A refined upper bound for the volume of links and the colored Jones polynomial

Anastasiia Tsvietkova

University of California, Davis

Joint work with Oliver Dasbach.

Geometry of links and the Jones polynomial

In 1976, Thurston suggested the Geometrization conjecture, and demonstrated that many 3-manifolds either have hyperbolic metric or can be decomposed into pieces with hyperbolic metric (the Hyperbolization theorem). It allowed to study manifolds from a new perspective: using geometry. By the virtue of Mostow-Prasad rigidity, for a manifold with finite volume hyperbolic metric is unique as long as it is complete, and gives rise to numerous invariants.
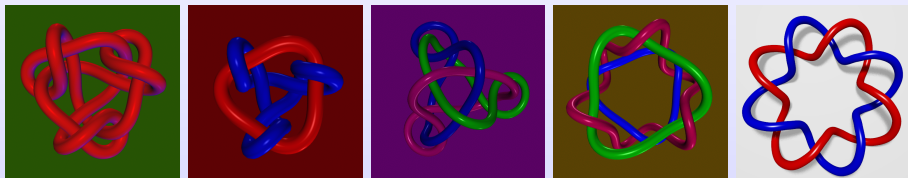

In 1984, Jones discovered the Jones polynomial for knots. To each oriented link, it assigns a Laurent polynomial with integer coefficients. The discovery stimulated a development of a new field of study: quantum invariants. Since quantum invariants were introduced into knot theory, there has been a strong interest in relating them to the intrinsic geometry of a link complement.

Links and their geometry

In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.
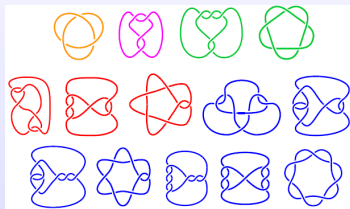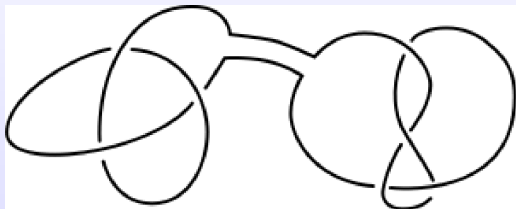
Links and their geometry

In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.
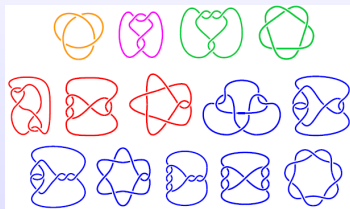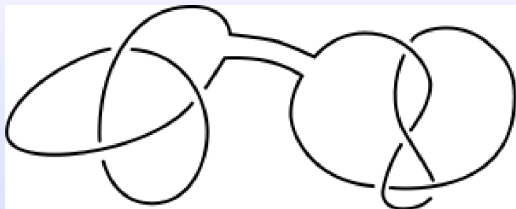
Every knot can be **uniquely decomposed** as a knot sum of **prime** knots (H. Schubert). It also true for non-split links, *i.e.* if there is no a 2–sphere in the complement separating the link (Hashizume).

Links and their geometry

In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.

Every knot can be **uniquely decomposed** as a knot sum of **prime** knots (H. Schubert). It also true for non-split links, *i.e.* if there is no a 2–sphere in the complement separating the link (Hashizume).

Hoste-Thistlethwaite-Weeks:
Of the 14 prime knots up to 7 crossings, only 3 are non-hyperbolic.

Links and their geometry

In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.
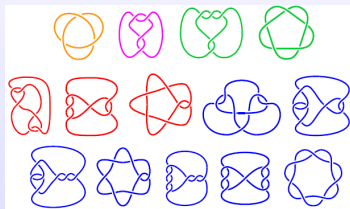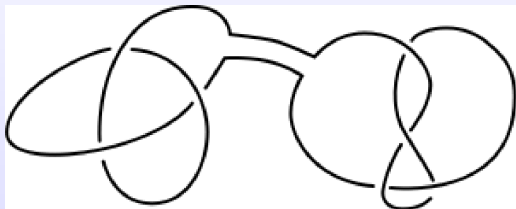

Every knot can be **uniquely decomposed** as a knot sum of **prime** knots (H. Schubert). It also true for non-split links, *i.e.* if there is no a 2–sphere in the complement separating the link (Hashizume).

Hoste-Thistlethwaite-Weeks:
Of the 14 prime knots up to 7 crossings, only 3 are non-hyperbolic.
Of the 1,701,935 prime knots up to 16 crossings, 32 are non-hyperbolic.

Links and their geometry

In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.

Every knot can be **uniquely decomposed** as a knot sum of **prime** knots (H. Schubert). It also true for non-split links, *i.e.* if there is no a 2–sphere in the complement separating the link (Hashizume).
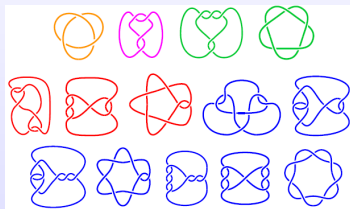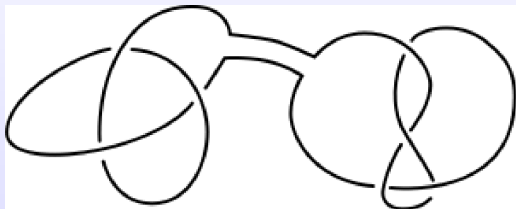
Hoste-Thistlethwaite-Weeks:
Of the 14 prime knots up to 7 crossings, only 3 are non-hyperbolic.
Of the 1,701,935 prime knots up to 16 crossings, 32 are non-hyperbolic.
Of the 8,053,378 prime knots with 17 crossings, 30 are non-hyperbolic.

