Non-orientable knot genus and the Jones polynomial

joint w/ Christine Lee (UT, Austin)

Geometric structures on 3-manifolds, IAS, Princeton, NJ.

Setting and outline of talk

$C(K)$= *crosscap number* (a. k. a. non-orientable genus) of a knot K= smallest genus over all non-orientable surfaces spanned by K.

Plan:

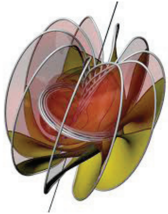
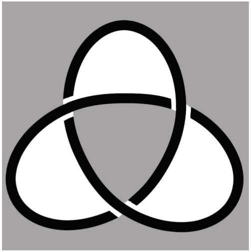
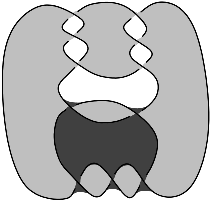
- Review what is known- Compare with the (oriented) *genus*.
- There is an algorithm to compute knot *genus*.
- No similar algorithm is known to compute crosscap number. Indicate progress/difficulties.
- Discuss calculations for knots up to 12 crossings.

Restrict to alternating knots:

- Classical genus results:
- Genus is calculated from alternating diagrams (*Seifert's algorithm*).
- Genus is calculated from the Alexander polynomial.
- Discuss non-orientable counterparts:
- Crosscap number is calculated from alternating diagrams (*state surfaces*).
- Crosscap number is estimated/calculated from the Jones polynomial.

Definitions etc

- Knots span surfaces: both orientable and non-orientable.

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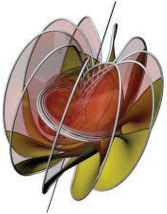
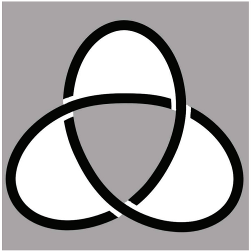
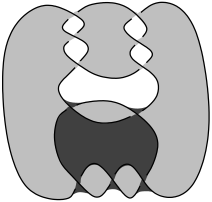

- **Definition.** S non-orientable surface spanned by a k-component link K. *crosscap number of S*

$$C(S) = 2 - \chi(S) - k.$$

- The *crosscap number of a link K* is the minimum crosscap number over all non-orientable surfaces spanned by K.
- Crosscap numbers first studied by B. E. Clark– made several observations (1978).

Facts, bounds and algorithms:

- Convention: $C(\mathrm{Unknot}) = 0$.
- $g(K)$= *orientable genus* of K. Then, $C(K) \leq 2g(K) + 1$.
- $C(K) = 1$ iff K is a $(2, p)$ torus knot or a $(2, p)$ cable.
- If K alternating, then $C(K) = 1$ iff K is a $(2, p)$ torus knot.
- (H. Murakami- Yasuhara) If $c(K)$=crossing number of K, then

$$C(K) \leq \left\lfloor \frac{c(K)}{2} \right\rfloor.$$

  and the bound is sharp.
- Crosscap numbers are known for families: (*e.g. 2-bridge knots, pretzel knots*)– Bessho, Hirasawa, Teragaito, Ichiharra, Mizushima.....

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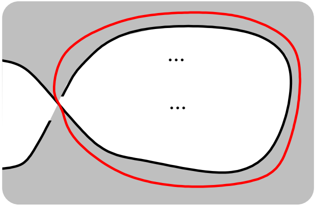
- $C(K)$ not known for a lot of knots up to 12 crossings ($g(K)$ is known).
- There is no known algorithm to calculate $C(K)$ (there is for $g(K)$)
- **Issue:** A surface realizing $C(K)$ need not be ∂- incompressible.

Facts, bounds and algorithms con't:

- **Pathology:** In fact, all surfaces realizing $C(K)$ may be obtained from oriented ones by adding a "*trivial crosscap*".
- This creates a ∂-compression disk in $M_K = S^3 \setminus K$. (Red line below).



- **Pathology Example:** The knot $K = 7_4$: We have $g(K) = 1$. Murasugi-Yasuhara calculated $C(K) = 3 = 2g(K) + 1$. skip
- All surfaces for 7_4, realizing $C(K) = 3$, are obtained from a genus 1 Seifert surface by adding a trivial crosscap.

Facts, bounds and algorithms: Normal surface theory

- Orientable genus $g(K)$:
- Normal surface theory algorithm and computational complexity (Hass-Lagarias-Pippenger -1999). An basic starting was that " Haken's normalization" process gives:

Theorem

Let $\mathcal{T}$ be a triangulation of a knot complement M_K. Then there is a fundamental, normal, orientable spanning surface of genus $g(K)$.

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- Basic steps of **Algorithm:** Given K,

Def. *fundamental*: not written as sum of non-empty normal surfaces.

