisation and application:

— The previous statement is general:

Iterating ad infinitum « system-probe interaction » plus « probe measurement » results in the collapse of the system probability distribution, provided that the interaction between the probes and the system preserves a specified basis of states, call « pointer » states.



These are identical phenomena.



— These are mesoscopic (quantum non-demolition) measurements (they become macroscopic only when an infinite number of probes have been involved)

A Bayesian point of view.

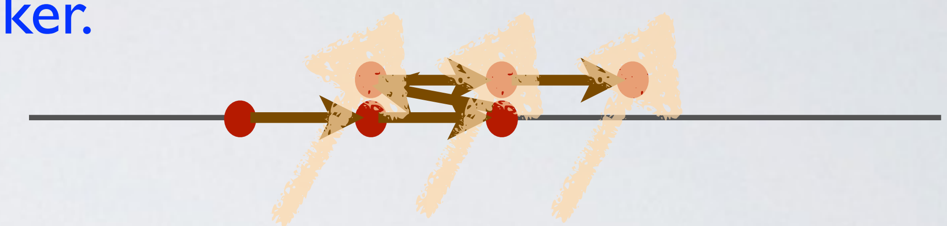
— Many applications: Measurement continuously in time,

(e.g. imaging « Bohr » virtual quantum jumps (experiments/theory)).

Quantum control, etc.…

Open quantum random walks (I)

— Back to the quantum parallelism on random walks but we add internal degree of freedom, « quantum gyroscope » to the walker.



* Hilbert space of states:
$\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p^{\otimes\infty}$

$\mathcal{H}_c = \mathbb{C}^2,\ \mathcal{H}_o = \mathbb{C}^{\mathbb{Z}}$

$\mathcal{H}_p^{\otimes\infty} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$

Coins are reset at each step.

p=probes (or coins);
c=color (or spin), internal d.o.f's;
o=position, orbital d.o.f.'s.

— Interaction (without measurements):

On state $|\psi\rangle_c \otimes |n\rangle_o \otimes |\phi\rangle_p$ in $\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p$, the unitary evolution

$$(B_+|\psi_c\rangle \otimes |n+1\rangle_o \otimes |+\rangle_p + (B_-|\psi\rangle_c) \otimes |n-1\rangle_o \otimes |-\rangle_p$$

Iterating interactions with different probes produce an «entangled» state, sum of states each indexed by a random walk (as before).

— The original definition of open QRW was the mean of this process (not keeping track of the measurements).

Attal, Petruccione, Sabot, Sinayskiy, 2012.

Open quantum random walks trajectories (II).

— Measurements and «quantum trajectories»:

$$(B_+|\psi_c\rangle \otimes |n+1\rangle_o \otimes |+\rangle_p + (B_-|\psi\rangle_c) \otimes |n-1\rangle_o \otimes |-\rangle_p$$

Measuring the probes, one may find + or - with probabilities,
and then «project» the state on |+> or |->:

$\propto (B_\pm|\psi\rangle_c) \otimes |n \pm 1\rangle_o$ with probability ${}_c\langle\psi|B_\pm^\dagger B_\pm|\psi\rangle_c$

The events, the output of the probe measurements,
are in one-to-one correspondence with random walks.

—These are classical random processes with values on the line x the internal states.

— For mixed (internal) state: ie. the system is not described by a vector but by a density matrix (a positive hermitian normalised matrix).

If after n step,	$(\rho_n,\ x_n)$
The updating is:	$(p_\pm(n)^{-1}\, B_\pm \rho_n B_\pm^\dagger, \quad x_n \pm 1)$
with probability:	$p_\pm(n) = \mathrm{tr}(B_\pm \rho_n B_\pm^\dagger)$

For a pure state:
$\rho_n = |\psi_n\rangle\langle\psi_n|$

The matrices B are the moduli parameters of the walks.
Events are random walks (output measurements),
but with probabilities induced by the gyroscope motion.