Colored Jones polynomial

For a link K, the colored Jones polynomial $J(K, n)(q)$ is a sequence of Laurent polynomials in q indexed by a natural number n (color), with integer coefficients. Color $n = 2$ gives the classical Jones polynomial. We will write $J(K, n)(q)$ as

$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$$

Colored Jones polynomial

For a link K, the colored Jones polynomial $J(K, n)(q)$ is a sequence of Laurent polynomials in q indexed by a natural number n (color), with integer coefficients. Color $n = 2$ gives the classical Jones polynomial. We will write $J(K, n)(q)$ as
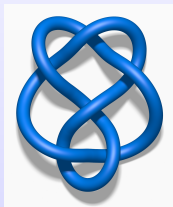
$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$$

Polynomial coefficients
$n = 2 : \{1, -2, 2, -2, 2, -1, 1\}$
$n = 3 : \{1, -2, 0, 4, ..., 3, -1, -1, 1\}$
$n = 4 : \{1, -2, 0, 2, ..., 0, -1, -1, 1\}$
....

Colored Jones polynomial

For a link K, the colored Jones polynomial $J(K, n)(q)$ is a sequence of Laurent polynomials in q indexed by a natural number n (color), with integer coefficients. Color $n = 2$ gives the classical Jones polynomial. We will write $J(K, n)(q)$ as

$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$$

Polynomial coefficients
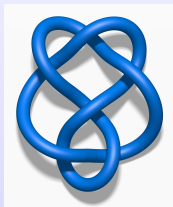$n = 2 : \{1, -2, 2, -2, 2, -1, 1\}$
$n = 3 : \{1, -2, 0, 4, ..., 3, -1, -1, 1\}$
$n = 4 : \{1, -2, 0, 2, ..., 0, -1, -1, 1\}$
....

There are various approaches to defining the colored Jones polynomial: e.g., via quantum groups and R-matrices (Turaev); via Kauffman bracket and the skein relation; via Chern-Simons theory (by Witten). None of these approaches provides a connection with the geometry of a link complement.

Links and their volume

To compute the hyperbolic volume of a link in S^3, one needs to find dihedral angles of every hyperbolic tetrahedron in a triangulation of the link complement. This can be done with the help of Thurston's gluing and completeness equations (used in the program SnapPea by J. Weeks) or with the help of alternative equations (Thistlethwaite-T.). If the angles are α_i, then the volume is the sum of $\Lambda(\alpha_i)$, where Λ is the Lobachevsky function. Either process does not give a simple expression for volume function in general.
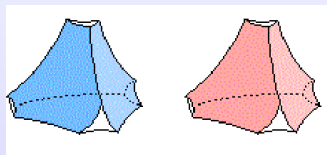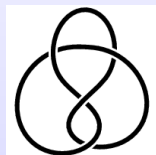
Gromov introduced a norm on the homology of a 3-manifold. The simplicial volume is v_3 times the Gromov norm, where v_3 is the volume of a regular ideal hyperbolic tetrahedron. For a hyperbolic 3-manifold, the simplicial volume is equal to the hyperbolic volume (Gromov-Thurston theorem). For an arbitrary 3-manifold, the simplicial volume is equal to the sum of the volumes of the hyperbolic pieces (another theorem) after a decomposition along essential spheres and tori.
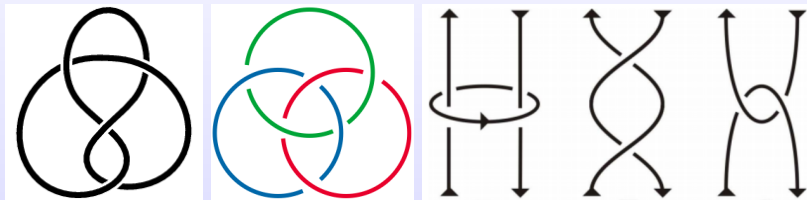
Volume Conjecture

(Kashaev 1997, J.Murakami-H.Murakami 2001).

For a hyperbolic link K, $\lim_{n \to \infty} \dfrac{2\pi \log |J(K, n)(e^{2\pi i/n})|}{n} = Vol(K)$. There is also a version of the conjecture that involves simplicial volume. In particular, simplicial volume of any knot is determined by its colored Jones polynomial.

Volume Conjecture

(Kashaev 1997, J.Murakami-H.Murakami 2001).

For a hyperbolic link K, $\lim_{n\to\infty} \frac{2\pi \log |J(K,n)(e^{2\pi i/n})|}{n} = Vol(K)$. There is also a version of the conjecture that involves simplicial volume. In particular, simplicial volume of any knot is determined by its colored Jones polynomial.

The conjecture was proved only for some hyperbolic knots and links: figure-eight knot (there are several proofs, e.g. by T. Ekholm), Borromean rings (Garoufalidis and Le), augmented octahedral links, Whitehead chains (van der Veen) and twisted Whitehead links (Zheng). Note: the links in these infinite families are commensurable with the Whitehead link, and so the volume is a rational multiple of the volume of the Whitehead link. Nothing is known about other infinite families and the conjecture - even a class of 2-bridge links remains a mystery in this regard.

Questions motivated by the Volume Conjecture

Is there a correlation of the polynomial coefficients with the hyperbolic volume?

Questions motivated by the Volume Conjecture

Is there a correlation of the polynomial coefficients with the hyperbolic volume? And what can be said about the coefficients themselves when $n \to \infty$?

Questions motivated by the Volume Conjecture
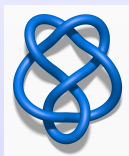
Is there a correlation of the polynomial coefficients with the hyperbolic volume? And what can be said about the coefficients themselves when $n \to \infty$?

(Thistlethwaite, 1986) For a link with a connected, irreducible, alternating diagram, the first and last coefficients (*i.e.* the coefficients of the terms of maximal and minimal degree) in the Jones polynomial are ± 1.

Questions motivated by the Volume Conjecture

Is there a correlation of the polynomial coefficients with the hyperbolic volume? And what can be said about the coefficients themselves when $n \to \infty$?

(Thistlethwaite, 1986) For a link with a connected, irreducible, alternating diagram, the first and last coefficients (*i.e.* the coefficients of the terms of maximal and minimal degree) in the Jones polynomial are ± 1.

