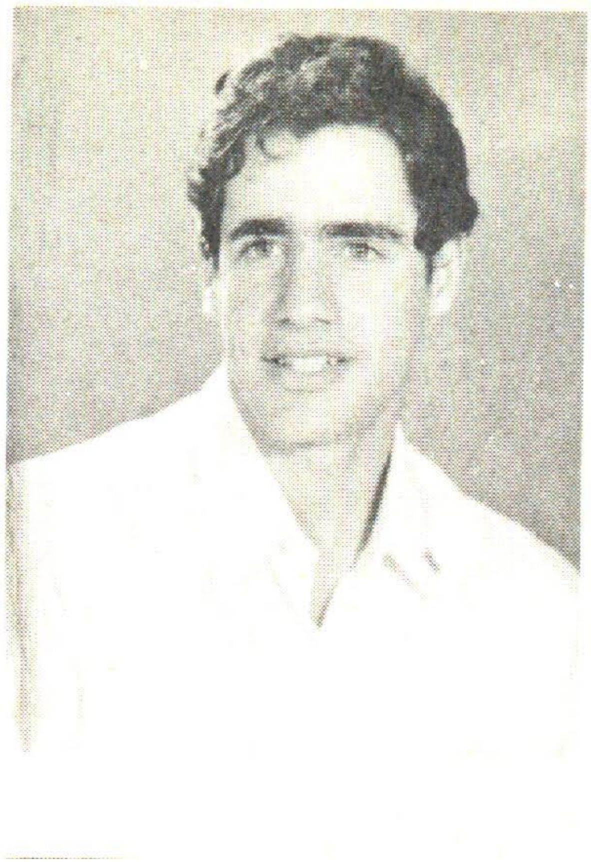
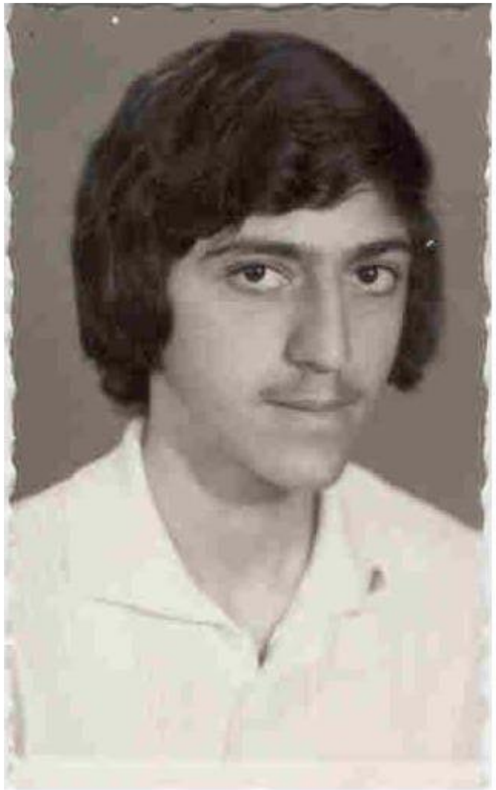
Avi, Graphs and Communication

Noga Alon, Tel Aviv U.