1. Obtain a "*suitable*" triangulation $\mathcal{T}$ of M_K.
2. Enumerate all fundamental normal surfaces in $\mathcal{T}$.
3. Identify the spanning orientable ones among surfaces in step 2.
4. Identify the smallest genus surface that appears in step 3.

Algorithms: Normal surface theory

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- Above discussed pathology creates complications:
- B. Burton and Burton-Ozlen (2012) made progress. First they note the following:

Theorem (Burton-Ozlen)

Let $\mathcal{T}$ be a triangulation of a knot complement M_K. Then, either

- *there is a fundamental, normal, non-orientable spanning surface with $C(S) = C(K)$; or*
- $C(K) \in \{2g(K),\ 2g(K) + 1\}$.

- They obtain an **Algorithm:** Given K, either

1. get a single value that is $C(K)$; or
2. narrow the values for $C(K)$ to two possible ones.

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2. narrow the values for $C(K)$ to two possible ones.

- Burton-Olzen upper bounds for $C(K)$ —Burton pushed further to get exact calculations.

Low crossing data: up to 12 crossings

Info copied from KnotInfo (Cha- Livingston).

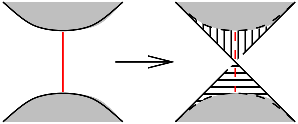
- $C(7_4) = 3$ (Murakami-Yasuhara)
- 2-bridge cases; $C(K)$ determined by Teragaito and Hirasawa
- Typically KnotInfo gives upper bounds that were obtained by finding non-orientable surfaces *state surfaces*.
- Burton-Ozlen: Used normal surfaces and integer programming to find non-orientable surfaces of small crosscap number. They got **new upper bounds for 778** of the knots in the table. Burton, using integer programming, was able to obtain exact values for several of these knots.
- (2012) Adams and Kindred: Method that determines the crosscap number of an alternating knot. They got previously unknown values for:

$$\mathbf{8_{10,15,16,17,18}} \quad \textbf{and} \quad \mathbf{9_{16,22,24,25,28,29,30,32,33,34,36,37,38,39,40,41}}.$$

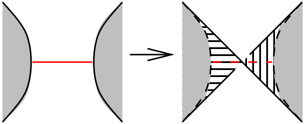
- (2014) K.- Lee: Bounds in terms of the Jones polynomial. Improved the bounds for **almost half** of the table knots, and precisely determined the number for **283** of the 12-crossing knots.

For a Kauffman state σ of a link diagram, form a *state surface* S_σ:

- Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



A–resolution



B–resolution

- **Special Cases:** Seifert state, checkerboard states of alternating knots.

State surfaces con't

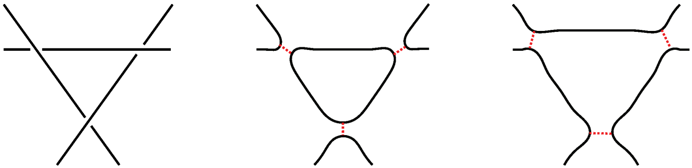
First considered (defined) by Przytycki.

- M. Ozawa gave diagrammatic criteria for state surfaces to be essential in link complement.
- Adams, Futer-K-Purcell, gave diagrammatic criteria for state surfaces to be *non-parabolic*.
- (F.-K.-P.) "Often" the geometric type (i.e. *semi fiber, quasi-Fuchsian*) is easily determineed by the link diagram.
- Also, (F.-K.-P.) used them to:
  - Relate *Colored Jones polynomials (CJP)* to *boundary slopes* of knots (Slope Conjecture)
  - Derive relations between CJP and hyperbolic geometry.
- Adams-Kindred, used them to give algorithm to calculate $C(K)$ of alternating links.

Alternating links

- [Murasugi, Gabai]. The Seifert state applied to a reduced alternating diagram $D(K)$ gives a minimum (orientable) genus surface.
- [Adams-Kindred (2013)]. Gave an algorithm to calculate $C(K)$ of alternating knots, from state surfaces.
- **Remark:** The set of Euler characteristics of state surfaces obtained from an alternating knot diagram is invariant under *flyping*– thus a knot invariant.
- **The Algorithm:** $D = D(K)$ alternating knot diagram. Think of D as a 4=valent graph.
- If D has regions that 1-gons or 2-gons resolve the corresponding crossings so that the region becomes a state circle.
- Suppose D has no 1-gons or 2-gons; then it has triangles.
- Pick a triangle region on D. Create two branches:

Algorithm con't:



- Repeat until each branch reaches a projection without crossings.
- Choose the resulting surfaces S that have maximal Euler characteristic.