Polynomial coefficients
$n = 2 : \{1, -2, 2, -2, 2, -1, 1\}$
$n = 3 : \{1, -2, 0, ..., -1, -1, 1\}$
$n = 4 : \{1, -2, 0, ..., -1, -1, 1\}$

Questions motivated by the Volume Conjecture

Is there a correlation of the polynomial coefficients with the hyperbolic volume? And what can be said about the coefficients themselves when $n \to \infty$?

(Thistlethwaite, 1986) For a link with a connected, irreducible, alternating diagram, the first and last coefficients (*i.e.* the coefficients of the terms of maximal and minimal degree) in the Jones polynomial are ± 1.

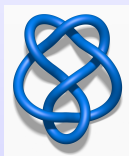Polynomial coefficients
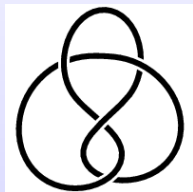$n = 2 : \{1, -2, 2, -2, 2, -1, 1\}$
$n = 3 : \{1, -2, 0, ..., -1, -1, 1\}$
$n = 4 : \{1, -2, 0, ..., -1, -1, 1\}$

(Dasbach and Lin, 2006) For an alternating link, the absolute values of the first three and last three coefficients of the colored Jones polynomial are independent of the color n when $n \geq 3$, and the second and penultimate coefficients are independent of n for $n \geq 2$. Moreover, the leading and trailing coefficients are known to be ± 1 for all n.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).
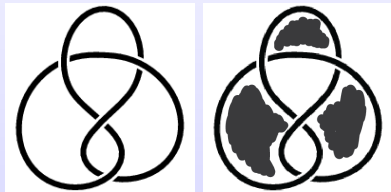
To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing. For a reduced graph, delete all multiple edges, leaving just one.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).

To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing. For a reduced graph, delete all multiple edges, leaving just one.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).
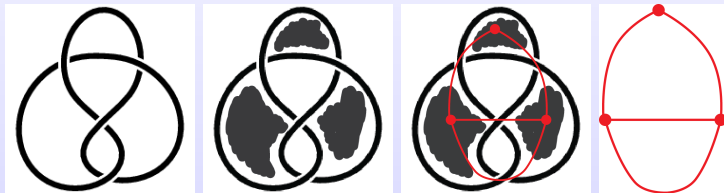
To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing. For a reduced graph, delete all multiple edges, leaving just one.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).
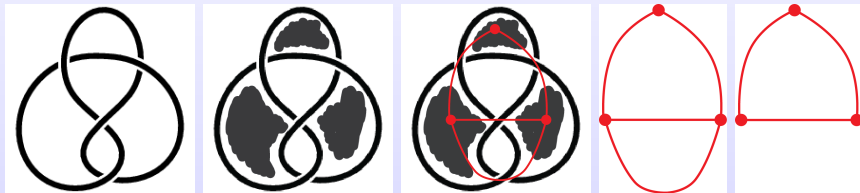
To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing. For a reduced graph, delete all multiple edges, leaving just one.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).

To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing. For a reduced graph, delete all multiple edges, leaving just one.
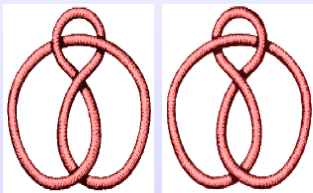

Note: computing further coefficients, starting from the third, is a harder task. The first two and last two coefficients of the colored Jones polynomial are determined by the coefficients of the classical Jones polynomial, unlike the rest. The classical Jones polynomial of an alternating link can be computed from the Tutte polynomial. However Tutte polynomial approach cannot be used for the colored Jones polynomial in general.

Further coefficients of the colored Jones polynomial

Therefore, one may use certain properties of a link diagram to compute (at least) the last and first three coefficients.

The tail (respectively head) of a polynomial is the sequence of its lowest (respectively highest) degree terms, up to some specified length. For a link L and its mirror image L^*, the colored Jones polynomial $J(L, n)(q) = J(L^*, n)(1/q)$, and therefore the head of L is the tail of L^*.

Further coefficients of the colored Jones polynomial

Therefore, one may use certain properties of a link diagram to compute (at least) the last and first three coefficients.

The tail (respectively head) of a polynomial is the sequence of its lowest (respectively highest) degree terms, up to some specified length. For a link L and its mirror image L^*, the colored Jones polynomial $J(L, n)(q) = J(L^*, n)(1/q)$, and therefore the head of L is the tail of L^*.
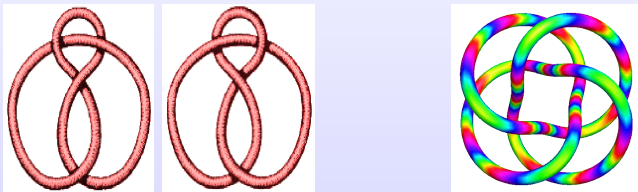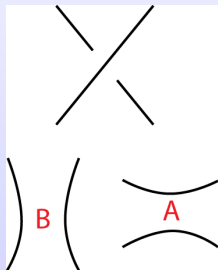

The head or tail does not always exist.

Example (Armond, Dasbach). The $(4, 3)$-torus knot has different tails and one head: one tail for even color n, and one tail for odd n. Its mirror image, respectively, has two heads.