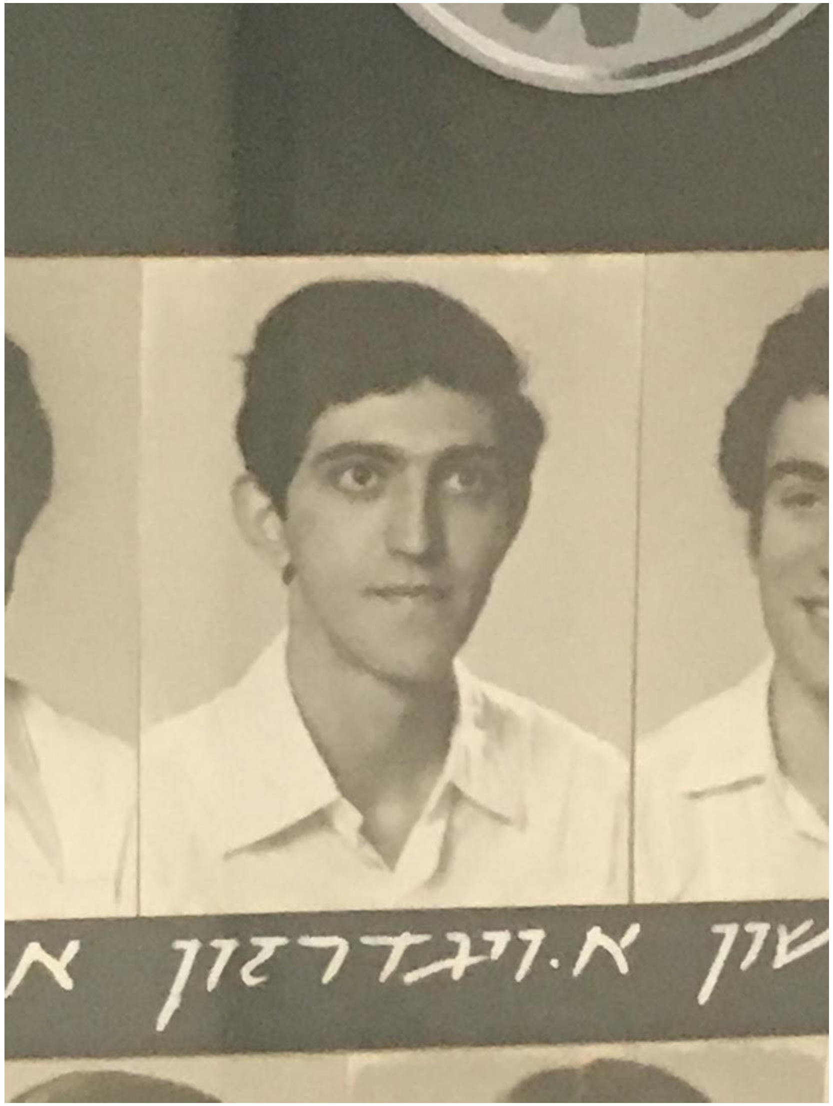
Princeton, Oct. 2016

I Avi





Graduating high school, Haifa, Israel



Technion, Haifa

dressed for dinner

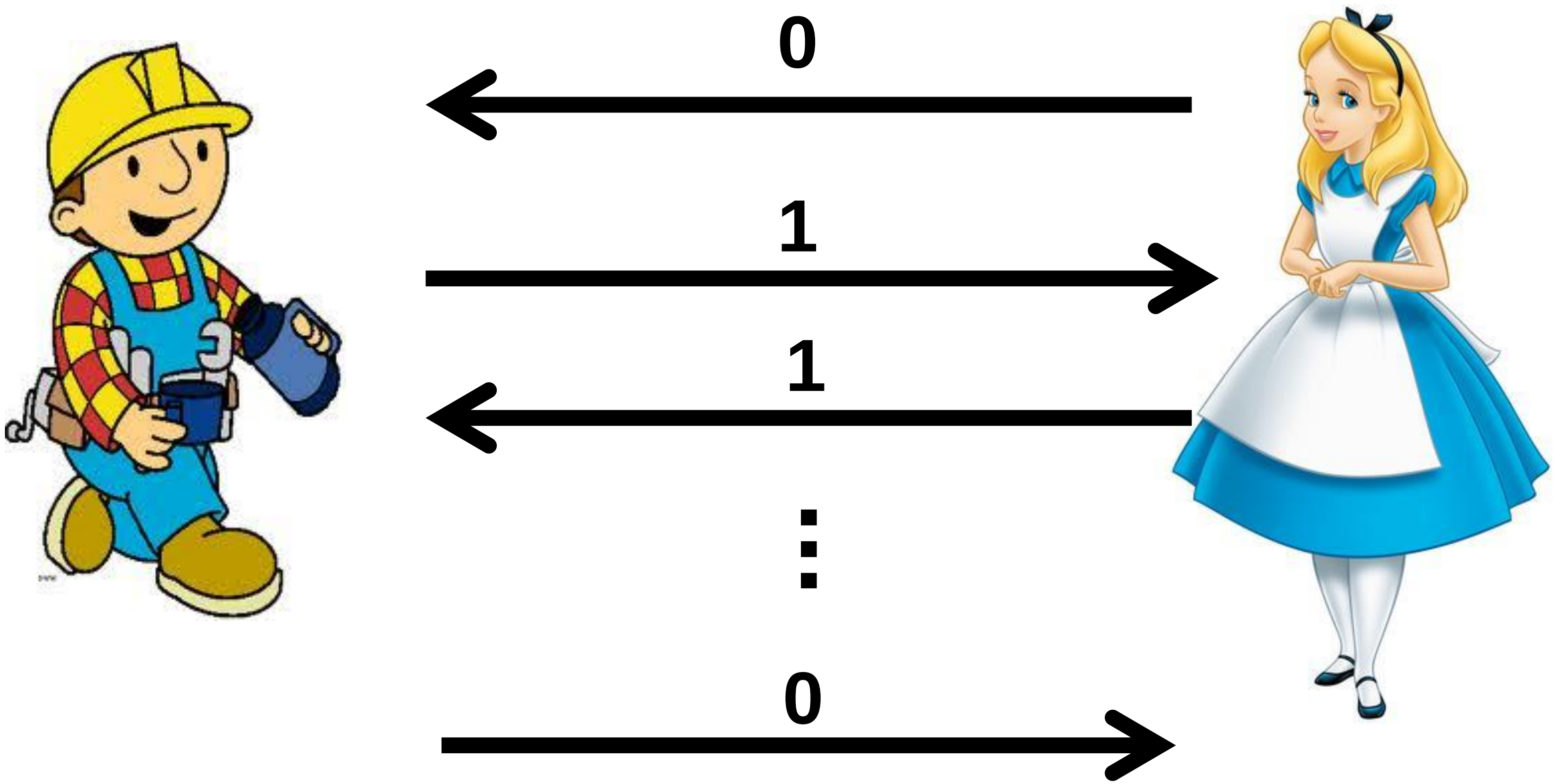
II Communication Complexity

Yao (79): **For a Boolean function**

$$f(x,y): \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$$

Bob knows x, Alice knows y, and they wish to compute f(x,y) by communicating the minimum possible number of bits.

How many bits are needed ?


0
1
1
0

A basic example: equality of n bits requires n bits of communication (in a deterministic protocol).

There are many variants: randomized (with public or private coins), non-deterministic, unbounded error, quantum, multiparty (number on the forehead or number in hand)

Avi has 12 papers in MathSciNet with the word "communication" in the title

and 12 more with "communication" in the abstract

III Testing equality in graphs

N. Alon, K. Efremenko, B. Sudakov

The Problem: G=(V,E) a connected undirected graph.
In each vertex there is a player with an n bit vector.
