Random matrix ensemble with locally-varying potential

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Unitary invariant ensemble

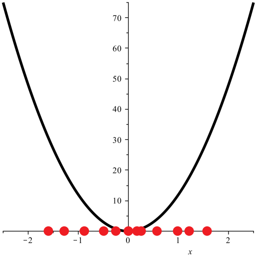
$$p(x_1, \cdots, x_N) = \frac{1}{Z_N} |\Delta_N(x)|^2 \prod_{j=1}^{N} e^{-NV(x_j)}$$

- Density of states: equilibrium measure. Finite support.
- Bulk universality: sine kernel

$$\frac{1}{N\rho_{eq}(x_0)} K_N \left(x_0 + \frac{\xi}{N\rho_{eq}(x_0)}, x_0 + \frac{\eta}{N\rho_{eq}(x_0)} \right) \to \mathbb{S}(\xi, \eta)$$

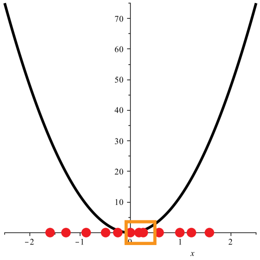
- [Pastur, Scherbina] [Bleher, Its] [Deift, Kriecherbauer, McLaughlin, Venekides, Zhou] [Lubinsky] [Bourgade, Erdös, Yau]
- Here $V(x)$ is macroscopic compared to the spacings of typical eigenvalues.

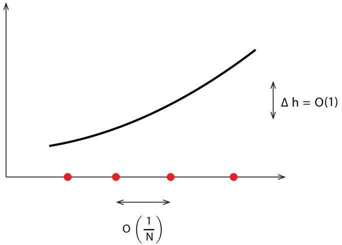
Unitary invariant ensemble



$NV(x)$

Unitary invariant ensemble

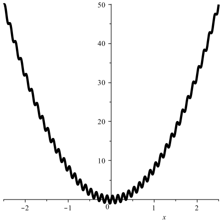


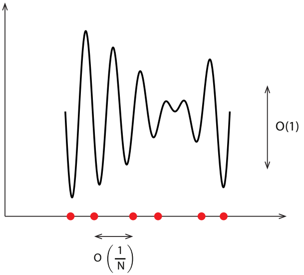


$$\Delta h = NV\left(a + \frac{y}{N}\right) - NV(a) \approx V'(a)y,$$ locally **linear**

Question

Q: What happens if locally non-linear?





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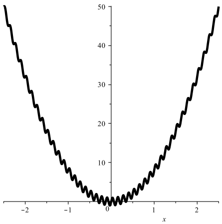

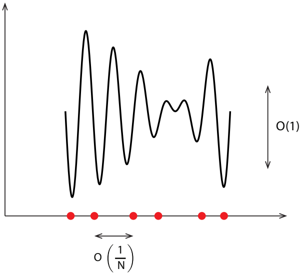

$NV(x) + \cos(Nx)$, or more generally $NV_1(x) + V_2(Nx)$

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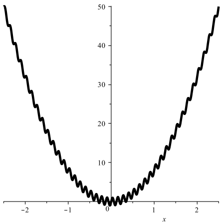
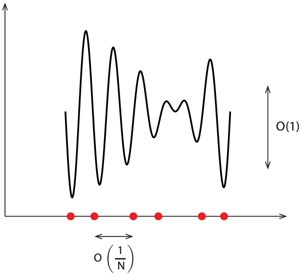




$NV(x) + \cos(Nx)$, or more generally $NV_1(x) + V_2(Nx)$

mixed scale: $NV(x) + \frac{N}{\Lambda}\cos(\Lambda x)$ or $NV_1(x) + \frac{N}{\Lambda}V_2(\Lambda x)$

Question

Potential: $NV_1(x) + \frac{N}{\Lambda} V_2(\Lambda x)$

density of state?

How is sine kernel changed?

We study the circular version without V_1 and a Jacobi version

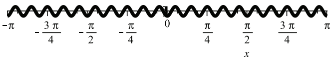
Circular ensemble with periodic potential

Fix $V(e^{i\theta})$.

$$p_{N,\Lambda}(e^{i\theta_1},\cdots,e^{i\theta_N}) = \frac{1}{Z_N}|\Delta_N(e^{i\theta})|^2\prod_{j=1}^{N} e^{-\frac{N}{\Lambda}V(e^{i\Lambda\theta_j})}$$

Unitary group $\mathcal{U}(N)$ with density $e^{-\frac{N}{\Lambda}\mathrm{Tr}(V(U^\Lambda))}$

Example: $V(e^{iN\theta}) = -c\cos(N\theta)$



"Motivation"

Gap size distribution of parked cars on London streets. Rawal, Rodgers 2005

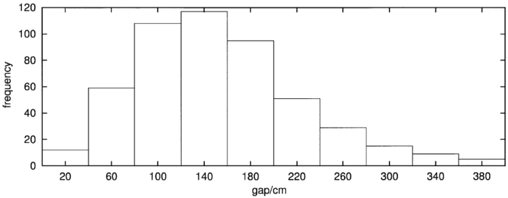


Fig. 1. The frequency distribution of gaps between parked cars.