Conjecture (Dasbach, Lin). The head and tail exist for alternating knots, *i.e.* all coefficients of $J(K, n)$ stabilize for n large enough.
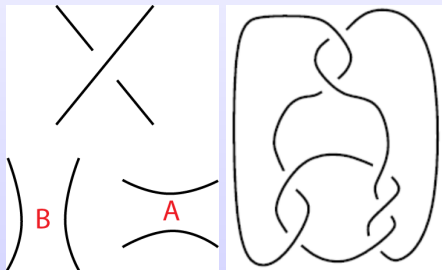
For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.
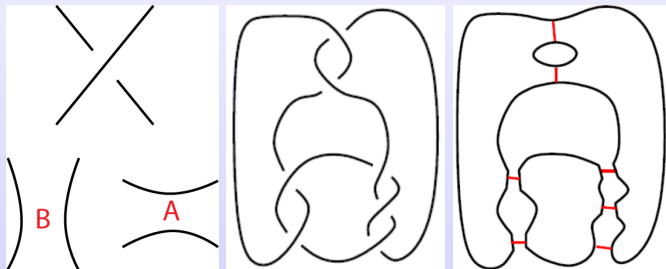
For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.
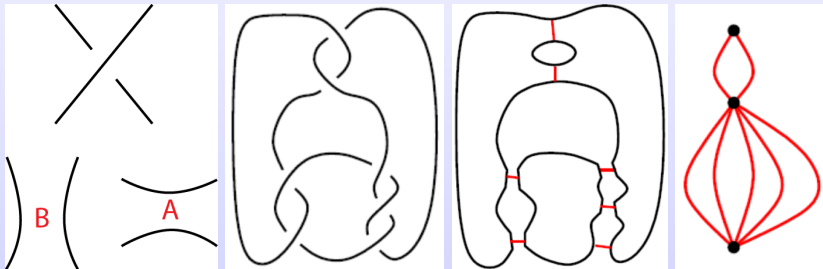
For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.
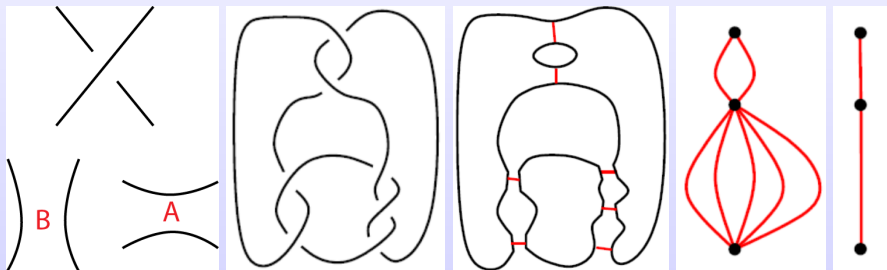
For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.
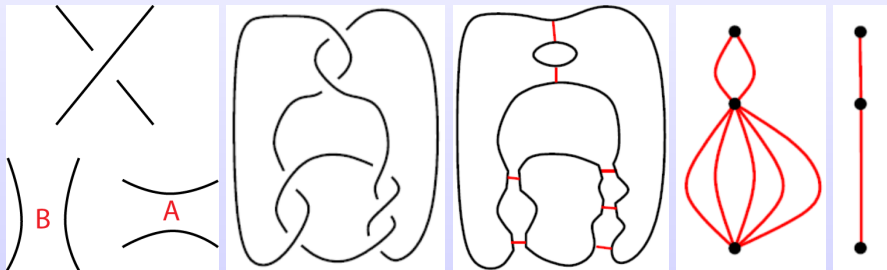
For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.

For which other links (beyond alternating) do head and tail exist? Checkerboard graphs and surfaces that are associated to alternating links have a natural generalization (suggested at different times by Stoimenov, Thistlethwaite).

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.

The existence of head and tail was proved by Armond for adequate links, using skein theoretical techniques (2011). Independently with different methods it was proved by Garoufalidis and Le for alternating links (2011).

Volume bounds for hyperbolic links

We saw that there is no simple expression for volume function in general. However, instead of the exact calculation, one may estimate volume from a diagram using volume bounds.

Theorem (Adams 1983; Lackenby 2004) For a hyperbolic link K with crossing number c, different from the figure-eight knot, $Vol(K) \leq (4c - 16)v_3$, where v_3 is the volume of a regular ideal hyperbolic tetrahedron.

Volume bounds for hyperbolic links

We saw that there is no simple expression for volume function in general. However, instead of the exact calculation, one may estimate volume from a diagram using volume bounds.

Theorem (Adams 1983; Lackenby 2004) For a hyperbolic link K with crossing number c, different from the figure-eight knot, $Vol(K) \leq (4c - 16)v_3$, where v_3 is the volume of a regular ideal hyperbolic tetrahedron.
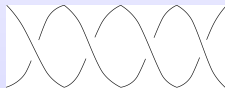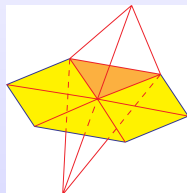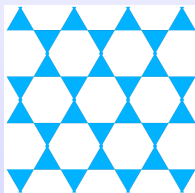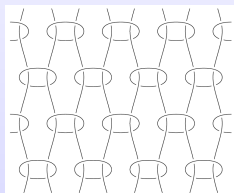

A twist is either a connected collection of bigons arranged in a row, which is not a part of a longer row of bigons, or a single crossing adjacent to no bigons. The twist number $t(D)$ is the number of twists in the diagram D.

Theorem (Lackenby 2004). For a hyperbolic link K with a prime alternating diagram D, $v_3(t(D) - 2)/2 \leq Vol(K) < v_3(16t(D) - 16)$.

Improved upper bound

Theorem (Agol and Thurston, 2004). Given a diagram D of a link K,
$Vol(K) \leq 10(t(D) - 1)v_3$.

This bound is asymptotically sharp and the constant 10 cannot be
improved for hyperbolic links in general. A chain fence link, which realizes
the upper bound exactly, and the corresponding (infinite) polyhedron:

Improved upper bound

Theorem (Agol and Thurston, 2004). Given a diagram D of a link K, $Vol(K) \leq 10(t(D) - 1)v_3$.
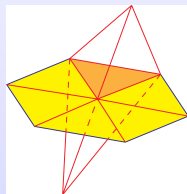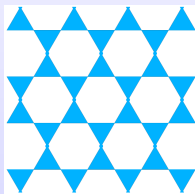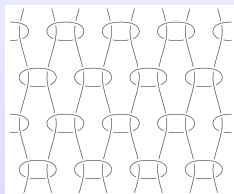
This bound is asymptotically sharp and the constant 10 cannot be improved for hyperbolic links in general. A chain fence link, which realizes the upper bound exactly, and the corresponding (infinite) polyhedron:

Bounds for some other families of hyperbolic links (e.g., lower bounds beyond alternating links) were obtained by Futer, Kalfagianni and Purcell.