The players wish to determine whether or not all their vectors are equal by sending messages along the edges of G.
What is the minimum possible number of bits and a communication protocol achieving it ?

This is interesting even for small simple G like K_3 or C_6

Let this minimum number of bits be f(n,G).
By subadditivity and Fekete the limit f(n,G)/n as n tends to infinity exists, denote it by f(G).

Easy: for each G on k vertices f(G) ≤ k-1 (each non-root in a spanning tree sends his vector to its parent and checks equality to the vectors of his children.)

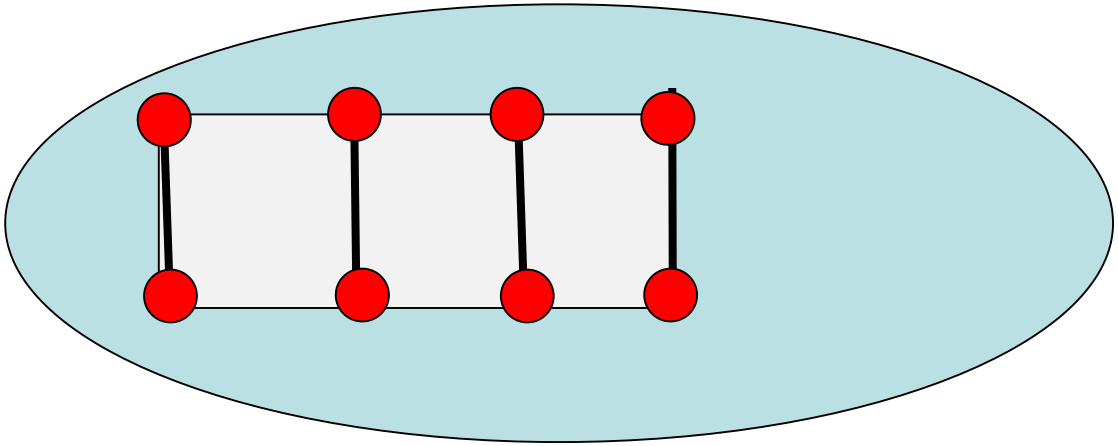
Prop: Any linear protocol (sending only linear functions of the inputs and the bits received) cannot use less than (k-1)n bits

Liang and Vaidya (11):
(i) $f(K_k) \geq k/2$
(ii) $f(K_k) < k-1$ for all $k \geq 3$

Brody (12) (using the graphs of **A-Moitra-Sudakov**): $f(K_3)=3/2$

These are graphs with $(1-o(1))\binom{n}{2}$ edges and n vertices that can be decomposed into pairwise edge disjoint induced matchings, each of size $n^{1-o(1)}$.

