Arthur packet: why and how?

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Langlands philosophy

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Automorphic forms are the spectrum of $L^2(X_{G,F})$ with respect to the action of

$$G_{\mathbb{A}_F} = \prod_v G_{F_v}.$$

In particular, they are all unitary!

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Langlands parameters are $\widehat{G}$-conjugacy classes of

$$\begin{array}{ccc} \phi : \ L_F & \longrightarrow & {}^L G \\ & \searrow \quad \swarrow & \\ & W_F & \end{array}$$

Local Langlands correspondence

Conjecture (LLC)

F is local, there is a "natural" way to associate each Langlands parameter ϕ with a finite set Π_ϕ of irreducible admissible representations, such that

$$\Pi_{irr}(G_F) = \bigsqcup_{\phi} \Pi_\phi.$$

We call Π_ϕ local L-packets.

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Some known cases of LLC:

- $GL(n)$: Harris-Taylor, Henniart
- $Sp(2n), O(n)$: Arthur
- $U(n)$: Mok, Kaletha-Minguez-Shin-White
- $GSp(2n)$, $GO(2n)$ ("up to twisting"): my thesis

Global Langlands correspondence

F is global,

$$\begin{array}{ccc} L_F & \xrightarrow{\phi} & {}^L G \\ \uparrow & & \uparrow \\ L_{F_v} & \xrightarrow{\phi_v} & {}^L G_v \end{array}$$

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Arthur's idea: enlarge certain unitary local L-packets, which will be called local Arthur packets.

Arthur packet

F is either local or global,

$$\begin{array}{ccc} \{\text{Arthur parameters of } G\} & \hookrightarrow & \{\text{Langlands parameters of } G\} \\ \| & \nearrow_{\psi \mapsto \phi_\psi} & \\ \left\{ {\psi : L_F \times SL(2,\mathbb{C}) \to {}^L G \atop \text{bounded on } L_F} \right\} / \widehat{G} - conj & & \end{array}$$

$$\phi_\psi(u) = \psi\left(u, \quad \begin{pmatrix} |u|^{1/2} & \\ & |u|^{-1/2} \end{pmatrix}\right), \quad u \in L_F$$

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F is local, there is a "natural" way to associate each Arthur parameter ψ with a finite set Π_ψ of irreducible unitary representations, such that $\Pi_\psi \supseteq \Pi_{\phi_\psi}$, and $\Pi_\psi = \Pi_{\phi_\psi}$ if ψ is trivial on $SL(2,\mathbb{C})$. Such sets are called local Arthur packets.

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Remark

- Local Arthur packets do not give classification of irreducible unitary representations and they can have intersection with each other!

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Some known cases of (local and global) Arthur packets:

- GL(n): Moeglin-Waldspurger
- Sp(2n), O(n): Arthur, Moeglin
- U(n): Mok, Kaletha-Minguez-Shin-White, Moeglin
- GSp(2n), GO(2n): Xu (in progress)

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Remark

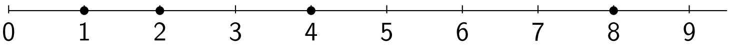
The global Arthur packet Π_ψ comes naturally from the stabilized form of the Arthur-Selberg trace formula. But we do not know the local nature of Arthur packet.

Inner structure of Arthur packet

Example

F is $\mathbb{Q}_p$, G is $Sp(2n)$.

L-packet



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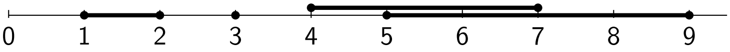

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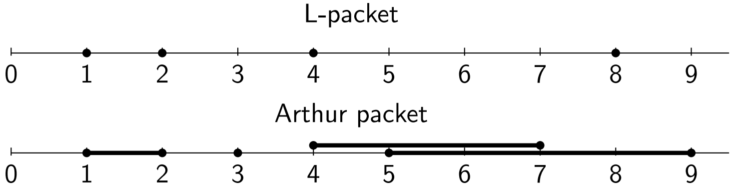

If there are no intersections, the packet structure is completely known by Moeglin. If there are intersections, it is still a mystery!

Problem: what is the combinatorics involved in computing local Arthur packets?