Numerical simulations:

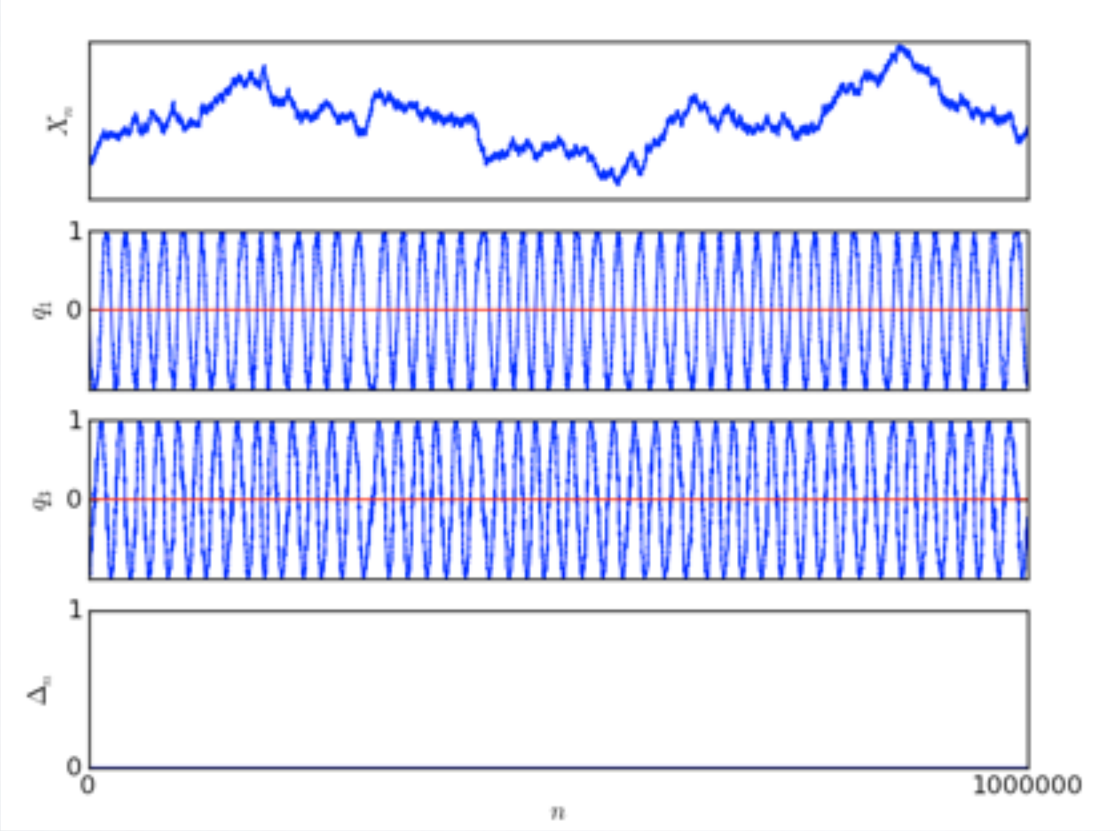
— Unitarity or normalisation of probabilities imposes: $B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{I}$

A choice for the simulations: $B_+ = \delta^{-1}\begin{pmatrix} u & r \\ s & v \end{pmatrix}$ and $B_- = \delta^{-1}\begin{pmatrix} -v & s \\ r & -u \end{pmatrix}$