Improved upper bound

Theorem (Agol and Thurston, 2004). Given a diagram D of a link K,
$$Vol(K) \leq 10(t(D) - 1)v_3.$$

This bound is asymptotically sharp and the constant 10 cannot be improved for hyperbolic links in general. A chain fence link, which realizes the upper bound exactly, and the corresponding (infinite) polyhedron:
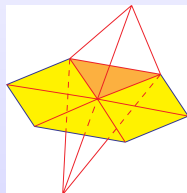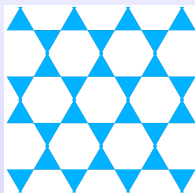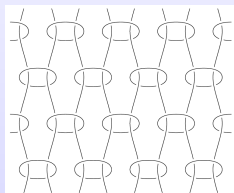

Bounds for some other families of hyperbolic links (e.g., lower bounds beyond alternating links) were obtained by Futer, Kalfagianni and Purcell.

Therefore, one may use certain properties of a link diagram to estimate the volume of the link complement.

Volumish Theorems

The above results were used to establish correlation of the first two and last two coefficients of the (colored) Jones polynomial with volume.

For color $n \geq 3$, let the colored Jones polynomial of K be
$$\pm(aq^{k_n} - bq^{k_n-1} + cq^{k_n-2}) + \ldots \pm (\gamma q^{k_n-r_n+2} - \beta q^{k_n-r_n+1} + \alpha q^{k_n-r_n})$$

Volumish Theorems

The above results were used to establish correlation of the first two and last two coefficients of the (colored) Jones polynomial with volume.

For color $n \geq 3$, let the colored Jones polynomial of K be
$$\pm(aq^{k_n} - bq^{k_n-1} + cq^{k_n-2}) + \ldots \pm (\gamma q^{k_n-r_n+2} - \beta q^{k_n-r_n+1} + \alpha q^{k_n-r_n})$$

Theorem (Dasbach and Lin, 2006) Let K be an alternating link, and b be the second coefficient of the colored Jones polynomial, and β - the penultimate coefficient. Then

$$v_8(max(|b|, |\beta|) - 1) \leq Vol(K) \leq 10v_3(|b| + |\beta| - 1),$$

where v_8 is volume of an ideal regular hyperbolic octahedron.

Volume bounds for various other families of hyperbolic links (beyond alternating) in terms of the first two and last two coefficients were obtained by Futer, Kalfagianni, Purcell and others. On the other hand, there is no single function of the second and the penultimate coefficient of the Jones polynomial that can control the volume of all hyperbolic knots (Futer-Kalfaganni-Purcell). In other words, these coefficients do not coarsely predict the volume of a knot.
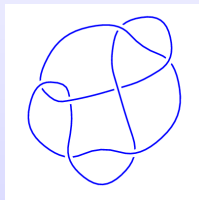
A refined approach to upper bound for volume

Let's look again at the upper bounds for the volume that were previously used to establish the correlation. There are links, for which the first gives a better estimate, and there are links, for which the second one does.
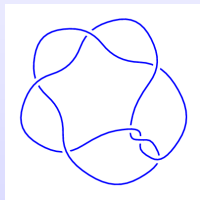
Adams, Lackenby
4 tetrahedra per crossing
$Vol(K) \leq (4c - 16)v_3$

Lackenby, Agol-Thurston
10 tetrahedra per twist
$Vol(K) \leq (10t - 10)v_3$

The constants 10 and 4 cannot be improved, but a "mix" of the two bounds might be closer to the actual volume function. The colored Jones polynomial point of view also suggests that a finer bound is needed to demonstrate closer correlation. Indeed, the twist number is the sum of just the second and penultimate coefficients of the colored Jones polynomial, and thus the above bounds cannot be used to involve further coefficients.

A refined approach to upper bound for volume

We will prove a refined upper bound from volume that has more parameters coming from a link diagram.

Let $t_i(D)$ be the number of twists that have exactly i crossings (i.e. $i - 1$ bigons), and $g_i(D)$ - the number of twists that have at least i crossings.
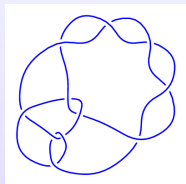
Twist number: $t = 5$
Twists with one crossing: $t_1 = 2$
Twists with two crossings: $t_2 = 2$
Twists with three crossings: $t_3 = 0$
Twists with at least four: $g_4 = 1$

A refined approach to upper bound for volume

We will prove a refined upper bound from volume that has more parameters coming from a link diagram.

Let $t_i(D)$ be the number of twists that have exactly i crossings (i.e. $i - 1$ bigons), and $g_i(D)$ - the number of twists that have at least i crossings.

Twist number: $t = 5$
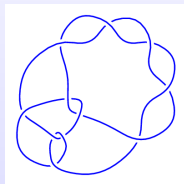Twists with one crossing: $t_1 = 2$
Twists with two crossings: $t_2 = 2$
Twists with three crossings: $t_3 = 0$
Twists with at least four: $g_4 = 1$

Theorem (Dasbach and T.). For a non-split link L with a diagram D, the simplicial volume $Vol(K) \leq (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$, where $A = 10$ if g_4 is non-zero, $A = 7$ if t_3 is non-zero, and $A = 6$ otherwise.

For links with one-crossing twists this is the Adams and Lackenby bound, and for links where all twists have at least four crossings, this is Lackenby and Agol-Thirston bound ($\pm$ a linear constant in both cases).