That is: nearly complete graphs which can be decomposed into nearly perfect induced matchings.

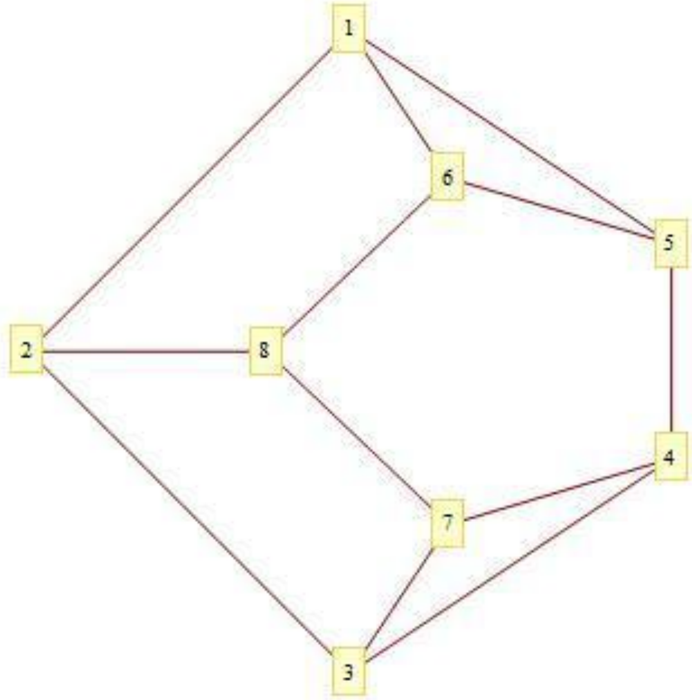


These graphs have been used in the design of efficient communication protocols for Radio Networks

A, Moitra and Sudakov(13), Brody and Håstad(13):

$f(K_k)=k/2$

What is f(G) for non-complete graphs G ?

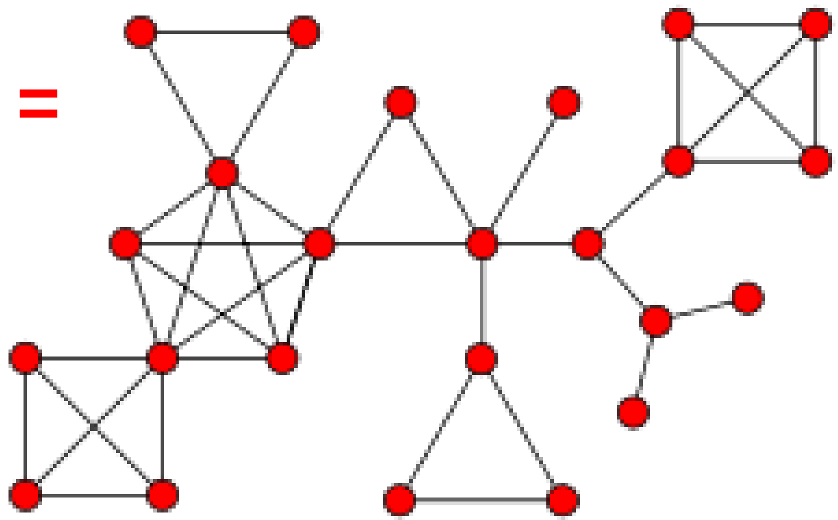


New Results (A,Efremenko,Sudakov 16+)

Prop: For every G with blocks $G_1, G_2, \dots, G_s$

$$f(G) = \sum_i f(G_i)$$

Example: G =



$$f(G) = \frac{7 \cdot 2 + 3 \cdot 3 + 2 \cdot 4 + 5}{2} = 18$$

Theorem (Upper bound): **If G has a spanning 2-edge connected subgraph with m edges then $f(G) \leq m/2$**

Let $c_2(G)$ denote the minimum number of edges in a 2-edge connected spanning subgraph of G (which may contain some edges twice).