"Motivation"

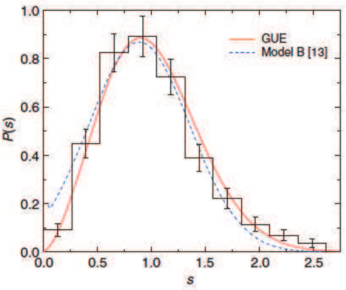
Random car parking model: random interval filling problem of Rényi 1958

Drop a needle of size 1 sequentially randomly without overlapping on a long interval.

Gap-size distribution: Non-vanishing density at $x = 0$

“Motivation”

Abul-Magd 2006. Perhaps GUE Wigner surmise?



"Motivation"

- Petr Seba. Data from streets in the Czech Republic; proposed different Markov model
- Anthony Fader: REU project. Streets in Ann Arbor, Michigan. No long enough streets without parking meters or driveways. Collected data from streets with parking meters and also from parking garages.

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- Parking meters = periodic potential?

Peter Forrester 1986

- $\Lambda = N$
- free energy, one-point and two-point correlation functions.
- $\beta = 1, 2, 4$
- Kosterlitz -Thouless conducting-insulating phase transition

Circular ensemble with periodic potential

$$p_{N,\Lambda}(e^{i\theta_1}, \cdots, e^{i\theta_N}) = \frac{1}{Z_N} |\Delta_N(e^{i\theta})|^2 \prod_{j=1}^{N} e^{-\frac{N}{\Lambda} V(e^{i\Lambda\theta_j})}$$

- Invariant under $\theta_j \mapsto \theta_j + \frac{2\pi}{\Lambda}$ for all j.
- Density of states $\rho_{N,\Lambda}(e^{i\theta})$ is periodic with period $\frac{2\pi}{\Lambda}$.
- Determinantal point process with kernel $K_{N,\Lambda}(e^{i\theta}, e^{i\varphi}) = K_{N,\Lambda}(e^{i(\theta+\frac{2\pi}{\Lambda})}, e^{i(\varphi+\frac{2\pi}{\Lambda})})$

Orthogonal polynomials

Determinantal point process

$$R_m^{(N,\Lambda)}(e^{i\theta_1}, \cdots, e^{i\theta_m}) = \det\left(K_{N,\Lambda}(e^{i\theta_i}, e^{i\theta_j})\right)_{i,j=1}^m$$

Kernel given in terms of Nth orthogonal polynomials with respect to $e^{-\frac{N}{\Lambda}V(e^{i\Lambda\theta})}d\theta$

$$\frac{p_N^{(\Lambda)}(e^{i\theta})\overline{p_N^{(\Lambda)}(e^{i\varphi})} - e^{iN(\theta-\varphi)}\overline{p_N^{(\Lambda)}(e^{i\theta})}p_N^{(\Lambda)}(e^{i\varphi})}{e^{i\theta} - e^{i\varphi}} e^{-\frac{N}{2\Lambda}V(e^{i\Lambda\theta}) - \frac{N}{2\Lambda}V(e^{i\Lambda\phi})}$$

Orthogonal polynomials

Lemma. Let $\pi_k^{(\Lambda)}(z)$ and $\pi_k(z)$ be monic OP's with respect to $w(e^{i\Lambda\theta})d\theta$ and $w(e^{i\theta})d\theta$, respectively. Then

$$\pi_N^{(N)}(z) = \pi_1(z^N)$$

Orthogonal polynomials

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$$\pi_N^{(N)}(z) = \pi_1(z^N)$$

More generally,

$$\pi_{bN+c}^{(aN)}(z) = z^c \pi_b^{(a)}(z^N)$$

Orthogonal polynomials

From Lemma,

$$K_{N,\Lambda}(e^{i\theta}, e^{i\varphi}) = \frac{\sin(\frac{N}{2}(\theta-\varphi))}{\sin(\frac{1}{2}(\theta-\varphi))} K_{1,\frac{\Lambda}{N}}(e^{iN\theta}, e^{iN\varphi}), \quad \text{if } \frac{\Lambda}{N} \in \mathbb{Z}$$

N particles and Λ period "$=$" 1 particle and $\frac{\Lambda}{N}$ period.