An improved Volumish theorem for alternating links

The refined upper bound for volume allowed to improve the Volumish theorem, showing for the first time that the first **three** and the last **three** coefficients correlate with the volume. For a link K, let the colored Jones polynomial (color n) be

$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n}).$$

An improved Volumish theorem for alternating links

The refined upper bound for volume allowed to improve the Volumish theorem, showing for the first time that the first **three** and the last **three** coefficients correlate with the volume. For a link K, let the colored Jones polynomial (color n) be

$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm (\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n}).$$
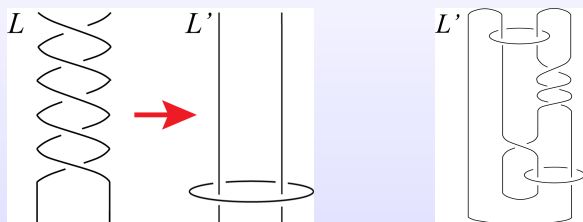
Theorem (Dasbach and T.). Let K be a hyperbolic alternating link. Then $Vol(K) \leq (6((c_2 + \gamma_2) - (c_3 + \gamma_3)) - 2(b_2 + \beta_2) - A) v_3$, where A is a linear constant taking values from 4 to 10 depending on the link diagram.

Note: it is not obvious from the expression, but it is an improvement over the bound in the Dasbach-Lin Volumish Theorem (can be proved using properties of $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$).

Example. The volume of the figure-8 is $2v_3$. The previous upper bound resulted in $10v_3$ for this link. The bound above gives $4v_3$.

Proof of the refined upper bound

First, pass from an original link L to the link L' whose volume is easier to bound. For this, augment every twist of L that has more than 3 crossings, and remove all crossings of that twist (shown below).

Claim 1. The (simplicial) volume of L is bounded above by the volume of L'.

Claim 2. The simplicial volume of L' is at most
$(10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$.

Proof of the refined upper bound

First, pass from an original link L to the link L' whose volume is easier to bound. For this, augment every twist of L that has more than 3 crossings, and remove all crossings of that twist (shown below).
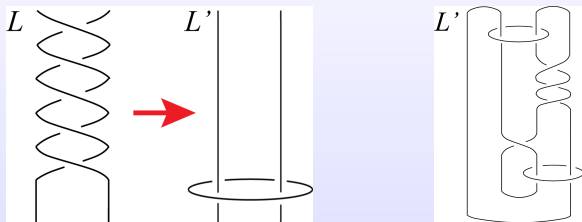

**Claim 1.** The (simplicial) volume of L is bounded above by the volume of L'.

**Claim 2.** The simplicial volume of L' is at most $(10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$.

Suppose we were only interested in hyperbolic volume. Even if we start with a hyperbolic link L, the above operation does not necessarily result in a hyperbolic link L' (unless L is an alternating link). Therefore, simplicial volume has to be used to prove the refined upper bound even for hyperbolic links.

Proof of the refined upper bound: main ideas

Claim 1. The volume of the original link L is bounded above by the volume of L'.

A *full twist* is a bigon. A *half-twist* is just a crossing. To proof the claim, we need to show that once we augment and remove crossings at long twists, we obtain a link with a larger volume. This holds for full twists, since a Dehn filling on a link A gives a link C, and the simplicial volume is reduced under the filling (for hyperbolic links, strictly reduced), as proved by Agol, Thurston. However this technique does not apply to half-twists. Rather, we have to generalize the following result from hyperbolic to simplicial volume.
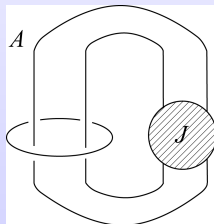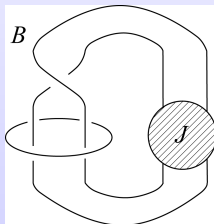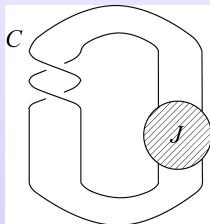

Theorem (Adams). Suppose J is an arbitrary 2-tangle, and the links A and B are hyperbolic. Then hyperbolic volumes of these links in S^3 are equal.

Proposition (generalization of Adams' result). Suppose J is an arbitrary 2-tangle, and the links A and B are non-split. Then simplicial volumes of these links in S^3 are equal.
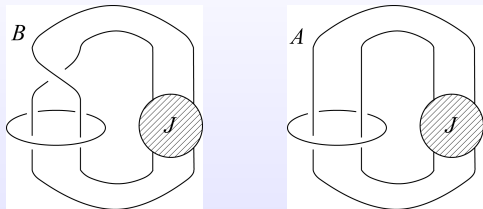

The original result holds due to the presence of a 3-punctured sphere in both link complements. To prove the generalization, we consider a decomposition of the link A by essential tori (one may think of the JSJ decomposition of an irreducible manifold).
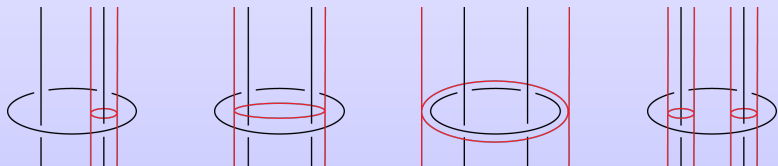
# Generalizing Adams' result to simplicial volume

An incompressible torus can intersect the three-punctured sphere only so that the meridian is parallel to one of the punctures. For most such intersection scenarios, one can find a three-punctured sphere along which the complement of the link A and the decomposing torus can be cut.
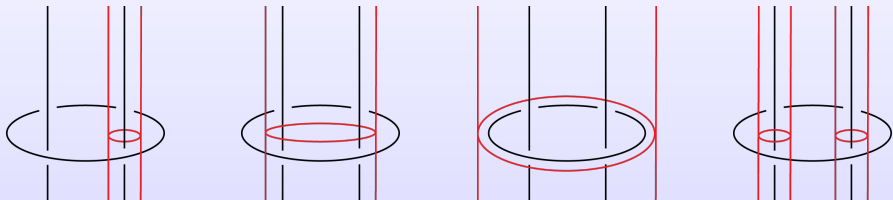
Then a half-twist is added to the link, and then both the link complement and the torus are re-glued back along the three-punctured sphere. The original Adams' result then implies that the volumes of the hyperbolic pieces are unchanged. For the pieces that are not hyperbolic, the simplicial volumes are 0. However, it is not always possible to re-glue the original torus back to itself.