with $\delta = \sqrt{u^2 + v^2 + r^2 + s^2}$

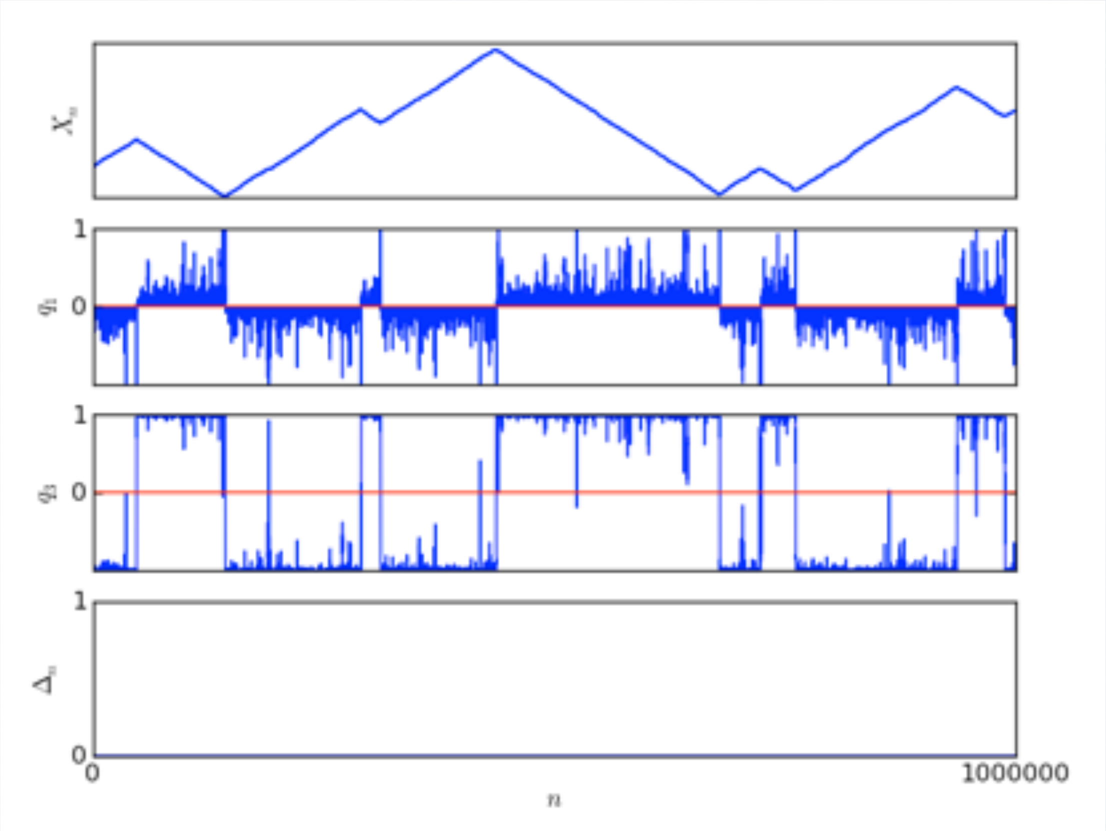
A Brownian like regime.

$u = 1.005,\ v = 1.00$ and $r = -s = 0.00015$



A ballistic like (but diffusive) regime.

$u = 1.1,\ v = 1.00$ and $r = -s = 0.00015$



— Kramer's like transition for a particle in a double well potential (... but not quite).

Scaling limit: «Open quantum Brownian motion».

— In the scaling limit one gets a time continuous process (with continuous measurement)

The scaling limit is $\epsilon \to 0$, $t = n\epsilon$ fixed, and $dx^2 \sim dt$.

— As for classical Brownian motion, scale time, distance and moduli simultaneously:

$$B_\pm = \frac{1}{\sqrt{2}}[\mathbb{I} \pm \sqrt{\epsilon} N + \epsilon(iH_\pm \pm M - \frac{1}{2}N^\dagger N) + o(\epsilon)]$$

with ϵ a small parameter and $H_\pm$, M hermitian but not N.

— In the scaling limit, the series of +/- of output probe measurements become a random process driven by a « classical » Brownian motion.

$$\begin{aligned} d\rho_t &= \big(i[H, \rho_t] + L_N(\rho_t)\big)\, dt + D_N(\rho_t)\, dB_t, \\ dX_t &= U_N(\rho_t)\, dt + dB_t, \end{aligned}$$

A Brownian motion, coding for all probe measurements.

with $L_N(\rho) := N\rho N^\dagger - \frac{1}{2}(N^\dagger N \rho + \rho N^\dagger N)$

$D_N(\rho) := N\rho + \rho N^\dagger - \rho\, U_N(\rho)$ and $U_N(\rho_t) := \mathrm{tr}(N\rho + \rho N^\dagger)$

— Both the density matrix and the position have stochastic evolution but the position drift is guided by the internal gyroscope.