Also, "$=$" $\frac{N}{\Lambda}$ particles and 1 period.

Density of states.

$$p_{N,\Lambda}(e^{i\theta_1}, \cdots, e^{i\theta_N}) = \frac{1}{Z_N} |\Delta_N(e^{i\theta})|^2 \prod_{j=1}^{N} e^{-\frac{N}{\Lambda} V(e^{i\Lambda\theta_j})}$$

- When $\Lambda = 1$: $w(\theta) = e^{-NV(e^{i\theta})}$, usual external potential
- When $\Lambda = \infty$: $w(\theta) = 1$, CUE (circular unitary ensemble)

$$\rho_{N,\Lambda}(e^{i\theta}) \to \begin{cases} \rho_{eq}(e^{i\theta}) & \text{when } \Lambda = 1 \\ 1 & \text{when } \Lambda = \infty \end{cases}$$

- Does $\rho_{N,\Lambda}$ converge when $N, \Lambda \to \infty$? No. But it is bounded.

Density of states I. $|\Delta_N(e^{i\theta})|^2 \prod_{j=1}^N e^{-\frac{N}{\Lambda}V(e^{i\Lambda\theta_j})}$

When $N = k\Lambda$:

$$\rho_{N,\Lambda}(e^{i\theta}) = f_k(e^{i\Lambda\theta})$$

where $f_k(e^{i\varphi})$ is the DOS for k particle system with $e^{-kV(e^{i\varphi})}$.

As $k \to \infty$, $f_k(e^{i\varphi}) \to \rho_{eq}(e^{i\varphi})$, equil. meas. for $V(e^{i\varphi})$. Indeed,

$$\rho_{N,\Lambda}(e^{i\theta}) \approx \rho_{eq}(e^{i\Lambda\theta}) \qquad N >> \Lambda$$

Especially, when $N = \Lambda$: 1 particle system

$$\rho_{N,N}(e^{i\theta}) \propto e^{-V(e^{i\Lambda\theta})}$$

Density of states II. $|\Delta_N(e^{i\theta})|^2 \prod_{j=1}^N e^{-\frac{N}{\Lambda} V(e^{i\Lambda\theta_j})}$

When $N = \frac{1}{k}\Lambda$,

$$\rho_{N,\Lambda}(e^{i\theta}) = \frac{e^{-\frac{1}{k}V(e^{i\Lambda\theta})}}{\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{1}{k}V(e^{i\theta})}d\theta}$$

As $k \to \infty$, RHS$\to 1$. Indeed,

$$\rho_{N,\Lambda}(e^{i\theta}) \to 1,$$

Density of states. $|\Delta_N(e^{i\theta})|^2 \prod_{j=1}^N e^{-\frac{N}{\Lambda} V(e^{i\Lambda\theta_j})}$

1. When $N >> \Lambda$,
$$\rho_{N,\Lambda}(e^{i\theta}) \approx \rho_{eq}(e^{i\Lambda\theta})$$
2. When $N = k\Lambda$,
$$\rho_{N,\Lambda}(e^{i\theta}) = f_k(e^{i\Lambda\theta})$$
3. When $N = \frac{1}{k}\Lambda$, with $I_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{1}{k}V(e^{i\theta})} d\theta$,
$$\rho_{N,\Lambda}(e^{i\theta}) = \frac{1}{I_0} e^{-\frac{1}{k}V(e^{i\Lambda\theta})}$$
4. When $N << \Lambda$,
$$\rho_{N,\Lambda}(e^{i\theta}) \to 1$$

Example. $V(e^{ix}) = -c\cos(x)$. Density of states

$\Lambda = N$ $\quad$ $N >> \Lambda$

$\Lambda = N$ $\quad$ $N >> \Lambda$

Bulk scaling limit. $|\Delta_N(e^{i\theta})|^2 \prod_{j=1}^N e^{-\frac{N}{\Lambda}V(e^{i\Lambda\theta_j})}$

When $\Lambda = 1$: Let $\rho_{eq}(e^{ix})$ be the density of the equilibrium measure for $e^{-V(e^{ix})}$. For a such that $\rho_{eq}(e^{ia}) > 0$,

$$\frac{2\pi}{\rho_{eq}(a)N} K_{N,1}(e^{i(a+\frac{2\pi}{\rho_{eq}(a)N}\xi)}, e^{i(a+\frac{2\pi}{\rho_{eq}(a)N}\eta)}) \to \mathbb{S}(\xi, \eta)$$