Generalizing Adams' result to simplicial volume

When it is not possible to re-glue the decomposing torus to itself, construct a new decomposition of the link B. The decomposition does not look similar to the decomposition of A at first, but the volumes of the pieces are the same.
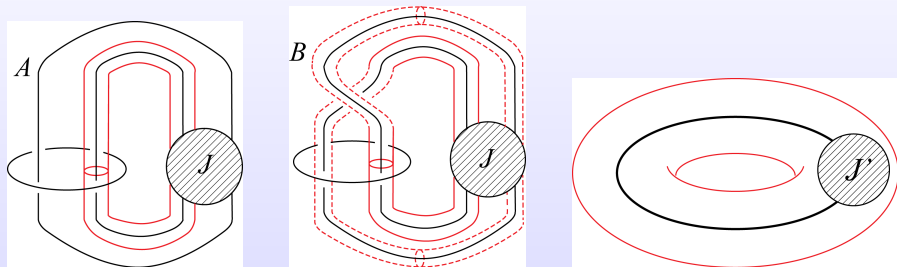
To illustrate this, consider the case depicted above (left), with the torus T_A in red. Let J' be a subtangle of J that is inside the torus. For the link B take another torus T_B (central figure). The new torus T_B partially coincides with T_A (as shown by solid red line) and is parallel to the boundary of B elsewhere (as shown by dotted red line). Inside the tori we have homeomorphic pieces of the decomposition with equal volumes (shown on the right). Similar analysis can be performed with the pieces outside the tori.

Proof of the refined upper bound: outline

Theorem (Dasbach and T.). For a non-split link L with a diagram D, the simplicial volume $Vol(K) \leq (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$, where $A = 10$ if g_4 is non-zero, $A = 7$ if t_3 is non-zero, and $A = 6$ otherwise.

Claim 1. The volume of L is bounded above by the volume of L'.

L'
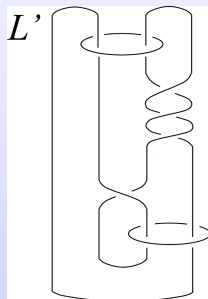
Proposition (generalization of Adams' result). Suppose J is an arbitrary 2-tangle, and the links A and B are non-split. Then simplicial volumes of these links in S^3 are equal. *Proved above*.

Note: L' is not necessarily non-split. Hence, before applying the Proposition, we need to consider a decomposition of L' by essential spheres.

Claim 2. The simplicial volume of the modified link, L', is at most $10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$.

Proof of the refined upper bound: ideas

Suppose we started with a non-split link L, but the partially augmented link L' is a split link. Then L' resulted from augmenting a composite link L with a decomposing 2-punctured sphere S.
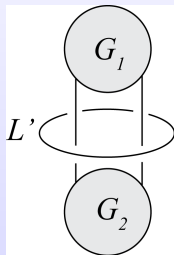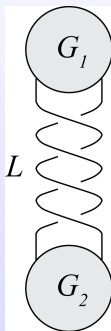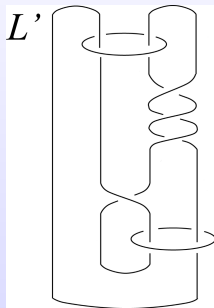
Proof of the refined upper bound: ideas

Suppose we started with a non-split link L, but the partially augmented link L' is a split link. Then L' resulted from augmenting a composite link L with a decomposing 2-punctured sphere S.
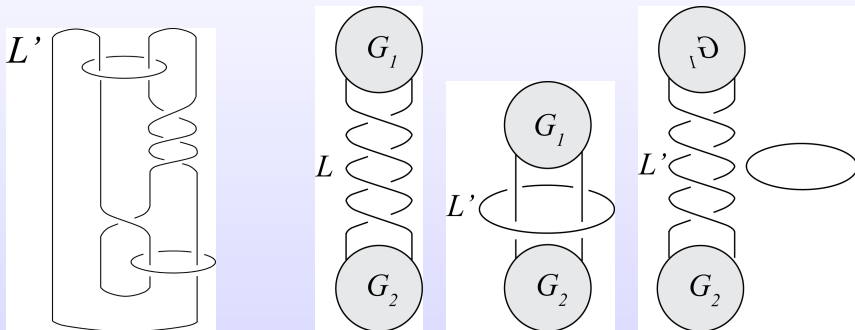
The augmentation takes place at a twist of L that is right next to S. Then L' splits into a link that looks like L or almost like L (a part of it is a mirror image of a part of L), and into several separate unknotted components. The separate unknotted components contribute 0 to the volume. And due to the additivity of simplicial volume under connected sum in dimension 3, the volume of L' is the same as the volume of L.

Proof of the refined upper bound: outline

Theorem (Dasbach and T.). For a non-split link L with a diagram D, the simplicial volume $Vol(K) \leq (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$, where $A = 10$ if g_4 is non-zero, $A = 7$ if t_3 is non-zero, and $A = 6$ otherwise.

> **Claim 1.** The volume of L is bounded above by the volume of L'. *Proved above.*

L'

Proposition (generalization of Adams' result). Suppose J is an arbitrary 2-tangle, and the links A and B are non-split. Then simplicial volumes of these links in S^3 are equal. *Proved above.*

Note: L' is not necessarily non-split. Hence, before considering a decomposition by tori, we need to consider a decomposition of L' by essential spheres. *Proved above.*

Claim 2. The simplicial volume of the modified link, L', is at most $10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$.

Claim 2. The simplicial volume of the modified link, L', is at most $(10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$.
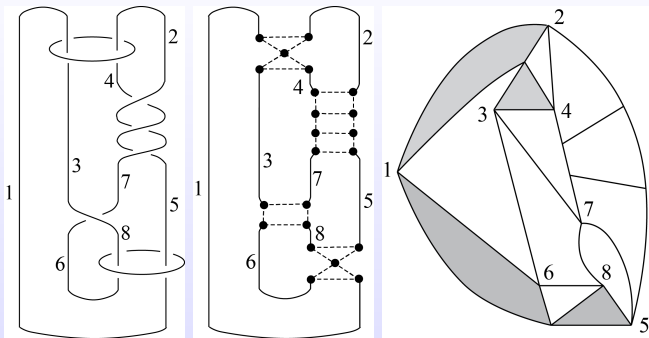
To prove it, perform a decomposition into two polyhedra, one above the projection plane, and one below. The decomposition is a "hybrid" of Menasco's decomposition for alternating links and Agol-Thurston decomposition for the links where every twist is augmented. In particular, there are two "bow-tie" triangular faces at every crossing circle, and a four-sided face at every twist. These faces share vertices (the vertices are labeled by numbers, and bow-ties are colored in gray on the left).