Transition between two regimes:

Take $H := \omega_0\,\sigma^2$ and $N = a\,\sigma^3$ with $\sigma^{1,2,3}$ the usual Pauli matrices.

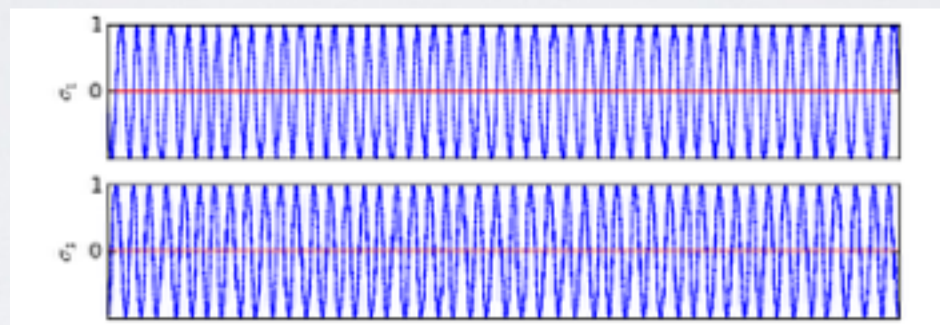
Describe ρ_t as a pure state: $\rho_t = \frac{1}{2}(\mathbb{I}+q_1\sigma^1+q_3\sigma^3)$ with $q_1 = \sin\theta$, $q_3 = \cos\theta$.

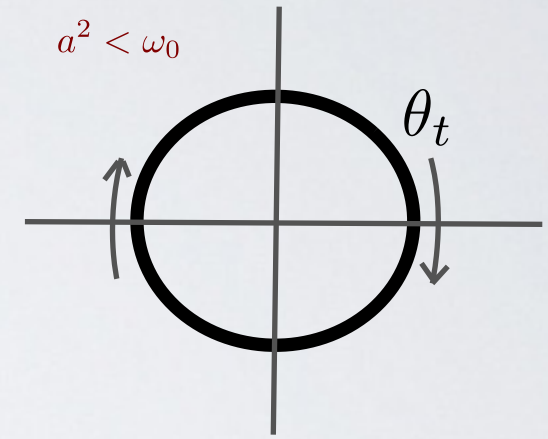
Eqs. $d\rho_t = \big(i[H,\rho_t] + L_N(\rho_t)\big)dt + D_N(\rho_t)\,dB_t,$ becomes:

$$d\theta_t = -2(\omega_0 + a^2 \sin\theta_t \cos\theta_t)dt - 2a\sin\theta_t\,dB_t$$

— Rabi like oscillations:

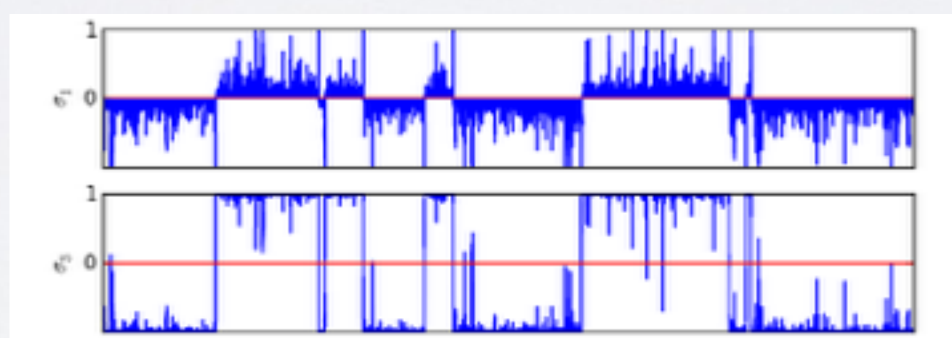
$a^2 < \omega_0$

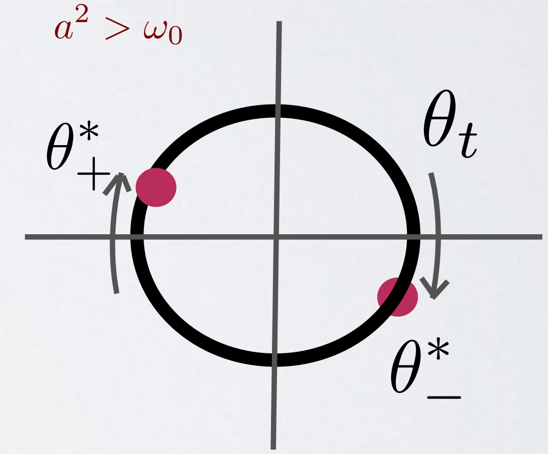




— Quantum jumps

$a^2 > \omega_0$

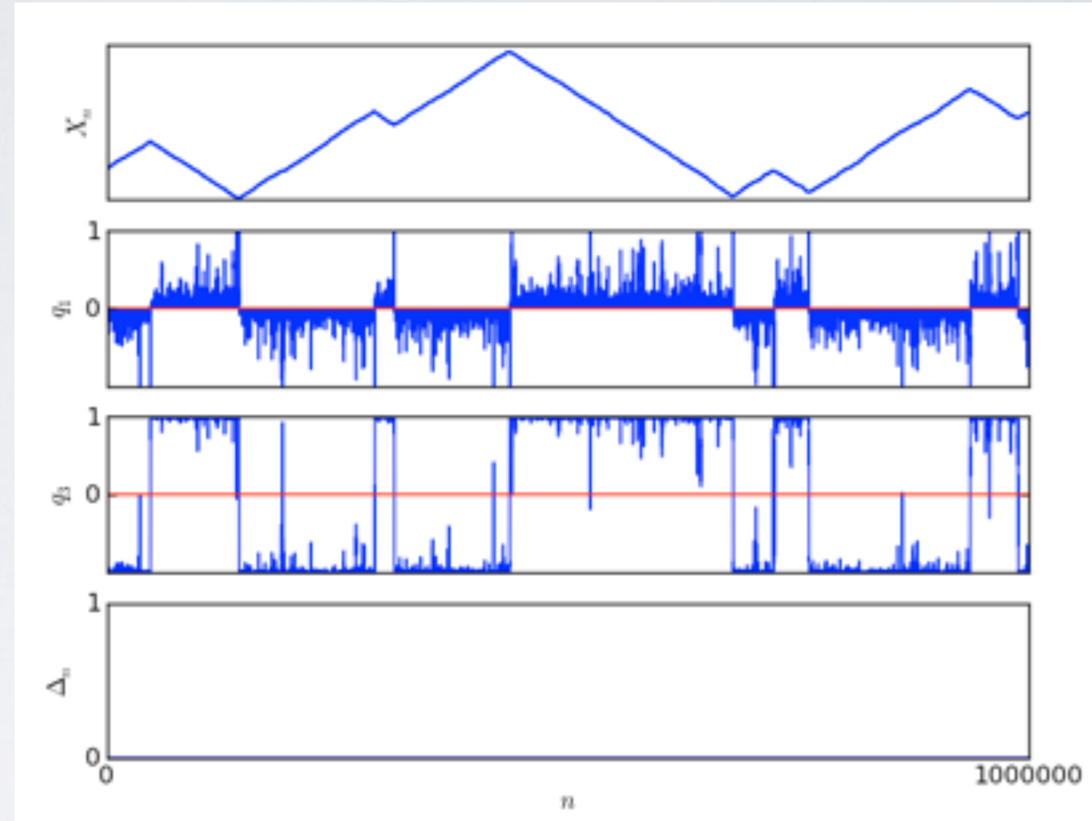




— Two potential minima, with Kramer's transition between them.
« but not quite », because jumps arise for « strong noise ».