Then $f(G) \leq c_2(G)/2$

Lower bound:

The **fractional cut-packing** number of G **fc(G)** is

$$\max \sum g(S,\bar{S})$$

where the sum ranges over all cuts $(S,\bar{S})$,

$$0 \le g(S,\bar{S}) \le 1$$

and for every edge e

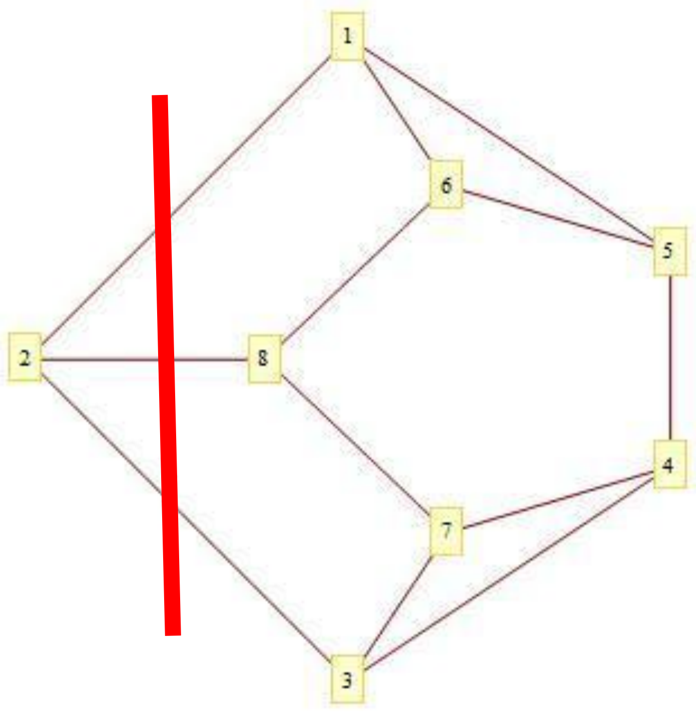
$$\sum_{e\in(S,\bar{S})} g(S,\bar{S}) \le 1$$

Theorem (lower bound):

For every G f(G)≥fc(G)

In particular, f(G) ≥ fc(G) ≥ k/2 for any k-vertex graph G

Indeed, g can assign value ½ to all the k cuts determined by a single vertex

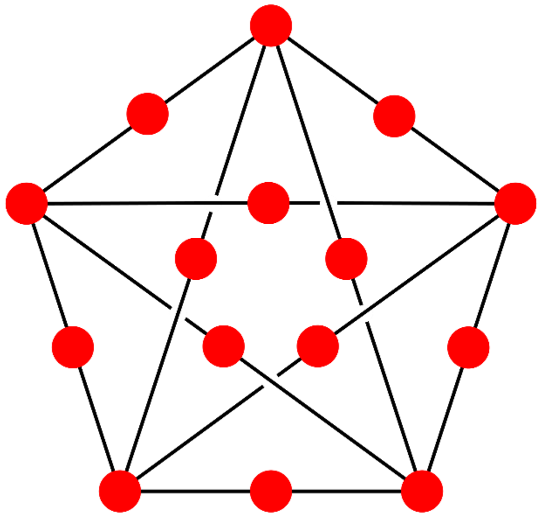


In particular:

(i) For every Hamiltonian G on k vertices f(G)=k/2

(ii) For $t \geq s \geq 2$ $f(K_{s,t}) = t$

(iii) For any 2-edge connected G with no two adjacent vertices of degree at least 3, f(G) is half the number of edes of G



IV Some proof ideas

Upper bound (for $G=C_4$)

Step 1: A Behrend type construction:

Lemma 1: in the Abelian group $A=Z_t^r$ with $r=\sqrt{\log m}$ and $t=2^{\sqrt{\log m}}$ there is a subset X of size at least