So, (with $a = 0$)

$$\frac{2\pi}{N} K_{N,1}(e^{i\frac{2\pi}{N}\xi}, e^{i\frac{2\pi}{N}\eta}) \to \mathbb{S}\big(\rho_{eq}(1)\xi, \rho_{eq}(1)\eta\big)$$

When $\Lambda = \infty$:

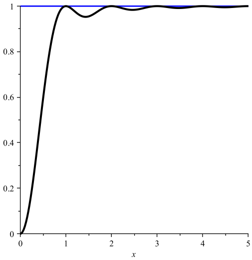
$$\frac{2\pi}{N} K_{N,1}(e^{i\frac{2\pi}{N}\xi}, e^{i\frac{2\pi}{N}\eta}) \to \mathbb{S}(\xi, \eta)$$

Bulk scaling limit. $\frac{2\pi}{N}K_{N,\Lambda}(e^{i\frac{2\pi}{N}\xi}, e^{i\frac{2\pi}{N}\eta})$ converges to

1. When $N >> \Lambda$,
$$\mathbb{S}\big(\rho_{eq}(1)\xi, \rho_{eq}(1)\eta\big)$$
2. When $N = k\Lambda$,
$$\mathbb{S}\Big(\frac{\xi}{k}, \frac{\eta}{k}\Big)K_k(e^{i\xi}, e^{i\eta})$$
3. When $N = \frac{1}{k}\Lambda$,
$$\mathbb{S}(\xi, \eta)\frac{e^{-\frac{1}{2k}V(e^{2\pi ik\xi})-\frac{1}{2k}V(e^{2\pi ik\eta})}}{\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{1}{k}V(e^{i\theta})}d\theta}$$
4. When $N << \Lambda$,
$$\mathbb{S}(\xi, \eta)$$

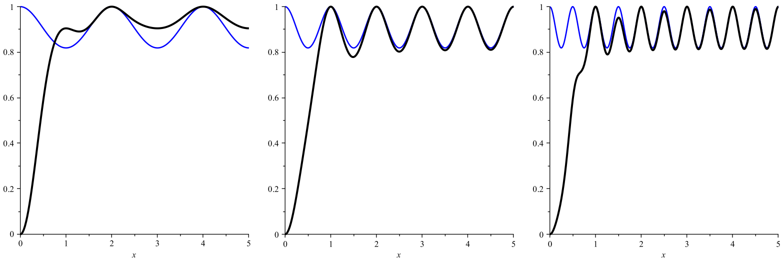
Two-point function. $R_2(0, x)$

$V(e^{ix}) = 0$ (CUE)



Two-point function. $R_2(0, x)$

$V(e^{ix}) = -\frac{1}{10}\cos(2\pi x)$



$N = 2\Lambda$ $\quad$ $N = \Lambda$ $\quad$ $N = \frac{1}{2}\Lambda$

Different scaling

Change

$$e^{-\frac{N}{\Lambda}V(e^{i\Lambda\theta})} \to e^{-NV(e^{i\Lambda\theta})}$$

Example: $V = \cos\theta$, $\Lambda = N$. Then

$$\rho_{N,N}(e^{i\theta}) \approx \begin{cases} 0, & \text{if } \theta \notin \frac{2\pi}{N}\mathbb{Z} \\ \infty, & \text{if } \theta \in \frac{2\pi}{N}\mathbb{Z} \end{cases}$$

Bulk scaling limit:

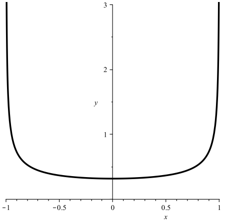
$$\frac{2\pi}{N} K_{N,N}(e^{i\frac{2\pi}{N}\xi}, e^{i\frac{2\pi}{N}\eta}) \to \begin{cases} 0, & \text{if } \xi - \eta \notin \mathbb{Z} \\ \infty, & \text{if } \xi - \eta \in \mathbb{Z} \end{cases}$$

Jacobi ensemble

Jacobi unitary ensemble

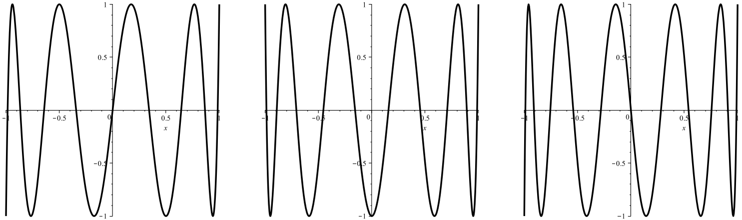
$$p(x_1, \cdots, x_N) = \frac{1}{Z_N} |\Delta_N(x)|^2 \prod_{j=1}^{N} \frac{1}{\sqrt{1-x_j^2}}$$

Density of states: $\frac{1}{\pi\sqrt{1-x^2}}$. Bulk universality.