Ballistically induced diffusion:

— Trajectories are ballistic,
with seesaw profiles induced
by gyroscope flips,
and large mean free paths.



$dX_t = U_N(\rho_t)\, dt + \, dB_t$ becomes:

$$dX_t = 2a \cos\theta_t \, dt + dB_t$$

with $2a\cos\theta^*_{\pm} \simeq \mp 2a$

The seesaw behavior is not due to collision but to the internal gyroscope behavior.

— But at very large time the position is Gaussian,
with large effective diffusion constant.

$$\mathbb{E}[X_t^2] = D_{\text{eff}}\, t \quad , \text{ with } D_{\text{eff}} = 1 + 4a^4/\omega_0^2$$

Question: Can we find similar phenomena (phenomenological description) producing a large effective diffusion constant (at large time)?

How to make the Open Quantum Brownian Motion conformally invariant ?

In progress

— The 2D Brownian motion is conformally invariant (Levy), up to random time parametrization: $dB_t\, d\bar{B}_t = 2dt$

— A way to render a 2D quantum walker conformally invariant is to couple it to an internal conformal field theory (CFT).
Which CFT? Which evolution equation for the internal state,
The walker position will satisfy a SDE with a drift of the form: $dX_t = U_N(\rho_t)\, dt + \, dB_t$

— To preserve conformal the drift has to be a (0,1)-form, because time scales under conformal transformation: $dZ_t = \bar{\Omega}^{\mathbb{D}}(Z_t, \bar{Z}_t)\, dt + dB_t.$

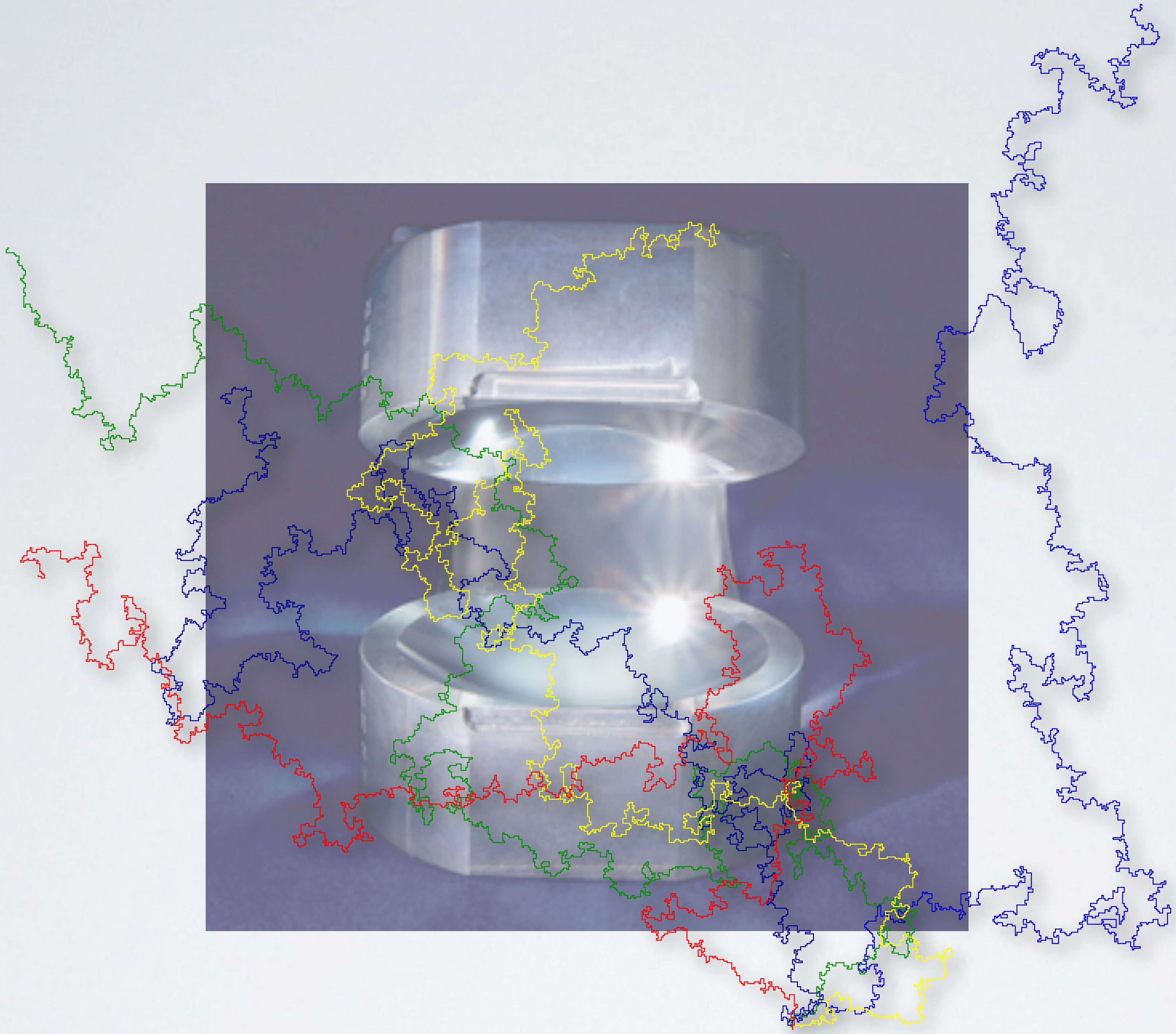
Hence, the CFT has to have a U(1) current and $\bar{\Omega}^{\mathbb{D}}(Z_t, \bar{Z}_t) := 4\sqrt{\kappa}\omega_t^{\mathbb{D}}\langle \bar{J}(\bar{Z}_t)\rangle$