$$m / 2^{O(\sqrt{\log m})} = m^{1-o(1)}$$

such that if $x_1+x_2+x_3=3x_4$ with x_i in X then

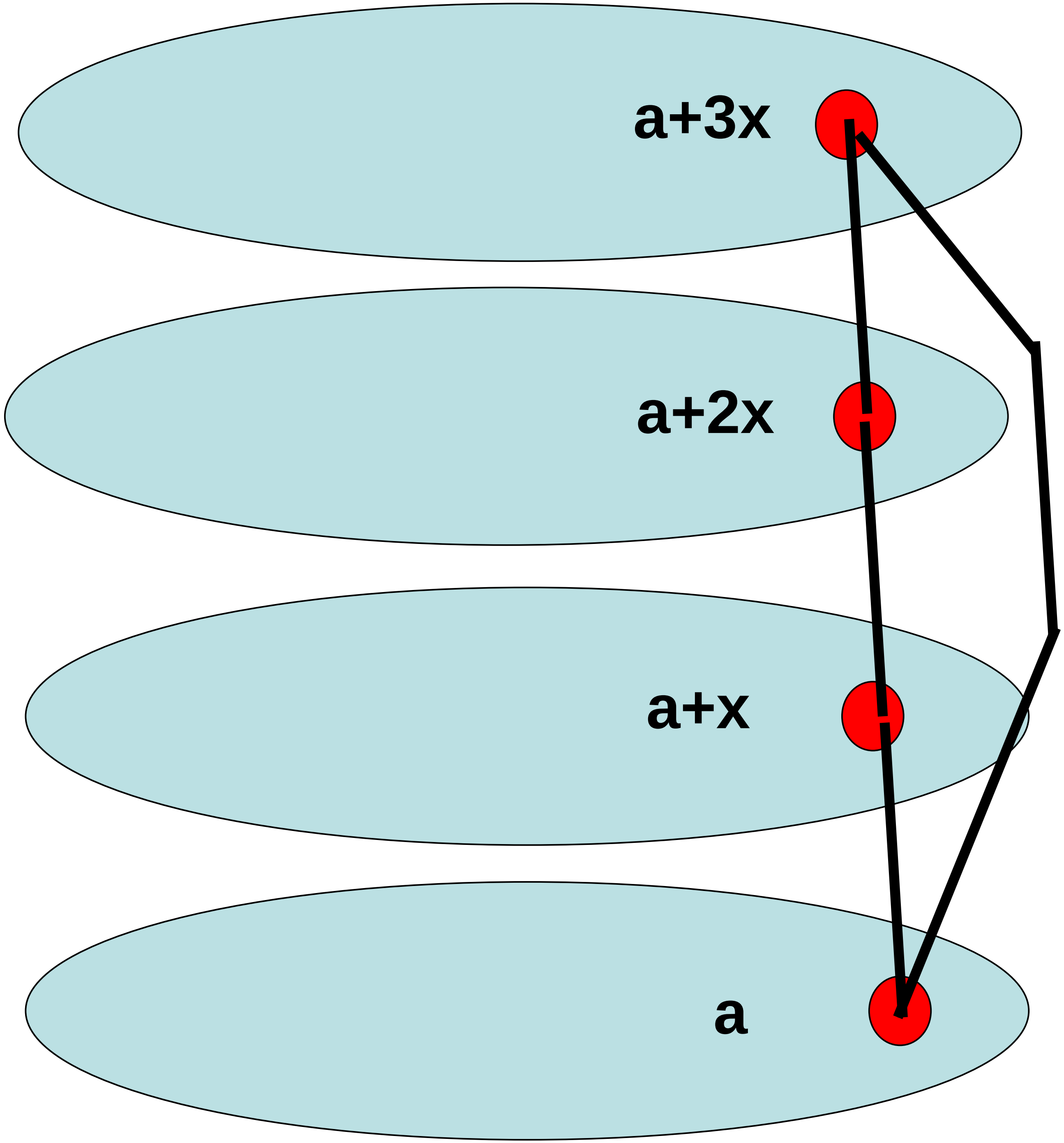
$$x_1=x_2=x_3=x_4$$

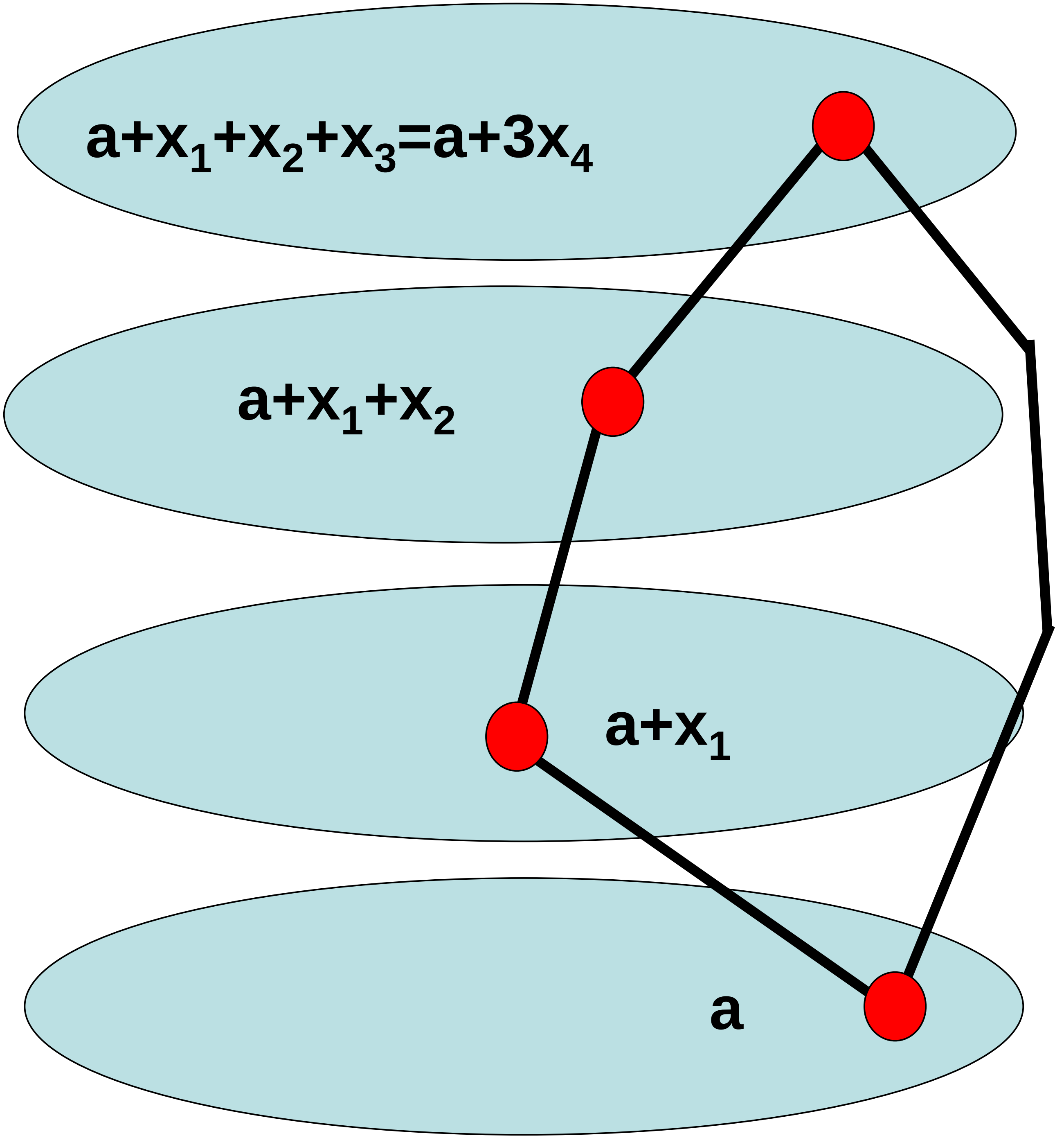
Note: the above estimate is tight by Croot, Lev, Pach (16), Ellenberg, Gijswijt (16)

Step 2: Constructing a graph (ala **Ruzsa-Szemerédi**):

Lemma 2: There exists a 4-partite graph F, with vertex classes V_1, V_2, V_3, V_4 , each with m vertices, which contains $m^{2-o(1)}$ edge disjoint copies of C_4 each with a vertex in every V_i so that every edge of F belongs to exactly one such C_4

Proof: each V_i is a copy of the abelian group $A=Z_t^r$, (where 2,3 do not divide t). For each a in A and x in X (from Lemma 1), take a copy of C_4 on a in V_1, a+x in V_2 , a+2x in V_3 and a+3x in V_4


a+3x
a+2x
a+x
a



$a+x_1+x_2+x_3=a+3x_4$
$a+x_1+x_2$
$a+x_1$
a

The number of copies of C_4 is $m|X|=m^{2-o(1)}$ and any copy of C_4 with a vertex in each V_i is one of those.