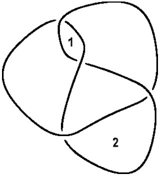
Theorem (Adams-Kindred, 2013)

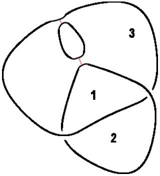
After applying the algorithm to an alternating diagram of k-component link K:

1. *If there is S as above that is non-orientable then $C(K) = 2 - \chi(S) - k$.*
2. *If all surfaces produced by the algorithm are orientable, S is a minimal genus Seifert surface of K and $C(K) = 2g(K) + 1$.*

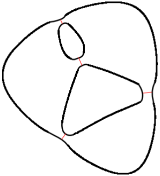
An example: Fig-8:

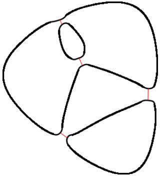
- Bigons labeled 1 and 2 and diagram resulting from applying the first step of the Algorithm. New bigon regions labeled 1, 2, and 3.





- State surfaces from different choices of bigon regions.





- Left one gives a non-orientable surface of maximal Euler characteristic $\chi(S) = -1$. Hence, $C(K) = C(S) = 2$.

Knot polynomial bounds

- **Genus:** (Crowell, Murasugi, 1960) For K alternating, $g(K)$ is half the degree span of the *Alexander polynomial of K*.

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- **Crosscap number:** Let

$$J_K(t) = \alpha_K t^n + \beta_K t^{n-1} + \ldots + \beta'_K t^{s+1} + \alpha'_K t^s$$

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- Set $T_K := |\beta_K| + |\beta'_K|$.

Theorem (K.-Lee, 2014)

Let K be a non-split, prime, non-torus, alternating link with k-components and with crosscap number $C(K)$. We have

$$\left\lceil \frac{T_K}{3} \right\rceil + 2 - k \;\leq\; C(K) \;\leq\; T_K + 2 - k,$$

Furthermore, both bounds are sharp.

Sharpness

- **Knots:** For K =alternating, non-torus knot we have

$$\left\lceil \frac{T_K}{3} \right\rceil + 1 \leq C(K) \leq \min\left\{ T_K + 1, \left\lfloor \frac{s_K}{2} \right\rfloor \right\}$$

where T_K as above and s_K =degree span of $J_K(t)$. Bounds are sharp.

- **Some examples:** Knotinfo $C(K)$ upper bound agrees with above lower bound. T_K value also from Knotinfo. We determine that $C(K) = 3$.

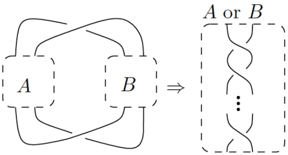
K	T_K	K	T_K	K	T_K	K	T_K
10_{85}	6	10_{93}	6	10_{100}	6	$11a_{74}$	5
$11a_{97}$	5	$11a_{223}$	5	$11a_{250}$	5	$11a_{259}$	5
$11a_{263}$	4	$11a_{279}$	6	$11a_{293}$	6	$11a_{313}$	6
$11a_{323}$	6	$11a_{330}$	6	$11a_{338}$	4	$11a_{346}$	6
$12a_{0636}$	5	$12a_{0641}$	4	$12a_{0753}$	5	$12a_{0827}$	5
$12a_{0845}$	5	$12a_{0970}$	6	$12a_{0984}$	6	$12a_{1017}$	6
$12a_{1031}$	5	$12a_{1095}$	6	$12a_{1107}$	6	$12a_{1114}$	6
$12a_{1142}$	5	$12a_{1171}$	6	$12a_{1179}$	6	$12a_{1205}$	6
$12a_{1220}$	6	$12a_{1240}$	6	$12a_{1243}$	4	$12a_{1247}$	6

Calculating T_K and s_K

- Let $D = D(K)$ reduced alternating knot diagram.
- (Murasugi, Kauffman '80s) We have $s_k = c(D) = c(K)$=number of crossings
- Let $\mathbb{G}_A$ and $\mathbb{G}_B$ the *reduced checkerboard graphs* (a.k.a. *simple Tait graphs*) of D.
- (Dasbach-Lin) We have

$$T_K = 2 - \chi(\mathbb{G}_A) - \chi(\mathbb{G}_B).$$

- If D is *twist reduced*, with twist number $t = t(D)$, then $T_k = t$.
- **Definition.** *twist region* = maximal string of bigons
  Twist reduced: A or B must be a string of bigons.



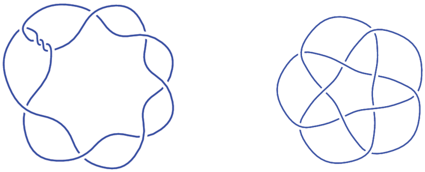
Twist number and crosscap number

Theorem (K.- Lee, 2014)

Let $D(K)$ a twist reduced, prime, alternating diagram with twist number $t \geq 2$ and crossing number c. We have sharp bounds:

$$1 + \left\lceil \frac{t}{3} \right\rceil \leq C(K) \leq \min\left\{ t+1, \left\lfloor \frac{c}{2} \right\rfloor \right\}.$$

- **Sharp upper bound:** $K = 10_3$(left)– $C(K) = 2g(K) + 1 = 3 = t + 1$.
- **Sharp lower bound:** $K = 10_{123}$– Both bounds give 5. We get $C(K) = 5$.