Jacobi ensemble

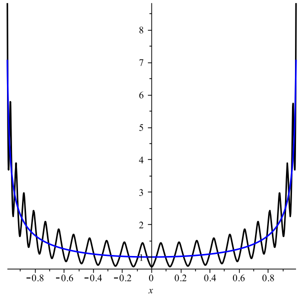
Let $T_k(x)$ be the Tchebyshev polynomial of first kind.
$T_k(x) = \cos(k\theta)$ where $x = \cos\theta$.



Jacobi ensemble

Fix $V(x)$, $x \in [-1,1]$, and consider

$$p_{N,\Lambda}(x_1,\cdots,x_N) = \frac{1}{Z_N}|\Delta_N(x)|^2 \prod_{j=1}^{N} \frac{e^{-\frac{N}{\Lambda}V(T_\Lambda(x_j))}}{\sqrt{1-x^2}}$$



$V(x) = -\frac{1}{2}x - \frac{1}{10}x^5$, $N = 30$, $\Lambda = 50$.

Jacobi ensemble

Set $N = \Lambda$. Density of states:

$$\rho_{N,N}(x) \approx \frac{a + bT_N(x)}{\pi\sqrt{1-x^2}} e^{-V(T_N(x))}$$

Bulk limit: for some $x_0(N) \to 0$,

$$\frac{\pi}{N} K_{N,N}\left(x_0(N) + \frac{\pi}{N}\xi, x_0(N) + \frac{\pi}{N}\eta\right)$$
$$\to \left(a\mathbb{S}(\xi,\eta) + b\frac{\sin(\pi\xi) - \sin(\pi\eta)}{\xi - \eta}\right) e^{-V(\sin(\pi\xi)) - V(\sin(\pi\eta))}$$

Jacobi ensemble, hard edge

JUE $\frac{1}{\sqrt{1-x^2}}$:

$$\frac{1}{2N^2} K_N^{JUE}\left(1-\frac{\xi}{2N^2}, 1-\frac{\eta}{2N^2}\right) \to \mathbb{B}_{-1/2}(\xi,\eta)$$

Bessel kernel

$$\mathbb{B}_a(\xi,\eta) = \frac{J_a(\sqrt{\xi})\sqrt{\eta}J_a(\sqrt{\eta}) - \sqrt{\xi}J_a'(\sqrt{\xi})J_a(\sqrt{\eta})}{\xi-\eta}$$

Eigenvalue scale $x = 1-\frac{\xi}{2N^2}$.

Jacobi ensemble, hard edge

locally-varying potential $\frac{e^{-\frac{N}{\Lambda}V(T_\Lambda(x_j))}}{\sqrt{1-x^2}}$ with $\Lambda = N$:

$$V(T_N(x)) \approx -V(\sqrt{\xi}) \quad \text{when } x = 1 - \frac{\xi}{2N^2}$$

Hard edge:

$$\frac{1}{2N^2} K_N^{JUE}\left(1 - \frac{\xi}{2N^2}, 1 - \frac{\eta}{2N^2}\right)$$
$$\to \left(a\mathbb{B}_{-1/2}(\xi,\eta) + bL(\xi,\eta)\right) e^{-\frac{1}{2}V(\sqrt{\xi})-\frac{1}{2}V(\sqrt{\eta})}$$

where

$$L(\xi,\eta) = \frac{\sqrt{\xi}\sin(\sqrt{\xi}) - \sqrt{\eta}\sin(\sqrt{\eta})}{(\xi\eta)^{1/4}(\xi-\eta)}$$

Summary

- Locally-varying potential $NV_1(x) + V_2(Nx)$
- Circular unitary ensemble with periodic potential
- Simple relation between orthogonal polynomials
- Bulk scaling: determinantal point process with kernel $A(x,y)B(x,y)$ structure.
- Jacobi unitary ensemble with potential $\frac{e^{-\frac{N}{\Lambda}V(T_\Lambda(x))}}{\sqrt{1-x^2}}$. Hard edge.
- Question: (i) other β (ii) Hermitian matrix (iii) soft edge