Remark:

For general 2 connected graphs instead of C_4 the construction applies the result of Whitney (32) about the existence of Ear Decomposition of such graphs

Remark: similar ideas are used in A(2001) to prove that the graph property of containing no copy of a fixed graph H can be tested (in the sense of Goldreich, Goldwasser and Ron 98) by examining random samples of size polynomial in the proximity parameter if and only if H is bipartite.

I couldn't attend the conference, and the paper was presented by Avi. Several people who have been there told me that this was my best FOCS presentation ever.

Step 3: The communication protocol (for C_4)

Let F be the graph from Lemma 2 with $m^{2-o(1)}$ being roughly 2^n . Thus log m is (1/2+o(1))n.

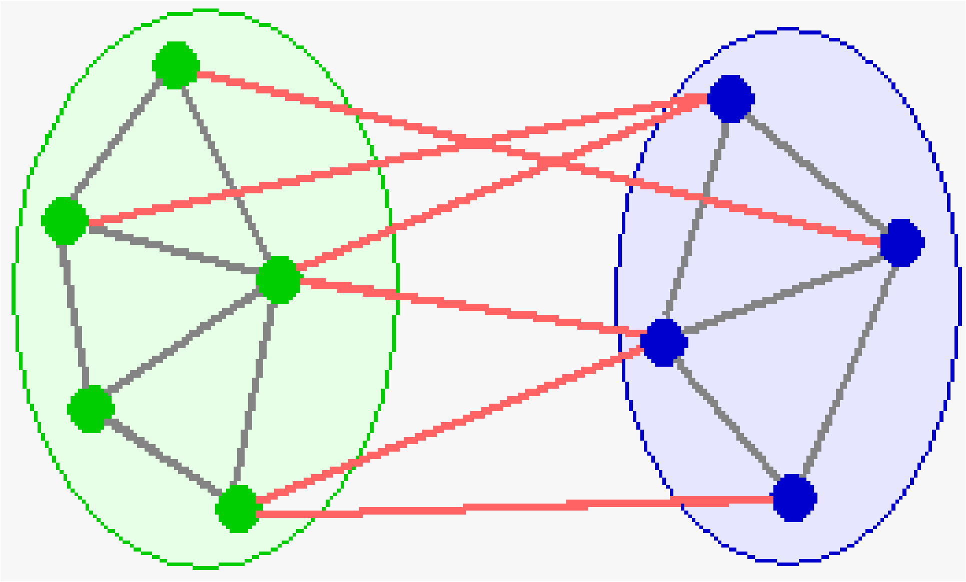
Identify the input vectors with the special copies of C_4 in F. If Player i has a copy H_i he sends to player (i+1)(mod 4) the vertex of his copy in V_i and checks if the vertex he got from player i-1 is indeed the one of his copy. If not he reports the inputs are not identical, if nobody reports, the copies are declared identical.

Total number of bits transmitted:
4 log m=(2+o(1)) n.

If the copies are identical, it is clear that nobody reports.

If nobody reports, and player i sends vertex u_i , then $u_1u_2u_3u_4$ is a special copy of C_4 in the graph F, and for all i the graph H_i contains the edge $u_{i-1}u_i$. By the property of F this means that all H_i are identical, as needed.

The lower bound is proved using the fact that in a valid communication proptocol at least n bits should be transmitted along any cut in the graph. This and the duality of linear programming show that f(G) ≥ fc (G).



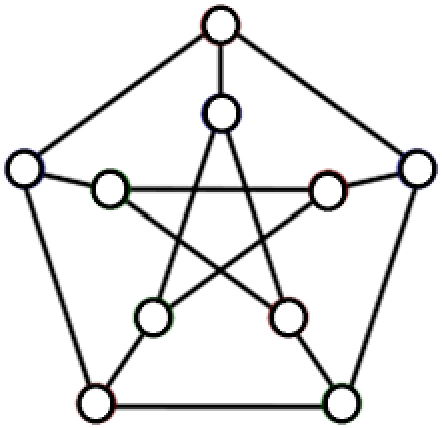
V Open Problems

Is $f(G)=0.5\ c_2(G)$ for all G ?

If not, is $f(G)=fc\ (G)$ for all G ?

Is $f(G)=0.5\ |V(G)|$ iff G is Hamiltonian ?

What is f(P) if P is the Petersen graph ?

P= 

Here $5 = fc(P) \leq f(P) \leq c_2(P) = 5.5$


HappyBirthdayCakePic.com
Happy Birthday
Avi

INSTITUTE
for ADVANCED STUDY