— We then have to write the corresponding SDE for the evolution of the internal CFT state $\omega_t^{\mathbb{D}}\langle \cdots \rangle$ according to the rule of the OQBM.

$$d\omega_t^{\mathbb{D}}\langle A\rangle = \sqrt{\kappa}\left[\omega_t^{\mathbb{D}}\langle J(Z_t)A + AJ(Z_t)\rangle - 2\,\omega_t^{\mathbb{D}}\langle J(Z_t)\rangle\,\omega_t^{\mathbb{D}}\langle A\rangle\right] dB_t + \text{c.c.},$$

$$dZ_t = 4\sqrt{\kappa}\omega_t^{\mathbb{D}}\langle \bar{J}(\bar{Z}_t)\rangle\, dt + dB_t$$

The SDE for the internal state has an additional drift term, $\kappa\,\omega_t^{\mathbb{D}}\langle L^*_{[Z_t]}(A)\rangle\, dt$
but in general it doe snot contribute.



Thank you.

Open Quantum Brownian motion:

--- Quantum trajectories: Probes are measured (with random outputs), and the system density matrix evolves randomly (according to the measurement outputs).

more below.....

.... and this can be generalised with many packets and/or entangled states.

--- Quantum dynamical map: Probes (reservoir) are *not* measured but trace out, and the system density matrix evolves with Lindblad equation

$$\partial_t \bar{\rho}_t = -\frac{1}{2}[P,[P,\bar{\rho}_t]] + i\big(N[P,\bar{\rho}_t] + [P,\bar{\rho}_t]N^\dagger\big) + i[H,\bar{\rho}_t] + L_N(\bar{\rho}_t)$$

for the density matrix $\bar{\rho}_t := \int dx\, \rho(x,t) \otimes |x\rangle_o\langle x|$

--- Quantum Stochastic equation: Probes (reservoir) are *not* measured but *not* trace out, and the *total* state evolves with a quantum-SDE.

$$dA_t = i[P + iN^\dagger, A_t]\, d\xi_t + i[P - iN, A_t]\, d\xi_t^\dagger + L_*(A_t)\, dt$$

for observable A, and dual Linbladian L.,
and (quantum) noises: $d\xi_t\, d\xi_t^\dagger = dt,\ d\xi_t^\dagger\, d\xi_t = 0$