To obtain the complement of L', fold rectangular dotted faces, glue together two triangular dotted faces at every bow-tie, and double along the rest of the faces.

There are more faces in this decomposition than in other decompositions. However the rectangular dotted faces are folded in the end and do not add to the count, but help the gluing.
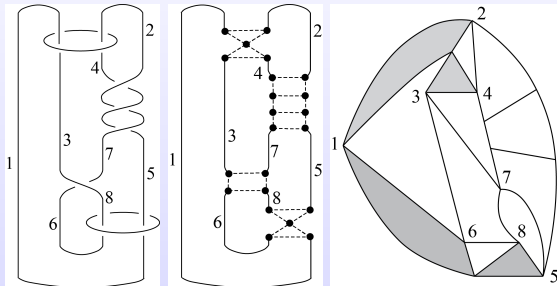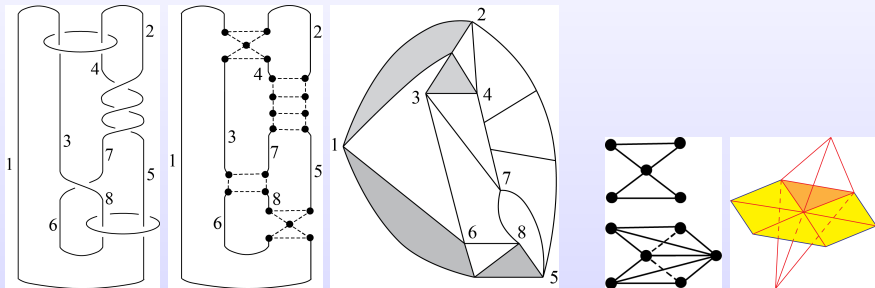
To obtain the complement of L', fold rectangular dotted faces, glue together two triangular dotted faces at every bow-tie, and double along the rest of the faces.

There are more faces in this decomposition than in other decompositions. However the rectangular dotted faces are folded in the end and do not add to the count, but help the gluing.

Perform a triangulation and count the maximal possible number of tetrahedra. Put two "extra" vertices in the link complement, one above the diagram, and one below. Connect them with the existing vertices so that each bow-tie face yield four tetrahedra (two above and two below), and other faces yield one tetrahedron per edge (stellar decomposition). This gives the upper bound. Lastly, collapsing the two extra vertices results in subtracting a linear constant.

Proof of the refined upper bound: outline

Theorem (Dasbach and T.). For a non-split link L with a diagram D, the simplicial volume $Vol(K) \leq (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$, where $A = 10$ if g_4 is non-zero, $A = 7$ if t_3 is non-zero, and $A = 6$ otherwise.

    Claim 1. The volume of L is bounded above by the volume of L'. *Proved above*.

L'

    Proposition (generalization of Adams' result). Suppose J is an arbitrary 2-tangle, and the links A and B are non-split. Then simplicial volumes of these links in S^3 are equal. *Proved above*.

    Note: L' is not necessarily non-split. Hence, before considering a decomposition by tori, we need to consider a decomposition of L' by essential spheres. *Proved above*.

    Claim 2. The simplicial volume of the modified link, L', is at most $10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$. *Proved above*.

Further questions

- The lower bound for the volume of alternating links was proved by Lackenby. Can one refine it or suggest another refined lower bound? Such a bound might be useful to show further correlation of the coefficients of the colored Jones polynomial and volume.

Further questions

- The lower bound for the volume of alternating links was proved by Lackenby. Can one refine it or suggest another refined lower bound? Such a bound might be useful to show further correlation of the coefficients of the colored Jones polynomial and volume.

- Can one involve three coefficients of the colored Jones polynomial for the links beyond alternating? Our refined upper bound for volume is for all links, but on the side of the Jones polynomial our current methods are limited to alternating links.

Further questions

- The lower bound for the volume of alternating links was proved by Lackenby. Can one refine it or suggest another refined lower bound? Such a bound might be useful to show further correlation of the coefficients of the colored Jones polynomial and volume.

- Can one involve three coefficients of the colored Jones polynomial for the links beyond alternating? Our refined upper bound for volume is for all links, but on the side of the Jones polynomial our current methods are limited to alternating links.

- For adequate links, other coefficients of the colored Jones polynomial stabilize for a color n large enough ("the head and tail" by Dasbach-Lin, Garoufalidis-Le, Armond). How are they related to the volume? Is a further refinement possible?

- How close can we get to the actual volume function using such methods? Futer, kalfagianni, Purcell: volume is not in general coarsely determined by any linear function of the second and penultimate coefficients. What about the third coefficient?

- How close can we get to the actual volume function using such methods? Futer, kalfagianni, Purcell: volume is not in general coarsely determined by any linear function of the second and penultimate coefficients. What about the third coefficient?
- Can one relate the coefficients to the volume through the exact computation rather than upper and lower bounds?

We obtained polynomials for two-bridge links that allow to compute volume exactly and from a link diagram (based on the alternative method for computing hyperbolic structure, Thistlethwaite-T.). E.g., for a twist knot with $k + 2$ crossings the tetrahedral shapes are (simple-looking) functions of the root of the polynomial

$$\sum_{j=0}^{n} \binom{2n-j}{j} w^{2j} + \sum_{j=0}^{n-1} \binom{2n-j-1}{k} w^{2j+1} = 0, \text{ where } n = k/2.$$

On the other side, Armond and Dasbach obtained formulas for the coefficients of the colored Jones polynomial of 2-bridge links.