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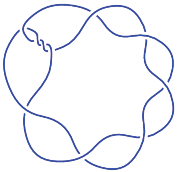
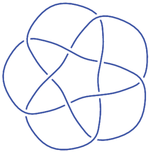




- **Note:** Upper bound of theorem follows easily. Discuss the lower bound.

Getting the lower bound: Outline

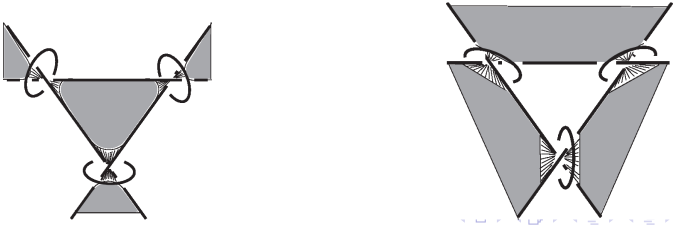
$D = D(K)$ prime,reduced, twist-reduced alternating diagram, with $t > 1$.

- **Step 1.** Show there is a spanning surface S from which $C(K)$ can be calculated, and an *augmented link* L, obtained from D, such that "augmentation components" added to D don't intersect S.
- **Step 2.** Use geometry of L (*angled polyhedral structures*) and normal surface theory to obtain a surface S', such that
  1. S' is a normal surface,
  2. $C(K)$ can be calculated from S'
- **Step 3.** To obtain the lower bound of $C(K)$ in terms of t, combine
  1. a combinatorial notion of area that satisfies Gauss-Bonnet (Casson),
  2. Estimates of slope lengths on cusps of augmented links (Futer-Purcell using work of Lackenby).

Step1: Augmenting

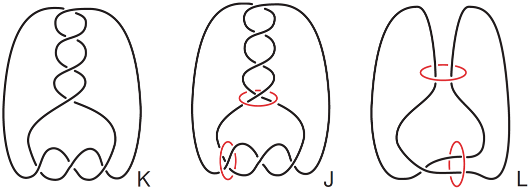
- Starting with $D = D(K)$ a prime,reduced, twist-reduced alternating diagram, we want to augment "around" the Adams-Kindred algorithm.
- Augmenting around bigon regions of D and creating a state surface disjoint from the augmentation component:

- Augmenting around triangle regions and creating a state surface disjoint from the augmentation components:



"Nice" polyhedral decomposition

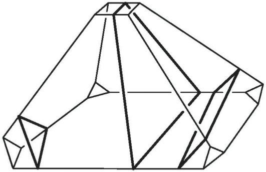
Alternating link K, *augmented* and *fully augmented* links J and a L.



- $M_L = S^3 \setminus L$ has a "nice" decomposition (Adams) into two convex ideal polyhedra P_1 and P_2 in the hyperbolic 3-space. (truncated vertices).
- Dihedral angles of P_i are $\pi/2$. Thus M_L is hyperbolic.
- Edges of $P_i \cap \partial M_L$ called *boundary edges*.
- Faces of $P_i \cap \partial M_L$ called *boundary faces*. They subdivide ∂M_L into rectangles.
- *Interior faces* of P_i admit checker-board coloring: opposite sides of boundary face gets same color interior faces.

Step 2:"Normalizing"

- **Recall:** For K=alternating, have augmented link L and surface S in M_L such that $C(S) = C(K)$.(S need not be ∂-incompressible).
- Going through the normalization process: There is a normal surface, S' in M_L so that either $C(K) = 1 - \chi(S')$ or $C(K) = 2 - \chi(S')$.
- *combinatorial area* $A_c(S')$= Sum of areas of all *normal disks* of S'.
- Normal disks look like:



- Combinatorial area of a normal disk D that crosses m interior edges of P_i:

$$A_c(D) = \frac{m\pi}{2} + \pi|D \cap \partial E(L)| - 2\pi.$$

Estimate of $-\chi(S')$

- We have
$$A_c(S') = -2\pi\chi(S')$$
- There is a notion of combinatorial length also due to Lackenby, such that
$$A_c(S') > \text{total length of } \partial S' \text{ on } \partial M_L.$$
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- Argument also goes through to estimate crosscap numbers of highly twisted knots in terms of twist number.
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- **Question:** Does the Jones polynomial (coarsely) determine the crosscap number of all knots?What about the Khovanov homology?